

12th Edition

A Problem Solving Approach to

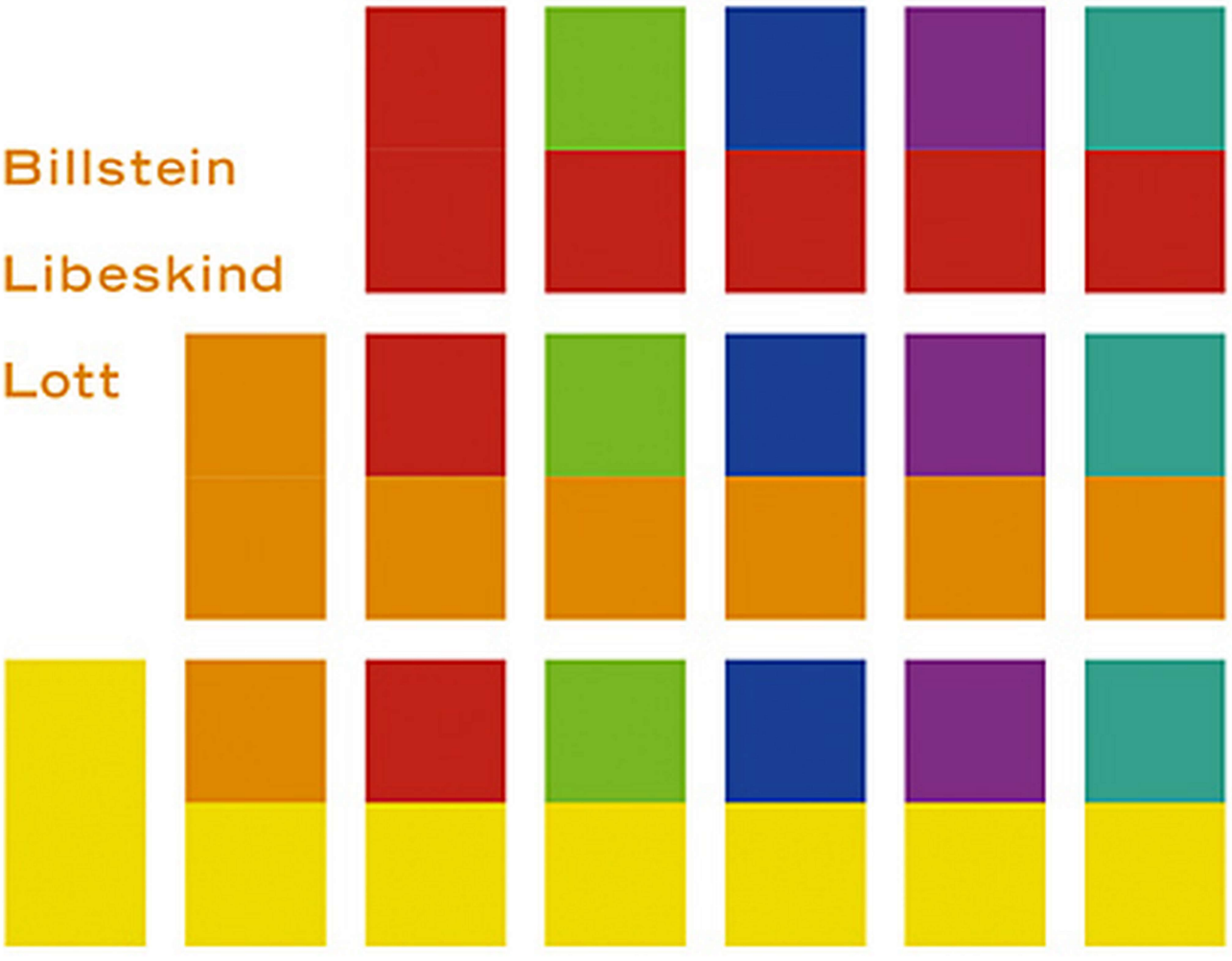
Mathematics

for Elementary School Teachers I

Billstein

Libeskind

Lott



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Chapter 1

An Introduction to Problem Solving

1-1 Mathematics and Problem Solving

1-2 Explorations with Patterns and Algebraic Thinking



Preliminary Problem

Jill received 10 boxes of coins, each box containing 10 identical looking coins. She knows that one box has 10 counterfeit coins, while all the other coins are genuine. She also knows that each fake coin weighs 1 ounce, while a real coin weighs 2 ounces. Jill has a scale and claims it is possible to determine which is the box with fake coins, in one weighing, as follows:

> "Number the boxes 1 through 10, and take 1 coin from the first box, 2 from the second, 3 from the third, and so on until 10 are taken from the last box. Next, I weigh all the coins taken out, and I can determine which box has the fake coins."

Explain why Jill's scheme would work.

If needed, see Hint on page 35.

Problem solving has long been central in the learning of mathematics at all levels. George Pólya (1887–1985), a great mathematician of the twentieth century, is the father of mathematical problem solving. He pointed out that "solving a problem means finding a way out of difficulty, a way around an obstacle, attaining an aim which was not immediately attainable." (Pólya 1981, p. ix)

IMAP Video
Watch Elsie talk about problem-solving with her students.

Polya developed a four-step problem solving process which has been adopted by many. A modified version is given here.

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

The *Common Core State Standards for Mathematics* (hereafter referred to as *Common Core Standards* and abbreviated as CCSS) were developed in 2010 through the work of the National Governors Association and the Council of Chief State School Officers. The *Common Core Standards* are built around its *Standards for Mathematical Practice* seen in Table 1.

Table 1

1. Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping in to a solution attempt.
2. Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations.
3. Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.
4. Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community.
5. Use appropriate tools strategically. Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software.
6. Attend to precision. Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem.
7. Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$.
8. Look for and express regularity in repeated reasoning. Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal.

The *Standards for Mathematical Practice* are used in this book to enhance Polya's four-step process as seen in Table 2.

Table 2

Expanded Four-Step Problem Solving Process with Input from *Standards for Mathematical Practice*

1. Understanding the problem.
- Start by explaining the personal meaning of a problem.
- Ask if the problem can be stated differently.
- Analyze goals to identify what is to be found and what is needed.
- Analyze the givens.
- Analyze the constraints.
- Ask what information is missing from the problem.
- Ask about missing or unneeded information in the problem.
- Make sense of quantities and their relationships in the problem situation.
- Look for discernable patterns or structures.

2. Devising a plan.
- Look for a pattern or a structure.
- Examine related or analogous problems and determine whether the same techniques applied to them can be applied to the current problem.
- Examine a simpler or special case of the problem to gain insight into the solution of the original problem.
- Make a table or list.
- Identify a subgoal.
- Make a diagram.
- Use guess and check.
- Work backward.
- Write an equation.
- Abstract a given situation and represent it symbolically.
- Plan a solution pathway.
- Make assumptions and approximations to simplify a complicated situation.
- Use clear definitions.

3. Carrying out the plan.
- State the meaning of any symbols used.
- Manipulate the representing symbols as if they have a life of their own.
- Implement the strategy or strategies in step 2 and perform any necessary actions or computations.
- Attend to the precision in language and mathematics used.
- Apply the mathematics to solve problems.
- Check each step of the plan along the way—this may be intuitive checking or formal proof of each step.
- Keep an accurate record of all work.
- Map relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
- Use appropriate tools strategically.
- Look for general methods and for shortcuts to calculations.
- Detect possible errors using estimation and other mathematical knowledge.
- Specify units of measure.

4. Looking back.
- Check the results in the original problem (in some cases this will require a proof).
- Interpret the solution in terms of the original problem: Does the answer make sense? Does it answer the question that was asked?
- Determine whether there is another method of finding the solution.
- Improve the model if it has not served its purpose.
- Maintain oversight of the process.
- Evaluate the reasonableness of intermediate results.
- Check answers with a different method.
- Continually ask: "Does this make sense?"
- Understand different approaches.
- Identify correspondences among different approaches.
- Justify conclusions.
- Communicate conclusions to others.
- Respond to arguments of others.
- If possible, determine other related or more general problems for which the technique will work.

Students learn mathematics as a result of solving problems. *Exercises* are routine practice for skill building and serve a purpose in learning mathematics, but problem solving must be a focus of school mathematics. A reasonable amount of tension and discomfort improves problem-solving performance.

Mathematical problem solving may occur when:

1. Students are presented with a situation that they understand but do not know how to proceed directly to a solution.
2. Students are interested in finding the solution and attempt to do so.
3. Students are required to use mathematical ideas to solve the problem.

We present many opportunities in this text to solve problems. Each chapter opens with a problem that can be solved using concepts developed in the chapter. We give a hint for the solution to the problem at the end of each chapter. Throughout the text, some problems are solved using a four-step process.

Working with others to solve problems enhances problem-solving ability and communication skills. We encourage *cooperative learning* and working in groups whenever possible. To encourage group work and help identify when cooperative learning could be useful, we identify activities and problems where group discussions are especially beneficial for learning mathematics.

1-1 Mathematics and Problem Solving

1-1 Objectives

Students will be able to understand and explain

- The four-step problem-solving process.
- How to solve problems using various problem-solving strategies.

If problems are approached in only one way, a mind-set may be formed. For example, consider the following:

> *Spell the word* spot *three times out loud. "S-P-O-T! S-P-O-T! S-P-O-T!" Now answer the question "What do we do when we come to a green light?" Write an answer.*

If we answer "Stop," we may be guilty of having formed a mind-set. We do not stop at a *green* light.

Consider the following problem: "A shepherd had 36 sheep. All but 10 died. How many lived?" If we answer "10," we are ready to try some problems. If not, we probably did not understand the question by not reading it carefully. *Understanding the problem* is the first step in the four-step problem-solving process.

IMAP Video

Watch Tonya's class model strategies for solving a problem.

Strategies for Problem Solving

We next provide a variety of problems with different contexts to provide experience in problem solving. Strategies are used to discover or construct the means to achieve a solution. For each strategy described, we give an example that can be solved with that strategy. Often, problems can be solved in more than one way. There is no one best strategy to use.

In many of the examples, we use the set of **natural numbers**, 1, 2, 3, ... Note that the first three dots, an *ellipsis*, are used to represent missing terms. The expanded problem-solving steps highlighting some strategies are shown next.

Historical Note



George Pólya (1887–1985) was born in Hungary, moved to the United States in 1940, and after a brief stay at Brown University, joined the faculty at Stanford University. A preeminent mathematician, he also focused on mathematics education. He published 10 books, including *How To Solve It* (1945), which has been translated into 23 languages.

Strategy: Look for a Pattern

IMAP Video
Watch a fourth grade class model Gauss's strategy.

Problem Solving Gauss's Problem

As a student, Carl Gauss and his fellow classmates were asked to find the sum of the first 100 natural numbers. The teacher expected to keep the class occupied for some time, but Gauss gave the answer almost immediately. How might he have done it?

Understanding the Problem The natural numbers are $1, 2, 3, 4, \ldots$. Thus, the problem is to find the sum $1 + 2 + 3 + 4 + \ldots + 100$.

Devising a Plan The strategy *look for a pattern* is useful here. One story about young Gauss reports that he listed the sum, and wrote the same sum backwards as in Figure 1. If $S = 1 + 2 + 3 + 4 + 5 + \ldots + 98 + 99 + 100$, then Gauss could have seen the following pattern.

$$\begin{aligned} S &= 1 + 2 + 3 + 4 + 5 + \ldots + 98 + 99 + 100 \\ +\ S &= 100 + 99 + 98 + 97 + 96 + \ldots + 3 + 2 + 1 \\ \hline 2S &= 101 + 101 + 101 + 101 + 101 + \ldots + 101 + 101 + 101 \end{aligned}$$

Figure 1

To discover the original sum from the last equation, Gauss could have divided the sum, $2S$, in Figure 1 by 2.

Carrying Out the Plan There are 100 sums of 101. Thus, $2S = 100 \cdot 101$ and $S = \dfrac{100 \cdot 101}{2} = 5050$.

Looking Back Note that the sum in each pair $(1, 100), (2, 99), (3, 98), \ldots, (100, 1)$ is always 101, and there are 100 pairs with this sum. This technique can be used to solve a more general problem of finding the sum of the first n natural numbers $1 + 2 + 3 + 4 + 5 + 6 + \ldots + n$. We use the same plan as before and notice the relationship in Figure 2. Because there are n sums of $n + 1$ we have $2S = n(n + 1)$ and $S = \dfrac{n(n + 1)}{2}$.

$$\begin{aligned} S &= 1 + 2 + 3 + 4 + \ldots + n \\ +\ S &= n + (n-1) + (n-2) + (n-3) + \ldots + 1 \\ \hline 2S &= (n+1) + (n+1) + (n+1) + (n+1) + \ldots + (n+1) \end{aligned}$$

Figure 2

A different strategy for finding a sum of consecutive natural numbers involves the strategy of *making a diagram* and thinking of the sum geometrically as a stack of blocks. This alternative method is explored in exercise 2 of Assessment 1-1A.

NOW TRY THIS 1

Explain whether the approach in Gauss's Problem of writing the sum backwards and applying the strategy "Look for a Pattern" will or will not work in finding the following sum: $1^2 + 2^2 + \ldots + 100^2$.

Historical Note



Carl Gauss (1777–1855), one of the greatest mathematicians of all time, was born to humble parents in Brunswick, Germany. He was an infant prodigy who later made contributions in many areas of science as well as mathematics. After Gauss's death, the King of Hanover honored him with a commemorative medal with the inscription "Prince of Mathematics." ●

Strategy: Examine a Related Problem

Problem Solving **Sums of Even Natural Numbers**

Find the sum of the even natural numbers less than or equal to 100. Generalize the result.

Understanding the Problem Even natural numbers are 2, 4, 6, 8, 10, ... The problem is to find the sum of these numbers: $2 + 4 + 6 + 8 + \ldots + 100$.

Devising a Plan Recognizing that the sum can be *related to Gauss's original problem* helps us devise a plan. Consider the following:

$$\begin{aligned} 2 + 4 + 6 + 8 + \ldots + 100 &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + \ldots + 2 \cdot 50 \\ &= 2(1 + 2 + 3 + 4 + \ldots + 50) \end{aligned}$$

Thus, we can use Gauss's method to find the sum of the first 50 natural numbers and then double that result.

Carrying Out the Plan We carry out the plan as follows:

$$\begin{aligned} 2 + 4 + 6 + 8 + \ldots + 100 &= 2(1 + 2 + 3 + 4 + \ldots + 50) \\ &= 2\left[\frac{50(50 + 1)}{2}\right] \\ &= 2550 \end{aligned}$$

Thus, the sum of the even natural numbers less than or equal to 100 is 2550.

Looking Back A different way to approach this problem is to realize that there are 25 sums of 102, as shown in Figure 3. (Why are there 25 sums to consider, and why is the sum in each pair always 102?)

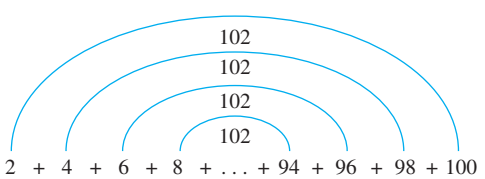


Figure 3

Thus, the sum is $25 \cdot 102 = 2550$.

The numbers 2, 4, 6, 8, . . . , 100 are an example of an *arithmetic sequence*—an ordered list of numbers, or *terms*, in which each term starting from the second one differs from the previous term by the same amount—the common difference. The common difference in the above sequence is 2.

NOW TRY THIS 2

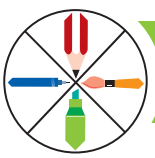
Find the sum of consecutive natural numbers shown: $25 + 26 + 27 + \ldots + 120$. Solve this problem in two different ways.

NOW TRY THIS 3

Each of 16 people in a round-robin handball tournament played each other person exactly once. How many games were played?

Strategies: Examine a Simpler Case; Make a Table

Often used strategies in problem solving are *examine a simpler case* and *make a table*. A table can be used to look for patterns that emerge in the problem, which in turn can lead to a solution. An example of these strategies is shown on the grade 4 student page below.



School Book Page Examine a Simpler Case

Lesson 15-5

Common Core

4.MD.2 Use the four operations to solve world problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Problem Solving

Solve a Simpler Problem and Make a Table

Each side of a triangle cracker below is one inch long. If there are 12 triangle crackers in a row, what is the perimeter of the figure?



Read and Understand

Change the problem into problems that are simpler to solve.

Look at 1 triangle, then 2 triangles, then 3 triangles.



perimeter = 3 inches

perimeter = 4 inches

perimeter = 5 inches

Plan

The perimeter is 2 more than the number of triangles.

Number of triangles	1	2	3
Perimeter (inches)	3	4	5

So, for 12 triangles the perimeter is 14 inches.

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Strategy: Identify a Subgoal

In attempting to devise a plan for solving a problem, a solution to a somewhat easier or more familiar related problem could make it easier. In such a case, finding the solution to the easier problem may become a *subgoal*. The magic square problem on page 8 shows an example of this.

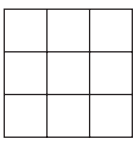


Figure 4

Problem Solving A Magic Square

Arrange the numbers 1 through 9 into a square subdivided into nine smaller squares like the one shown in Figure 4 so that the sum of every row, column, and major diagonal is the same. The result is a *magic square*.

Understanding the Problem Each of the nine numbers $1, 2, 3, \ldots, 9$ must be placed in the small squares, a different number in each square, so that the sums of the numbers in each row, in each column, and in each of the two major diagonals are the same.

Devising a Plan If we knew the fixed sum of the numbers in each row, column, and diagonal, we would have a better idea of which numbers can appear together in a single row, column, or diagonal. Thus *the subgoal* is to find that fixed sum. The sum of the nine numbers, $1 + 2 + 3 + \ldots + 9$, equals 3 times the sum in one row. (Why?) Consequently, the fixed sum can be found using the process developed by Gauss. We have $\frac{1 + 2 + 3 + \ldots + 9}{3} = \frac{(9 \cdot 10) \div 2}{3} = 15$, so the sum in each row, column, and diagonal must be 15. Next, we need to decide what numbers could occupy the various squares. The number in the center space will appear in four sums, each adding to 15 (two diagonals, the second row, and the second column). Each number in the corners will appear in three sums of 15. (Why?) If we write 15 as a sum of three different numbers 1 through 9 in all possible ways, we could then count how many sums contain each of the numbers 1 through 9. The numbers that appear in at least four sums are candidates for placement in the center square, whereas the numbers that appear in at least three sums are candidates for the corner squares. Thus the new *subgoal* is to write 15 in as many ways as possible as a sum of three different numbers from $1, 2, 3, \ldots, 9$.

Carrying Out the Plan The sums of 15 can be written systematically as follows:

$$9 + 5 + 1$$
$$9 + 4 + 2$$
$$8 + 6 + 1$$
$$8 + 5 + 2$$
$$8 + 4 + 3$$
$$7 + 6 + 2$$
$$7 + 5 + 3$$
$$6 + 5 + 4$$

Note that the order of the numbers in sums like $9 + 5 + 1$ is irrelevant because the order in which additions are done does not matter. In the list, 1 appears in only two sums, 2 in three sums, 3 in two sums, and so on. Table 3 summarizes this information.

Table 3

Number	1	2	3	4	5	6	7	8	9
Number of sums containing the number	2	3	2	3	4	3	2	3	2

The only number that appears in four sums is 5; hence, 5 must be in the center of the square. (Why?) Because 2, 4, 6, and 8 appear 3 times each, they must go in the corners. Suppose we choose 2 for the upper left corner. Then 8 must be in the lower right corner. This is shown in Figure 5(a). Now we could place 6 in the lower left corner or upper right corner. If we choose the upper right corner, we obtain the result in Figure 5(b). The magic square can now be completed, as shown in Figure 5(c).

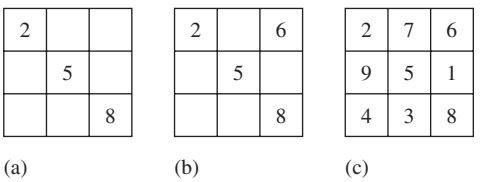


Figure 5

Looking Back We have seen that 5 was the only number among the given numbers that could appear in the center. However, we had various choices for a corner, and so it seems that the magic square we found is not the only one possible. Can you find all the others?

Another way to see that 5 could be in the center square is to consider the sums $1 + 9, 2 + 8, 3 + 7, 4 + 6$, as shown in Figure 6. We could add 5 to each to obtain 15.

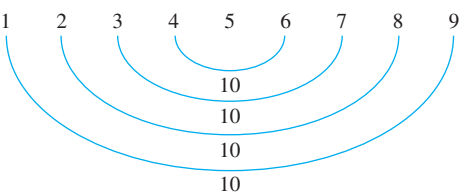


Figure 6

Strategy: Make a Diagram

In the following problem, *making a diagram* helps us to understand the problem and work toward a solution.

Problem Solving 50-m Race Problem

Bill and Jim ran a 50-m race three times. The speed of the runners did not vary. In the first race, Jim was at the 45-m mark when Bill crossed the finish line.

a. In the second race, Jim started 5 m ahead of Bill, who lined up at the starting line. Who won?
b. In the third race, Jim started at the starting line and Bill started 5 m behind. Who won?

Understanding the Problem When Bill and Jim ran a 50-m race, Bill won by 5 m; that is, whenever Bill covered 50 m, at the same time Jim covered only 45 m. If Bill started at the starting line and Jim started at the 5-m line or if Jim started at the starting line and Bill started 5 m behind, we are to determine who would win in each case.

Devising a Plan A strategy to determine the winner under each condition is to *make a diagram*. A diagram for the first 50-m race is given in Figure 7(a). In this case, Bill won by 5 m. In the second race, Jim had a 5-m head start and hence when Bill ran 50 m to the finish line, Jim ran only 45 m. Because Jim is 45 m from the finish line, he reached the finish line at the same time as Bill did. This is shown in Figure 7(b). In the third race, because Bill started 5 m behind, we use Figure 7(a) but move Bill back 5 m, as shown in Figure 7(c). From the diagram we determine the results in each case.

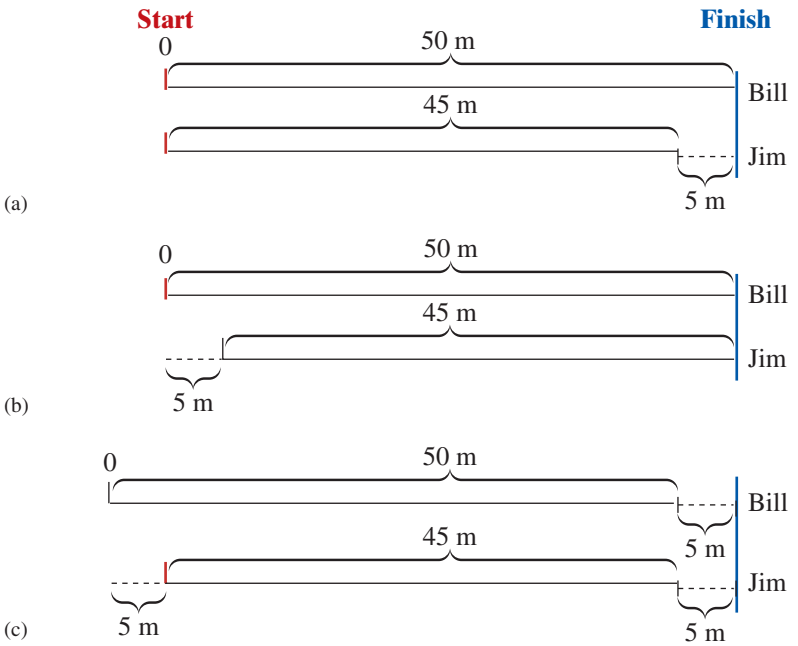


Figure 7

Carrying Out the Plan From Figure 7(b) we see that if Jim had a 5-m head start, then the race ends in a tie. If Bill started 5 m behind Jim, then at 45 m they would be tied. Because Bill is faster than Jim, Bill would cover the last 5 m faster than Jim and win the race.

Looking Back The diagrams show that the solution makes sense and is appropriate. Other problems can be investigated involving racing and handicaps. For example, if Bill and Jim run on a 50-m oval track, how many laps will it take for Bill to lead Jim by one full lap? (Assume the same speeds as earlier.)

NOW TRY THIS 4

An elevator stopped at the middle floor of a building. It then moved up 4 floors, stopped, moved down 6 floors, stopped, and then moved up 10 floors and stopped. The elevator was now 3 floors from the top floor. How many floors does the building have?

IMAP Video
Watch Arriel use the guess and check strategy.

Strategy: Use Guess and Check

In the strategy of *guess and check*, we first guess at an answer using as reasonable a guess as possible. Then we check to see whether the guess is correct. If not, the next step is to learn as much as possible about the answer based on the guess before making a next guess. This strategy can be regarded as a form of trial and error, in which the information about the error helps us choose what to try next. The guess-and-check strategy is often used when a student does not know how to solve the problem more efficiently or if the student does not yet have the tools to solve the problem in a faster way. Research has suggested that students in grades 1–3 rely primarily on a *guess-and-check* strategy when faced with a mathematical problem. In grades 6–12 this tendency decreases. Older students benefit more from the observed "errors" after a guess when formulating a new "trial."

The grade 4 student page shown on page 11 gives an example of this strategy, referred to as "try, check, revise."

School Book Page — Try, Check, Revise

Try, Check, Revise

Suzanne spent $27, not including tax, on dog supplies. She bought two of one item and one of another item. What did she buy?

$8 + $8 + $15 = $31
$7 + $7 + $12 = $26
$6 + $6 + $15 = $27

Use Try, Check, Revise when quantities are being combined to find a total, but you don't know which quantities.



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Strategy: Work Backward

In some problems, it is easier to start with the result and to *work backward*. This is demonstrated on the grade 6 student page below.

School Book Page — Work Backward

Test-Taking Strategies

Working Backward

The problem-solving strategy *Work Backward* is useful when taking multiple-choice tests. Work backward by testing each choice in the original problem. You will eliminate incorrect answers. Eventually you will find the correct answer.

EXAMPLE

A fruit stand is selling 8 bananas for $1.25. At this rate, how much will 24 bananas cost?

(A) $1.50 (B) $2.50 (C) $3.75 (D) $5.00

Use mental math to test the choices that are easy to use.

$2.50 is twice $1.25. Twice 8 is only 16, so choice B is not the answer.

$5.00 is four times $1.25. Three times 8 is 24, so choice D is not the correct answer.

Since 24 is between 16 and 32, the cost must be between $2.50 and $5.00. The correct answer is choice C.

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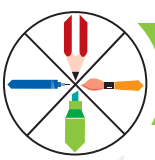
Strategy: Write an Equation

Even though algebraic thinking is involved in the strategy *writing an equation* and may evoke thoughts of traditional algebra, a closer look reveals that algebraic thinking starts very early in students' school lives. For example, finding the missing subtrahend in a problem like

$$\begin{array}{r} 14 \\ -\square \\ \hline 3 \end{array}$$

could be thought of algebraically as $14 - \square = 3$, or as $3 + \square = 14$. In a traditional algebra course, this might be seen as $14 - x = 3$ or $3 + x = 14$ with 11 as a solution. We use such algebraic thinking long before formal algebra is taught. And the strategy of *writing an equation* is used in this text before more formal approaches to algebra are seen in Chapter 8, *Real Numbers and Algebra*.

A student example of *writing an equation* to solve a problem is seen on the grade 6 student page below.



School Book Page Writing an Equation

GPS Guided Problem Solving

Writing Equations to Solve Problems

Around the World On March 4, 2005, Steve Fossett set a record by completing a nonstop solo jet flight around the world. He landed with 1,515 pounds of fuel. A fuel leak caused a loss of 2,600 pounds of fuel. He started with 18,100 pounds of fuel. How much fuel did the jet use on the flight?

What You Might Think	What You Might Write
What do I know? What am I trying to find?	• Fuel remaining = 1,515 pounds • Fuel lost = 2,600 pounds • Total fuel at start = 18,100 pounds • How much fuel (in pounds) was actually used on the flight?
How can I show the main idea?	Draw a diagram.
What equation can I write to show the problem?	$1{,}515 + 2{,}600 + x = 18{,}100$
How do I solve the problem?	$4{,}115 + x = 18{,}100$ $4{,}115 - 4{,}115 + x = 18{,}100 - 4{,}115$ $x = 13{,}985$
What is the answer?	A total of 13,985 pounds of fuel was used on the flight.

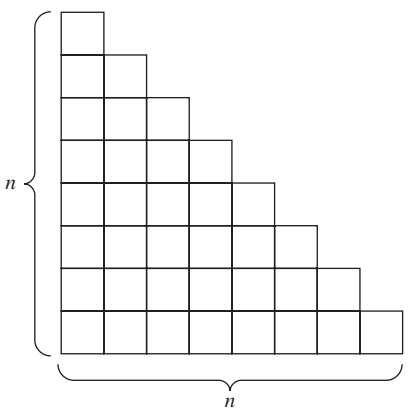
18,100 lb		
1,515 lb	2,600 lb	x

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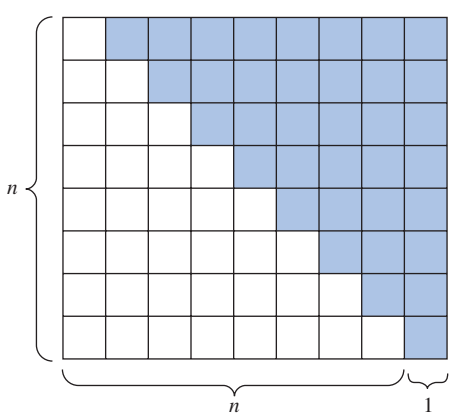
Assessment 1-1A

1. Use the approach in Gauss's Problem to find the following sums of arithmetic sequences (do not use formulas):
- **a.** $1 + 2 + 3 + 4 + \ldots + 99$
- **b.** $1 + 3 + 5 + 7 + \ldots + 1001$
- **c.** $3 + 6 + 9 + 12 + \ldots + 300$
- **d.** $4 + 8 + 12 + 16 + \ldots + 400$

2. Use the ideas in drawings (a) and (b) to find the solution to Gauss's Problem for the sum $1 + 2 + 3 + \ldots + n$. Explain your reasoning.



(a)



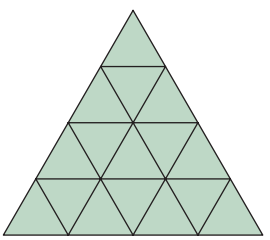
(b)

3. Find the sum $36 + 37 + 38 + 39 + \ldots + 146 + 147$.

4. Cookies are sold singly or in packages of 2 or 6. With this packaging, how many ways can you buy
- **a.** 10 cookies?
- **b.** a dozen cookies?

5. In a big, red box, there are 7 smaller blue boxes. In each of the blue boxes, there are 7 black boxes. In each of the black boxes, there are 7 yellow boxes. In each of those yellow boxes, there are 7 tiny, gold boxes. How many boxes are there altogether? Explain your answer.

6. How many triangles are in the following figure?



7. Without computing each sum of the arithmetic sequence, find which is greater, O or E, and by how much.

$$O = 1 + 3 + 5 + 7 + \ldots + 97$$
$$E = 2 + 4 + 6 + 8 + \ldots + 98$$

8. Alababa, Bubba, Cory, and Dandy are in a horse race. Bubba is the slowest; Cory is faster than Alababa but slower than Dandy. Name the finishing order of the horses.

9. How many ways can you make change for a $50 bill using $5, $10, and $20 bills?

10. The following is a magic square (all rows, columns, and diagonals sum to the same number). Find the value of each letter.

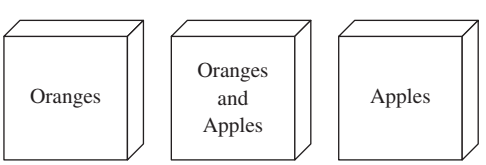
17	a	7
12	22	b
c	d	27

11. Debbie and Amy began reading a novel on the same day. Debbie reads 9 pages a day and Amy reads 6 pages a day. If Debbie is on page 72, on what page is Amy?

12. The 14 digits of a credit card are written in the boxes shown. If the sum of any three consecutive digits is 20, what is the value of A?

A		7									7		4

13. Three closed boxes (A, B, and C) of fruit arrive as a gift from a friend. Each box is mislabeled. How could you choose only one fruit from one box to decide how the boxes should be labeled?

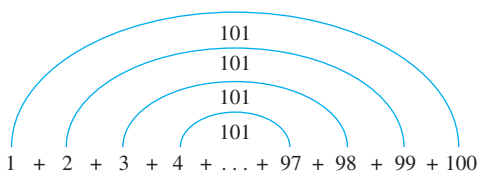


14. An electrician charges $50 per hour and spends $15 a day on gasoline. If she netted $1315 in 4 days, how many hours did she work?

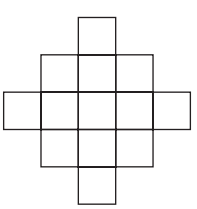
15. Kathy stood on the middle rung of a ladder. She climbed up three rungs, moved down five rungs, and then climbed up seven rungs. Then she climbed up the remaining six rungs to the top of the ladder. How many rungs are there in the whole ladder?

Assessment 1-1B

1. Use the approach in Gauss's Problem to find the following sums of arithmetic sequences (do not use formulas):
 a. $1 + 2 + 3 + 4 + \ldots + 49$
 b. $1 + 3 + 5 + 7 + \ldots + 2009$
 c. $6 + 12 + 18 + \ldots + 600$
 d. $1000 + 995 + 990 + \ldots + 5$
2. Use the diagram below to explain how to find the sum of
 a. the first 100 natural numbers.



 b. $1 + 2 + 3 + 4 + \ldots + 201$
3. Find the sum of $58 + 59 + 60 + 61 + \ldots + 203$.
4. Eve Merriam* entitled her children's book *12 Ways to Get to 11* (1993). Using only addition and natural numbers, describe 12 ways that one can arrive at the sum of 11.
5. Explain why in a drawer containing only two different colors of socks one must draw only three socks to find a matching pair.
6. How many squares are in the following figure?



7. If $P = 1 + 3 + 5 + 7 + \ldots + 99$ and $Q = 5 + 7 + 9 + \ldots + 101$ are sums of arithmetic sequences, determine which is greater, P or Q, and by how much.
8. The sign says that you are leaving Missoula, Butte is 120 mi away, and Bozeman is 200 mi away. There is a rest stop halfway between Butte and Bozeman. How far is the rest stop from Missoula if both Butte and Bozeman are in the same direction?
9. Marc goes to the store with exactly $1.00 in change. He has at least one of each coin less than a half-dollar coin, but he does not have a half-dollar coin.
 a. What is the least number of coins he could have?
 b. What is the greatest number of coins he could have?
10. Find a 3-by-3 magic square using the numbers 3, 5, 7, 9, 11, 13, 15, 17, and 19.
11. Eight marbles look alike, but one is slightly heavier than the others. Using a balance scale, explain how you can determine the heavier one in exactly three weighings.
12. Recall the song "The Twelve Days of Christmas":

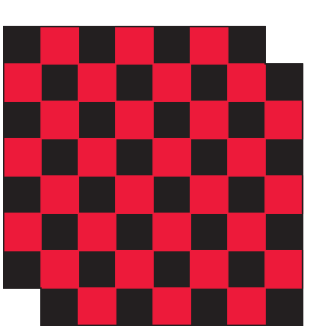
 On the first day of Christmas my true love gave to me a partridge in a pear tree.
 On the second day of Christmas my true love gave to me two turtle doves and a partridge in a pear tree.
 On the third day of Christmas my true love gave to me three French hens, two turtle doves, and a partridge in a pear tree.

 This pattern continues for 9 more days. After 12 days,
 a. which gifts did my true love give the most? (Yes, you have to remember the song.)
 b. how many total gifts did my true love give to me?
13. a. Suppose you have quarters, dimes, and pennies with a total value of $1.19. How many of each coin can you have without being able to make change for a dollar?
 b. Tell why one of the combinations of coin you have in part (a) is the least number of coins that you can have without being able to make change for a dollar.
14. Suppose you buy lunch for the math club. You have enough money to buy 20 salads or 15 sandwiches. The group wants 12 sandwiches. How many salads can you buy?
15. One winter night the temperature fell 15 degrees between midnight and 5 A.M. By 9 A.M., the temperature had doubled from what it was at 5 A.M. By noon, it had risen another 10 degrees to 32 degrees. What was the temperature at midnight?

Mathematical Connections 1-1

Reasoning

1. Create a 3-by-3 magic square using nine of the ten numbers 20, 21, 22, 23, 24, 25, 26, 27, 28, and 29. Explain your solution and reasoning. List the strategies you have used.
2. In the checkerboard, two squares on opposite corners have been removed. A domino can cover two adjacent squares on the board. Can dominoes be arranged in such a way that all the remaining squares on the board can be covered with no dominoes overlapping or hanging off the board? If not, why not? (Hint: Each domino must cover one black and one red square. Compare this with the number of each color of squares on the board.)



3. a. If 10 people shake hands with one another exactly once, how many handshakes take place?
 b. Find as many ways as possible to do the problem.
 c. Generalize the solution for n people.
4. Consider a game where you have two distinct piles of objects. Two players alternate moves, each player taking any number (not 0) of objects desired from a single pile. The player who

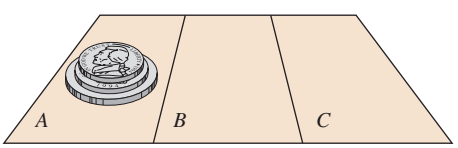
*Merriam, E. *12 Ways to Get to 11*. New York: Aladdin Paperbacks, 1993.

takes the last object (so nothing is left in either pile) is the winner. If there are a objects in one pile and b objects in the second, we write (a, b). Given the game $(1, 1)$, the first player loses because if she takes 1 from a pile, the second player takes 1 from the other pile. Answer the following:

a. Show that the first player can win the game $(1, 2)$ as well as $(1, 100)$.
b. Who will win the game $(1, a)$ if $a > 1$? Why?
c. Which player will win the games $(2, 2)$, $(3, 3)$, $(4, 4)$? Why?
d. Which games can the first player always win? Why?

5. Place a half-dollar, a quarter, and a nickel in position A as shown in the figure below. Try to move these coins, one at a time, to position C. At no time may a larger coin be placed on a smaller coin. Coins may be placed in position B.

a. How many moves does it take to get them to position C?
b. Now add a penny to the pile and see how many moves are required. This is a simple case of the famous Tower of Hanoi problem, in which ancient Brahman priests were required to move a pile of 64 disks of decreasing size, after which the world would end. How long would it take at a rate of one move per second?



Open-Ended

6. Choose a problem-solving strategy and make up a problem that would use this strategy. Write the solution using Pólya's four-step approach.
7. The distance around the world is approximately 40,000 km. Approximately how many people of average size would it take to stretch around the world if they were holding hands?

Cooperative Learning

8. Work in pairs on the following versions of a game called NIM. A calculator is needed for each pair.

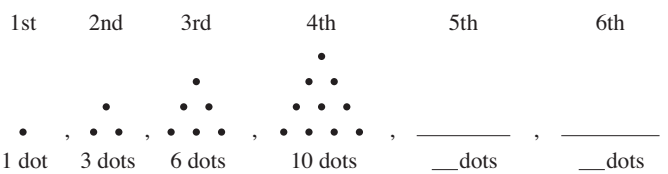
a. Player 1 presses [1] and [+] or [2] and [+]. Player 2 does the same. The players take turns until the target number of 21 is reached. The first player to make the display read 21 is the winner. Determine a strategy for deciding who always wins.
b. Try a game of NIM using the digits 1, 2, 3, and 4, with a target number of 104. The first player to reach 104 wins. What is the winning strategy?
c. Try a game of NIM using the digits 3, 5, and 7, with a target number of 73. The first player to exceed 73 loses. What is the winning strategy?
d. Now play Reverse NIM with the keys [1] and [2]. Instead of [+], use [−]. Put 21 on the display. Let the target number be 0. Determine a strategy for winning Reverse NIM.
e. Try Reverse NIM using the digits 1, 2, and 3 and starting with 24 on the display. The target number is 0. What is the winning strategy?
f. Try Reverse NIM using the digits 3, 5, and 7 and starting with 73 on the display. The first player to display a negative number loses. What is the winning strategy?

Connecting Mathematics to the Classroom

9. John asks why the last step of Pólya's four-step problem-solving process, *looking back*, is necessary since he has already given the answer. What could you tell him?
10. A student asks why he just can't make "random guesses" rather than "intelligent guesses" when using the guess-and-check problem-solving strategy. How do you respond?
11. Rob says that it is possible to create a magic square with the numbers 1, 3, 4, 5, 6, 7, 8, 9, and 10. How do you respond?

National Assessments

National Assessment of Educational Progress (NAEP) Questions



A pattern of dots is shown above. How many dots would be in the 6th picture?

Answer: _____

Explain how you found your answer.

NAEP, Grade 4, 2009

Susie said, "I have 83¢ but fewer than 10 coins." Show in the chart how many of each coin she could have to total 83¢.

Total number of coins	 25¢	 10¢	 5¢	 1¢

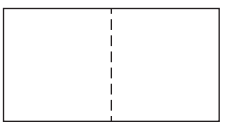
NAEP, Grade 4, 2009

Ms. Kim has 45 stickers that she wants to give out to 6 students. The students are sitting in a circle. Ms. Kim gives out one sticker at a time and keeps going around the circle until all the stickers are gone. How many of the students will get more than 7 stickers?

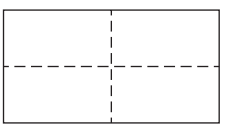
A. 2
B. 3
C. 5
D. 6

NAEP, Grade 4, 2011

Sam folds a piece of paper in half once. There are 2 sections.



Sam folds the paper in half again. There are 4 sections.



Sam folds the paper in half again. There are 8 sections.

Sam folds the paper in half 2 more times.

Which list shows the number of sections each time Sam folds the paper?

A. 2, 4, 8, 10, 12
B. 2, 4, 8, 12, 24
C. 2, 4, 8, 16, 24
D. 2, 4, 8, 16, 32

NAEP, Grade 4, 2011

1-2 Explorations with Patterns

1-2 Objectives

Students will be able to understand and explain

- Finding patterns and determining if a pattern holds.
- Deductive and inductive reasoning and when to use them.
- Different types of sequences, such as arithmetic, geometric, and Fibonacci.
- Finding the *n*th term of certain sequences.
- Using differences to find a pattern for neither arithmetic nor geometric sequences.

Mathematics has been described as the study of patterns. Patterns are everywhere—in wallpaper, tiles, traffic, and even television schedules. Police investigators study case files to find the *modus operandi*, or pattern of operation, when a series of crimes are committed. Scientists look for patterns to isolate variables so that they can reach valid conclusions in their research.

Non-numerical patterns abound. For young children, a pattern could appear in non-numerical form as shown in Now Try This 5.

NOW TRY THIS 5

a. Find three more terms to continue a pattern:

$$\bigcirc, \triangle, \triangle, \bigcirc, \triangle, \triangle, \bigcirc, __, __, __$$

b. Describe in words the pattern found in part (a).

Patterns can be surprising as seen in the following example.

Example 1

a. Describe any patterns seen in the following:

$$1 + 0 \cdot 9 = 1$$
$$2 + 1 \cdot 9 = 11$$
$$3 + 12 \cdot 9 = 111$$
$$4 + 123 \cdot 9 = 1111$$
$$5 + 1234 \cdot 9 = 11111$$

b. Do the patterns continue? Why or why not?

E-Manipulative Activity
Additional practice with patterns not involving numbers can be found in the *Patterns* activity on the E-Manipulatives disk (or MML). The activity involves completing simple and complex patterns of symbols and colored blocks.

Solution

a. There are several possible patterns. For example, the numbers on the far left are natural numbers. The pattern starts with 1 and continues to the next greater natural number in each successive line. The numbers "in the middle" are products of two numbers, the second of which is 9; the left-most number in the first product is 0; after that the left-most number in each product is formed using successive natural numbers as digits, including an additional digit in each successive line. The five computations given above result in the numbers that are formed using 1s and include an additional 1 in each successive line.

b. The pattern in the complete equation appears to continue for a number of cases, but it does not continue in general; for example,

$$13 + 123456789101112 \cdot 9 = 1{,}111{,}111{,}101{,}910{,}021.$$

This pattern breaks down when the pattern of digits in the number being multiplied by 9 contains previously used digits.

As seen in Example 1, determining a pattern on the basis of a few cases is not reliable. For all patterns found, we should either show the pattern does not hold in general or justify that the pattern always works. Reasoning is used in both cases.

Reasoning

CCSS **Some books list various types of reasoning as a problem-solving strategies. However, we think that reasoning underlies problem solving. The *Common Core Standards for Mathematical Practice* lists the following:**

- **Reason abstractly and quantitatively.**
- **Construct viable arguments and critique the reasoning of others.**
- **Look for and express regularity in repeated reasoning. (p. 1)**

For students to recognize reasoning and proof as fundamental aspects of mathematics, it is necessary that they use both reasoning and proof in their studies. However, it must be recognized that the level of use depends on the grade level of the students and their understanding of mathematics. For example, from very early ages, students use *inductive reasoning* to look for regularities in patterns based on a very few cases and to develop **conjectures**—statements or conclusions that have not been proven. **Inductive reasoning** is the method of making generalizations based on observations and patterns. Such reasoning may or may not be valid, and conjectures based on inductive reasoning may or may not be true. The validity, or truth, of conjectures in mathematics relies on **deductive reasoning**—the use of definitions, undefined terms, mathematical axioms that are assumed to be true, and previously proved theorems, together with logic to prove these conjectures.

Throughout mathematics, there is a fine interweaving of inductive reasoning and conjecturing to develop conclusions thought to be true. Deductive reasoning is required to prove those statements. In this section we show how inductive reasoning may lead to false conclusions or false conjectures. We show how deductive reasoning is used to prove true conjectures.

Inductive and Deductive Reasoning

Scientists make observations and propose general laws based on patterns. Statisticians use patterns when they draw conclusions based on collected data. This process of *inductive reasoning* may lead to new discoveries; its weakness is that conclusions are drawn only from the collected evidence. If not all cases have been checked, another case may prove the conclusion false. For example, considering only that $0^2 = 0$ and that $1^2 = 1$, we might conjecture that *every number*

squared is equal to itself. However, $2^2 \neq 2$. Thus we found an example that contradicts the conjecture. Such an example is a **counterexample**; it shows that the conjecture is false. Sometimes finding a counterexample is difficult, but not finding one immediately does not prove a conjecture is true. A dramatic example of a conjecture that holds true for a very large number of cases but still fails to be true for all cases involves the concept of *perfect squares.* A natural number that is a square of some natural number is a perfect square. For example, 9 is a perfect square because $9 = 3^2$. The conjecture that $1 + 1141n^2$ is never a perfect square is true for every natural number n until $n = 30{,}693{,}385{,}322{,}765{,}657{,}197{,}397{,}208$ when it fails.

Next, consider a pattern that does work and helps solve a problem. How can you find the sum of three consecutive natural numbers without performing the addition? Three examples are given below.

$$\begin{aligned} &14 + 15 + 16; && \text{Sum} = 45 \\ &19 + 20 + 21; && \text{Sum} = 60 \\ &99 + 100 + 101; && \text{Sum} = 300 \end{aligned}$$

After studying the sums, a pattern of multiplying the middle number by 3 emerges. The pattern suggests other mathematical questions to consider. For example,

1. Does this work for any 3 consecutive natural numbers?
2. How can we find the sum of any odd number of consecutive natural numbers?
3. What happens if there is an even number of consecutive natural numbers?

To answer the first question, we give a proof showing that the sum of three consecutive natural numbers is equal to 3 times the middle number. This proof is an example of *deductive reasoning*.

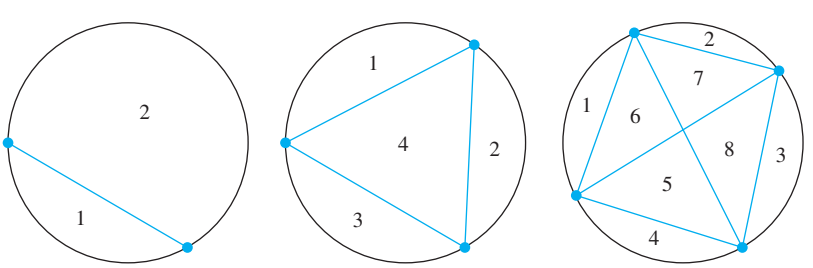
Proof

Let n be the first of three consecutive natural numbers. Then the three numbers are n, $n + 1$, and $n + 2$. The sum of these three numbers is $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$. Therefore, the sum of the three consecutive natural numbers is three times the middle number.

A somewhat different way is to let the middle number be m. Then the three consecutive numbers are $m - 1, m$ and $m + 1$. Their sum is $(m - 1) + m + (m + 1) = 3m$, that is, 3 times the middle number. ■

The Danger of Making Conjectures Based on a Few Cases

The following discussion illustrates the danger of making a conjecture based on a few cases. In Figure 8, we choose points on a circle and connect them to form distinct, nonoverlapping regions. In this figure, 2 points determine 2 regions, 3 points determine 4 regions, and 4 points determine 8 regions. What is the maximum number of regions that would be determined by 10 points?



(a) 2 points, 2 regions (b) 3 points, 4 regions (c) 4 points, 8 regions

Figure 8

The data from Figure 8 are recorded in Table 4. It appears that each time the number of points increases by 1, the number of regions doubles. If this were true, then for 5 points we would

have determined the number of regions by doubling the number of regions with 4 points, or $2 \cdot 8 = 16 = 2^4$, and so on. If we base our conjecture on this pattern, we might believe that for 10 points, we would have 2^9, or 512 regions. (Why?)

Table 4

Number of points	2	3	4	5	6	. . .	10
Maximum number of regions	2	4	8				?

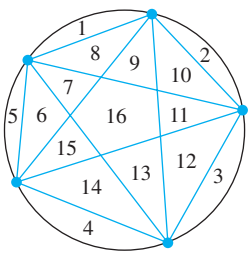


Figure 9

An initial check for this conjecture is to see whether we obtain 16 regions for 5 points. The diagram for 5 points in Figure 9 confirms our guess of 16 regions. For 6 points, the pattern predicts that the number of regions is 32. Choosing the points so that they are neither symmetrically arranged nor equally spaced and counting the regions carefully, we get 31 regions, not 32 as predicted. No matter how the points are located on the circle, the guess of 32 regions is not correct. This counterexample tells us that the doubling pattern is not correct. Note that it does not tell us whether there are 512 regions with 10 points; it tells us only that this conjecture is not true.

NOW TRY THIS 6

A *prime number* is a natural number with exactly two distinct positive numbers, 1 and the number itself, that divide it with 0 remainder; for example, 2, 3, 5, 7, 11, and 13 are primes. One day Amy makes a *conjecture* that the formula $y = x^2 + x + 11$ will produce only prime numbers if she substitutes the natural numbers, $1, 2, 3, 4, 5, \ldots$ for x. She shows her work in Table 5 for $x = 1, 2, 3, 4$.

Table 5

x	1	2	3	4
y	13	17	23	31

a. What type of reasoning is Amy using?
b. Try the next several natural numbers and see whether they seem to work.
c. Show that Amy's conjecture is false for $x = 11$.

Arithmetic Sequences

A **sequence** is an ordered arrangement of numbers, figures, or objects. A sequence has items or *terms* identified as *1st*, *2nd*, *3rd*, and so on. Often, sequences can be classified by their properties. For example, what property do the following first three sequences have that the fourth does not?

a. $1, 2, 3, 4, 5, 6, \ldots$
b. $0, 5, 10, 15, 20, 25, \ldots$
c. $2, 6, 10, 14, 18, 22, \ldots$
d. $1, 11, 111, 1111, 11111, 111111, \ldots$

In each of the first three sequences, each term—starting from the second term—is obtained from the preceding term by adding a fixed number, the **common difference** or **difference**. In part (a) the difference is 1, in part (b) the difference is 5, and in part (c) the difference is 4. Sequences such as the first three are arithmetic sequences. An **arithmetic sequence** is a sequence in which each successive term from the second term on is obtained from the previous term by the addition

or subtraction of a fixed number. The sequence in part (d) is not arithmetic because there is no single fixed number that can be added to or subtracted from the previous term to obtain the next term.

It is convenient to denote the terms of a sequence by a single letter with a subscript. For example in the sequence (b) above $a_1 = 0, a_2 = 5, a_3 = 10$, and so on. Arithmetic sequences can be generated from objects, as shown in Example 2.

Example 2

Find a numerical pattern in the number of matchsticks required to continue the sequence shown in Figure 10 if each subsequent figure has one more square.

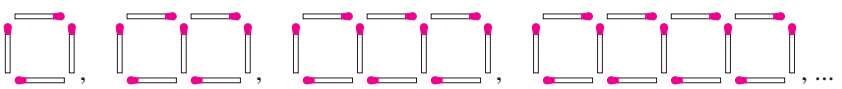


Figure 10

Solution Assume the matchsticks are arranged, so each figure has one more square on the right than the preceding figure. Note that the addition of a square to an arrangement requires the addition of three matchsticks each time. Thus, with this assumption, the numerical pattern obtained is 4, 7, 10, 13, 16, 19, . . . , an arithmetic sequence starting at 4 and having a difference of 3.

An informal description of an arithmetic sequence is one that can be described as an "add d" pattern, where d is the common difference. In Example 2, $d = 3$. In the language of children, the pattern in Example 2 is "add 3." This is an example of a **recursive pattern**. In a recursive pattern, after one or more consecutive terms are given, each successive term of the sequence is obtained from the previous term(s). For example, 11, 14, 17, . . . is another "add 3" sequence starting with 11.

A recursive pattern is typically used in a spreadsheet, as seen in Table 6, where the index column tracks the order of the terms. The headers for the columns are A, B, and so on. The first entry in the A column (in the A1 cell) is 4; to find the term in the A2 cell, we use the number in the A1 cell and add 3. The pattern is continued using the *Fill Down* command. In spreadsheet language, the formula = A1 + 3 finds any term after the first by adding 3 to the previous term and using *Fill Down*. A formula based on a recursive pattern is a **recursive formula**. (For more explicit directions on using a spreadsheet, see the Technology Manual, which can be found online at www.pearsonhighered.com/Billstein12einfo.)

Table 6

Index Column →

	A	B
1	4	
2	7	
3	10	
4	13	
5	16	
6	19	
7	22	
8	25	
9	28	
10		
11		
12		
13		

If we want to find the number of matchsticks in the 100th figure in Example 2, we use the spreadsheet or we find an explicit formula or a general rule for finding the number of matchsticks when given the position of the term, the term number. The problem-solving strategy *making a table* is again helpful here.

The spreadsheet in Table 6 provides an easy way to *make a table*. The index column gives the term numbers, and column A gives the terms of the sequence. If we are building such a table without a spreadsheet, it might look like Table 7. Notice that each term is a sum of 4 and a certain number of 3s. We see that the number of 3s is 1 less than the term number. This pattern should continue, since the first term is $4 + 0 \cdot 3$ and each time we increase the number of the term by 1, we add one *more* 3. Thus, it seems that the 100th term is $4 + (100 - 1)3$; and, in general, the ***n*th term** is $4 + (n - 1)3$. Note that $4 + (n - 1)3$ could be written as $3n + 1$.

Table 7

Term Number	Term
1	$4 = 4 + 0 \cdot 3$
2	$7 = 4 + 3 = 4 + 1 \cdot 3$
3	$10 = (4 + 1 \cdot 3) + 3 = 4 + 2 \cdot 3$
4	$13 = (4 + 2 \cdot 3) + 3 = 4 + 3 \cdot 3$
.	.
.	.
.	.
n	$4 + (n - 1)3 = 3n + 1$

Still a different approach to finding the number of matchsticks in the 100th term of Figure 10 might be as follows: If the matchstick figure has 100 squares, we could find the total number of matchsticks by adding the number of horizontal and vertical sticks. There are $2 \cdot 100$ placed horizontally. (Why?) Notice that in the first figure, there are 2 matchsticks placed vertically; in the second, 3; and in the third, 4. In the 100th figure, there should be $100 + 1$ vertical matchsticks. Altogether, there will be $2 \cdot 100 + (100 + 1)$, or 301, matchsticks in the 100th figure. Similarly, in the nth figure, there would be $2n$ horizontal and $n + 1$ vertical matchsticks, for a total of $3n + 1$. This discussion is summarized in Table 8.

Table 8

Term Number	Number of Matchsticks Horizontally	Number of Matchsticks Vertically	Total
1	2	2	4
2	4	3	7
3	6	4	10
4	8	5	13
.	.	.	.
.	.	.	.
.	.	.	.
100	200	101	301
.	.	.	.
.	.	.	.
.	.	.	.
n	$2n$	$n + 1$	$2n + (n + 1) = 3n + 1$

If we are given the value of the term, we can use the formula $3n + 1$ for the nth term in Table 8 to *work backward* and find the term number. For example, given the term 1798, we can write an equation: $3n + 1 = 1798$. Therefore, $3n = 1797$ and $n = 599$. Consequently, 1798 is the 599th term. We could obtain the same answer by solving $4 + (n - 1)3 = 1798$ for n.

In the matchstick problem, we found the nth term of a sequence. If the nth term of a sequence is given, we can find any term of the sequence, as shown in Example 3.

Example 3

Find the first four terms of a sequence, the nth term of which is given by the following, and determine whether the sequence seems to be arithmetic:

a. $4n + 3$ **b.** $n^2 - 1$

Solution

a.

Term Number	Term
1	$4 \cdot 1 + 3 = 7$
2	$4 \cdot 2 + 3 = 11$
3	$4 \cdot 3 + 3 = 15$
4	$4 \cdot 4 + 3 = 19$

Hence, the first four terms of the sequence are 7, 11, 15, 19. This sequence seems arithmetic, with difference 4.

b.

Term Number	Term
1	$1^2 - 1 = 0$
2	$2^2 - 1 = 3$
3	$3^2 - 1 = 8$
4	$4^2 - 1 = 15$

Thus, the first four terms of the sequence are 0, 3, 8, 15. This sequence is not arithmetic, because it has no common difference.

We generalize our work with arithmetic sequences in Chapter 8.

Example 4

The diagrams in Figure 11 show the molecular structure of alkanes, a class of hydrocarbons. C represents a carbon atom and H a hydrogen atom. A connecting segment shows a chemical bond.

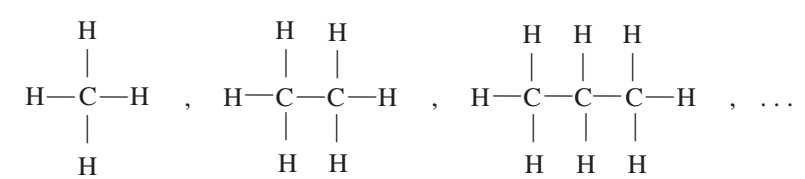


Figure 11

a. Hectane is an alkane with 100 carbon atoms. How many hydrogen atoms does it have?
b. Write a general rule for alkanes C_nH_m showing the relationship between m and n.

Solution

a. To determine the relationship between the number of carbon and hydrogen atoms, we first study the drawing of the alkanes and disregard the extreme left and right hydrogen atoms in each. With this restriction, we see that for every carbon atom, there are two hydrogen atoms. Therefore, there are twice as many hydrogen atoms as carbon atoms plus the two hydrogen atoms at the extremes. For example, when there are 3 carbon atoms, there are $(2 \cdot 3) + 2 = 8$ hydrogen atoms. This notion is summarized in Table 9. If we extend the table for 4 carbon atoms, we get $(2 \cdot 4) + 2$, or 10, hydrogen atoms. For 100 carbon atoms, there are $(2 \cdot 100) + 2$, or 202, hydrogen atoms.

Table 9

No. of Carbon Atoms	No. of Hydrogen Atoms
1	$2 \cdot 1 + 2 = 4$
2	$2 \cdot 2 + 2 = 6$
3	$2 \cdot 3 + 2 = 8$
·	·
·	·
·	·
100	$2 \cdot 100 + 2 = 202$
·	·
·	·
·	·
n	$2n + 2 = m$

b. In general, for n carbon atoms there would be n hydrogen atoms attached above, n attached below, and 2 attached on the sides. Hence, the total number of hydrogen atoms m would be $2n + 2$. It follows that the number of hydrogen atoms is $m = 2n + 2$.

Example 5

A theater is arranged so that there are 20 seats in the first row and 4 additional seats in each consecutive row to the back of the theater, where there are 144 seats. How many rows are there in the theater?

Solution Two strategies lend themselves to this problem. One is to *build a table* and to consider the entries as seen in Table 10.

Table 10

Row Number	Number of Seats
1	20
2	$20 + 4$
3	$20 + 2 \cdot 4$
4	$20 + 3 \cdot 4$
5	$20 + 4 \cdot 4$
·	·
·	·
·	·
n	$20 + (n - 1)4$

Observe that in Table 10 when we write the number of seats as 20 plus the number of additional 4 seats in consecutive rows, the number of 4s added is one less than the number of the row. We know that in the last row there are 144 seats. Thus, we have the following:

$144 = 20 + (n - 1)4$. Subtracting 20 from each side of the equation, we get $124 = (n - 1)4$ or $n - 1 = 31$.

Therefore, $n = 32$, and there are 32 rows in the theater.

A different way to solve the problem is to use a spreadsheet as seen in Table 11, where the number of the row is seen in the index column and the entry in cell A1 indicates 20 seats in that row. Filling down the A column by writing the recursive formula = A1 + 4 in cell A2 and using the *Fill Down* command, we find 144 seats in row 32. Thus, there are 32 rows in the theater.

Table 11

	A	B
1	20	
2	24	
3	28	
4	32	
5	36	
6	40	
7	44	
8	48	
9	52	
10	56	
11	60	
12	64	
13	68	
14	72	
15	76	
16	80	
17	84	
18	88	

Spreadsheet continued.

19	92
20	96
21	100
22	104
23	108
24	112
25	116
26	120
27	124
28	128
29	132
30	136
31	140
32	144
33	148
34	152
35	156
36	160
37	164

Fibonacci Sequence

Dan Brown's popular book *The Da Vinci Code** brought renewed interest to one of the most famous sequences of all time, the ***Fibonacci sequence***.

The Fibonacci sequence is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots .$$

This sequence is not *arithmetic* as there is no fixed difference, d. The first two terms of the Fibonacci sequence are 1, 1 and each subsequent term is the sum of the previous two. If we denote the terms of the Fibonacci sequence by $F_1, F_2, F_3, \ldots$, we have

$$F_1 = F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \text{and so on.}$$

Also $F_3 = F_2 + F_1, F_4 = F_3 + F_2, \ldots$, and in general $F_n = F_{n-1} + F_{n-2}$ for $n = 3, 4, 5, \ldots$. The numbers in the sequence are known as *Fibonacci numbers*.

*Brown, Dan. *The Da Vinci Code*. Doubleday, 2003.

Historical Note



Leonardo de Pisa was born around 1170. His real family name was Bonaccio but he preferred the nickname Fibonacci, derived from *filius Bonacci*, meaning "son of Bonacci." In his book *Liber Abaci* (1202) he described the now-famous rabbit problem, whose solution, the sequence 1, 1, 2, 3, 5, 8, 13, 21, . . . , became known as the *Fibonacci sequence*.

Example 6

Consider the two rows of hexagonal cells in Figure 12. The cells in the upper row are numbered by the even natural numbers and the ones in the lower row by the odd natural numbers. You can start at cell 1 or 2 and move to a neighboring cell with a higher number on it. How many different ways are possible to get from start to cell number 7?

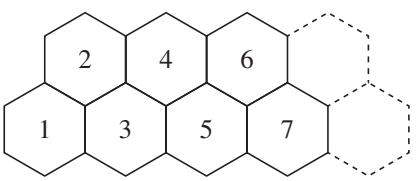


Figure 12

Solution We first use the strategy of *solving a simpler problem*. Since we can start at cell 1 or cell 2, there is only one way to get to cell 1, but two ways to get to cell 2: by starting at cell 2 or via cell 1. There are three ways to get to cell 3: 1–3, 1–2–3, 2–3. There are five ways to cell 4: 2–4, 2–3–4, 1–2–3–4, 1–3–4, 1–2–4.

We can find the number of ways to get to other cells, record the results (see Table 12), and *search for a pattern.*

Table 12

Cell Number	Number of Ways to Get to the Cell
1	1
2	2
3	3
4	5
5	8
6	13

The pattern seems to be similar to the one in the Fibonacci sequence; that is, each term starting from the 3rd is the sum of the preceding two terms. If that is the case, then to reach cell number 7, the number of ways will be 8 + 13 or 21.

But how can we be sure that the pattern exhibited in Table 12 continues as in the Fibonacci sequence? To reach any cell, for example cell 7 in Figure 12, we need to pass through either cell 5 or cell 6 (the two adjacent cells with lower numbers). Thus, the number of ways to reach cell 7 will be the number of ways to reach cell 5, that is, 8 ways, plus the number of ways to reach cell 6, that is, 13 ways. In general, to reach cell number n, we will have to go through cell number $n - 1$ or cell number $n - 2$, that is, we will add the number of ways to reach cell $n - 2$ to the number of ways to reach cell $n - 1$.

NOW TRY THIS 7

In Figure 13, we want to know how many different paths there are from point A to point B if one is allowed to walk along the sides or on the indicated diagonals of the squares in the directions indicated by the arrows. Discover a pattern and explain why the pattern continues.

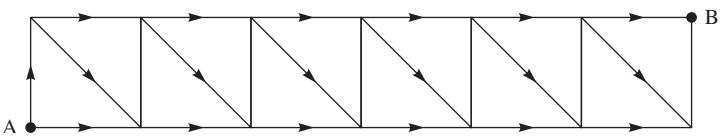


Figure 13

Geometric Sequences

A child has 2 biological parents, 4 grandparents, 8 great grandparents, 16 great-great grandparents, and so on. The number of generational ancestors form the **geometric sequence** 2, 4, 8, 16, 32, ... Each successive term of a geometric sequence is obtained from its predecessor by multiplying by a fixed nonzero number, the **ratio**. In this example, both the first term and the ratio are 2. (The ratio is 2 because each person has two parents.) To find the nth term examine the pattern in Table 13.

Table 13

Term Number	Term
1	$2 = 2^1$
2	$4 = 2 \cdot 2 = 2^2$
3	$8 = (2 \cdot 2) \cdot 2 = 2^3$
4	$16 = (2 \cdot 2 \cdot 2) \cdot 2 = 2^4$
5	$32 = (2 \cdot 2 \cdot 2 \cdot 2) \cdot 2 = 2^5$
.	.
.	.
.	.

In Table 13, when the given term is written as a power of 2, the term number is the **exponent**. Following this pattern, the 10th term is 2^{10}, or 1024, the 100th term is 2^{100}, and the nth term is 2^n. Thus, the number of ancestors in the nth previous generation is 2^n. The notation used in Table 13 can be generalized as follows.

Definition of a^n

If n is a natural number, then $a^n = \overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}^{n \text{ factors}}$.

If $n = 0$ and $a \neq 0$, then $a^0 = 1$. (This will be motivated in Chapter 2.)

Geometric sequences play an important role in everyday life. For example, suppose we have \$1000 in a bank that pays 5% interest annually. (Note that $5\% = 0.05$.) If no money is added or withdrawn, then at the end of the first year we have all of the money we deposited plus 5% more.

Year 1: $\$1000 + 0.05(\$1000) = \$1000(1 + 0.05) = \$1000(1.05) = \$1050$

If no money is added or taken out, then at the end of the second year we would have 5% more money than the previous year.

Year 2: $\$1050 + 0.05(\$1050) = \$1050(1 + 0.05) = \$1050(1.05) = \$1102.50$

The amount of money in the account after any number of years can be found by noting that every dollar invested for one year becomes $1 + 0.05 \cdot 1$, or 1.05 dollars. Therefore, the amount in each year is obtained by multiplying the amount from the previous year by 1.05. The amounts in the bank after each year form a geometric sequence because the amount in each year (starting from year 2) is obtained by multiplying the amount in the previous year by the same number, 1.05. This is summarized in Table 14.

Table 14

Term Number (Year)	Term (Amount at the End of Each Year)
1	$\$1000(1.05)^1$
2	$\$1000(1.05)^2$
3	$\$1000(1.05)^3$
4	$\$1000(1.05)^4$
·	·
·	·
·	·
n	$\$1000(1.05)^n$

NOW TRY THIS 8

a. Two bacteria are in a dish. The number of bacteria triples every hour. Following this pattern, find the number of bacteria in the dish after 10 hours and after n hours.

b. Suppose that instead of increasing geometrically as in part (a), the number of bacteria increases arithmetically by 3 each hour. Compare the growth after 10 hours and after n hours. Comment on the difference in growth of a geometric sequence versus an arithmetic sequence.

Other Sequences

Figurate numbers, based on geometrical patterns, provide examples of sequences that are neither arithmetic nor geometric. Such numbers can be represented by dots arranged in the shape of certain geometric figures. The number 1 is the beginning of most patterns involving figurate numbers. The arrays in Figure 14 represent the first four terms of the sequence of **triangular numbers**.

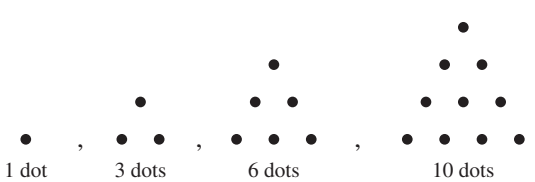


Figure 14

The triangular numbers can be written numerically as $1, 3, 6, 10, 15, \ldots$. The sequence $1, 3, 6, 10, 15, \ldots$ is not an arithmetic sequence because there is no common difference, as Figure 15 shows. It is not a geometric sequence because there is no common ratio. It is not the Fibonacci sequence.

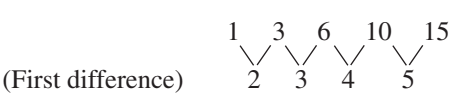


Figure 15

However, the sequence of differences, 2, 3, 4, 5, . . . , appears to form an arithmetic sequence with difference 1, as Figure 16 shows. The next successive terms for the original sequence are shown in color in Figure 16.

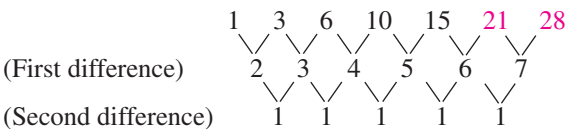


Figure 16

Table 15 suggests a pattern for finding the next terms and the nth term for the triangular numbers. The second term is obtained from the first term by adding 2; the third term is obtained from the second term by adding 3; and so on.

Table 15

Term Number	Term
1	1
2	$3 = 1 + 2$
3	$6 = 1 + 2 + 3$
4	$10 = 1 + 2 + 3 + 4$
5	$15 = 1 + 2 + 3 + 4 + 5$
·	·
·	·
·	·
10	$55 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

We could approach the problem differently without looking for differences. Because the nth triangular number has n dots in the nth row, it is equal to the sum of the dots in the previous triangular number (the $(n - 1)$st one) plus the n dots in the nth row. Following this pattern, the 10th term is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$, or 55, and the nth term is $1 + 2 + 3 + 4 + 5 + \ldots + (n - 1) + n$. This problem is similar to Gauss's Problem in Section 1-1. Because of the work done in Section 1-1, we know that this sum can be expressed as

$$\frac{n(n + 1)}{2}.$$

Next consider the first four *square numbers* in Figure 17. These square numbers, $1, 4, 9, 16$ can be written as $1^2, 2^2, 3^2, 4^2$. Continuing, the number of dots in the 10th array would be 10^2, the number of dots in the 100th array would be 100^2, and the number of dots in the nth array would be n^2. The sequence of square numbers is neither arithmetic nor geometric. Investigate whether the sequence of first differences is an arithmetic sequence and tell why.

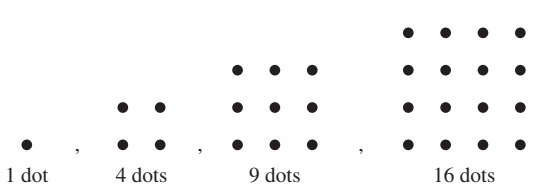


Figure 17

When asked to find a pattern for a given sequence, we first look for some easily recognizable pattern and determine whether the sequence is arithmetic or geometric. If a pattern is unclear, taking successive differences may help. *It is possible that none of the methods described reveals a pattern.*

NOW TRY THIS 9

Consider the rectangular numbers in Figure 18 in which the number of columns and the number of rows increase by 1 with each successive "rectangle." What is the 10th rectangular number, and what is the nth rectangular number?

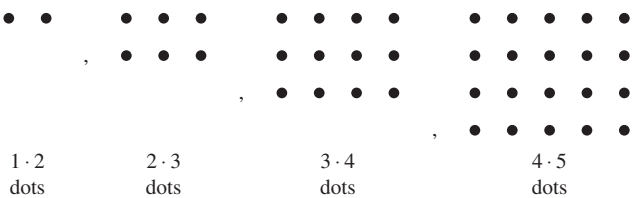


Figure 18

Example 7

Use differences to find a pattern. Then assuming that the pattern discovered continues, find the seventh term in each of the following sequences:

a. 5, 6, 14, 29, 51, 80, . . .

b. 2, 3, 9, 23, 48, 87, . . .

Solution

a. Figure 19 shows the sequence of first differences.

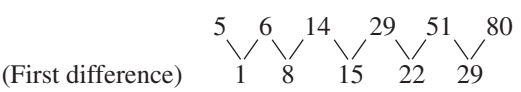


Figure 19

To discover a pattern for the original sequence, we try to find a pattern for the sequence of differences 1, 8, 15, 22, 29, ... This sequence is an arithmetic sequence with fixed difference 7 as seen in Figure 20.

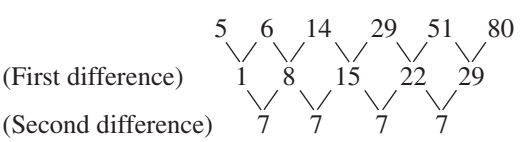


Figure 20

Thus, the sixth term in the first difference row is 29 + 7, or 36, and the seventh term in the original sequence is 80 + 36, or 116. What number follows 116?

b. Because the second difference is not a fixed number, we go on to the third difference as in Figure 21.

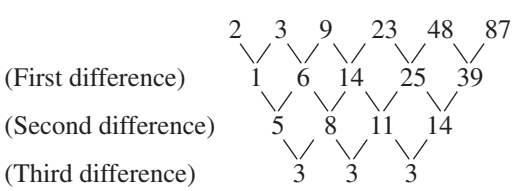


Figure 21

The third difference is a fixed number; therefore, the second difference is an arithmetic sequence. The fifth term in the second-difference sequence is 14 + 3, or 17; the sixth term in the first-difference sequence is 39 + 17, or 56; and the seventh term in the original sequence is 87 + 56, or 143.

Example 8

Figure 22 shows the first three figures of arrays of matchsticks with the number of matchsticks written below the figures. If the next figure consists of a 4-by-4 square arrangement, and each subsequent figure has one more row and one more column of matchsticks squares than the preceding figure, without actually counting, find the number of matchsticks in

a. the 7th figure
b. the nth figure.

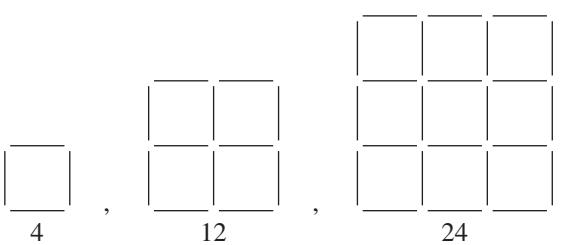


Figure 22

Solution

First Approach

Building the 4th and 5th figures, we find that the number of matchsticks in the figures is 40 and 60 correspondingly.

To discover a pattern for the original sequence, we use differences. Figure 23 shows first and second differences.

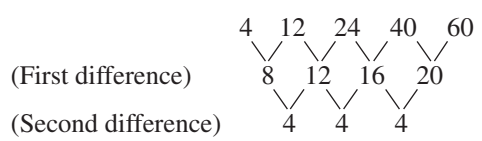


Figure 23

Thus, the next two terms in the first difference are $20 + 4$ or 24 and $24 + 4 = 28$. Hence the next two terms in the original sequence are $60 + 24$ or 84 and $84 + 28$ or 112. Thus, the answer to part (a) is 112.

To answer part (b), we use the differences in Figure 21 to write the number of matchsticks in each figure in Table 16.

Table 16

Figure Number	Number of Matchsticks
1	$4 \cdot 1$
2	$12 = 4 + 8 = 4(1 + 2)$
3	$24 = 4 + 8 + 12 = 4(1 + 2 + 3)$
4	$40 = 4 + 8 + 12 + 16 = 4(1 + 2 + 3 + 4)$
5	$60 = 4 + 8 + 12 + 16 + 20 = 4(1 + 2 + 3 + 4 + 5)$
⋮	⋮
n	$4(1 + 2 + 3 + 4 + 5 + \ldots + n)$

Now using Gauss's Problem, we find that the number of matchsticks in the nth figure is

$$4(1 + 2 + 3 + 4 + 5 + \ldots + n) = 4\frac{n(n + 1)}{2} = 2n(n + 1).$$

Notice that the above solution is based on *inductive reasoning*. Based on a few cases, we decided that the pattern of second differences continues and is always 4. However, we cannot be certain that the second differences continue to be 4. For that reason, we give another approach.

Second Approach

We count the matchsticks in rows and columns. Notice that the number of columns is the same as the number of rows. Thus, we will find the number of matchsticks in the rows and multiply the result by 2. In the first figure, we have 2 rows and 1 matchstick in each. Because we are adding one row and one column to get the subsequent figure, we can write the number of matchsticks in each figure as shown in Table 17.

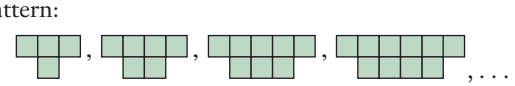
Table 17

Figure Number	Number of Rows	Number in Each Row	Total in the Rows
1	2	1	$1 \cdot 2$
2	3	2	$2 \cdot 3$
3	4	3	$3 \cdot 4$
4	5	4	$4 \cdot 5$
5	6	5	$5 \cdot 6$
⋮	⋮	⋮	⋮
n	$n + 1$	n	$n(n + 1)$

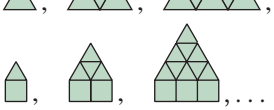
Because the number of matchsticks in the columns is the same as in the rows, the total number is $2n(n + 1)$. If $n = 7$, then we have the answer for part (a) as $2 \cdot 7(7 + 1)$ or 112.

Assessment 1-2 A

1. For each of the following sequences of figures, determine a possible pattern and draw the next figure according to that pattern:

a. 

b. 

c. 

2. Each of the following sequences is either arithmetic or geometric. Identify the sequences and list the next three terms for each.

a. $1, 5, 9, 13, \ldots$

b. $70, 120, 170, \ldots$

c. $1, 3, 9, \ldots$

d. $10, 10^3, 10^5, 10^7, \ldots$

e. $193 + 7 \cdot 2^{30}, 193 + 8 \cdot 2^{30}, 193 + 9 \cdot 2^{30}, \ldots$

3. Find the 100th term and the nth term for each of the sequences in exercise 2.

4. Use a traditional clock face to determine the next three terms in the following sequence.

$$1, 6, 11, 4, 9, \ldots$$

5. The pattern $1, 8, 27, 64, 125, \ldots$ is a cubic pattern named because $1 = 1 \cdot 1 \cdot 1$ or 1^3, $8 = 2 \cdot 2 \cdot 2$ or 2^3, and so on.

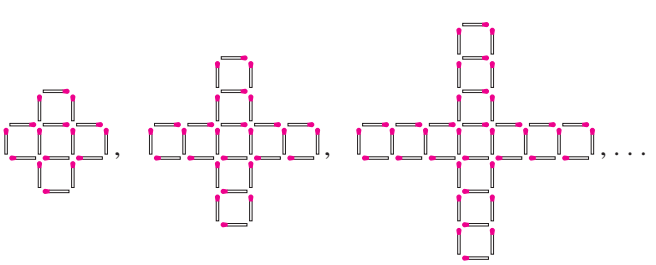
a. What is the least 4-digit number greater than 1000 in this pattern?

b. What is the greatest 3-digit number in this pattern?

c. What is the greatest number in this pattern that is less than 10^4?

d. If this pattern was produced in a normal spreadsheet, what is the number in cell A14?

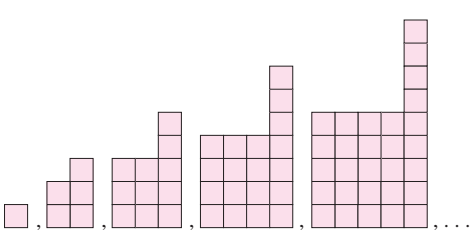
6. The first windmill has 5 matchstick squares, the second has 9, and the third has 13, as shown. How many matchstick squares are in **(a)** the 10th windmill? **(b)** the nth windmill? **(c)** How many matchsticks will it take to build the nth windmill?



7. In the following sequence, the figures are made of cubes that are glued together. If the exposed surface needs to be painted, how many squares will be painted in **(a)** the 15th figure? **(b)** the nth figure?



8. The school population for a certain school is predicted to increase by 60 students per year for the next 12 years. If the current enrollment is 700 students, what will the enrollment be after 12 years?
9. Joe's annual income has been increasing each year by the same dollar amount. The first year his income was \$24,000, and the ninth year his income was \$31,680. In which year was his income \$45,120?
10. The first difference of a sequence is the arithmetic sequence 2, 4, 6, 8,10, ... Find the first six terms of the original sequence in each of the following cases:
 a. The first term of the original sequence is 3.
 b. The sum of the first two terms of the original sequence is 10.
 c. The fifth term of the original sequence is 35.
11. List the next three terms to continue a pattern in each of the following. (Finding differences may be helpful.)
 a. 5, 6, 14, 32, 64, 115, 191
 b. 0, 2, 6, 12, 20, 30, 42
12. How many terms are there in each of the following sequences?
 a. $51, 52, 53, 54, \ldots, 251$
 b. $1, 2, 2^2, 2^3, \ldots, 2^{60}$
 c. $10, 20, 30, 40, \ldots, 2000$
 d. $1, 2, 4, 8, 16, 32, \ldots, 1024$
13. Find the first five terms in sequences with the following nth terms.
 a. $n^2 + 2$
 b. $5n + 1$
 c. $10^n - 1$
 d. $3n - 2$
14. Find a counterexample for each of the following:
 a. If n is a natural number, then $(n + 5)/5 = n + 1$.
 b. If n is a natural number, then $(n + 4)^2 = n^2 + 16$.
15. Assume that the following patterns are built of square tiles and the pattern continues. Answer the questions that follow.

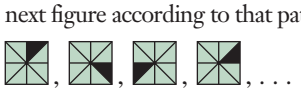
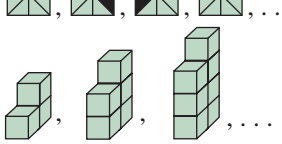
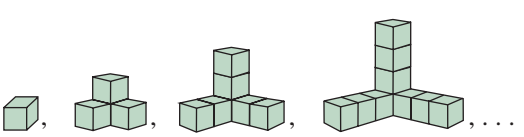


 a. How many square tiles are there in the sixth figure?
 b. How many square tiles are in the nth figure?
 c. Is there a figure that has exactly 1259 square tiles? If so, which one? (*Hint*: To determine if there is a figure in the sequence containing exactly 1259 square tiles, first think about the greatest square number less than 1259.)
16. Consider the sequences given in the table below. Find the least number, n, such that the nth term of the geometric sequence is greater than the corresponding term in the arithmetic sequence.

Term Number	1	2	3	4	5	6	. . .	n
Arithmetic	400	600	800	1000	1200	1400	. . .	
Geometric	2	4	8	16	32	64	. . .	

17. A sheet of paper is cut into 5 same-size parts. Each of the parts is then cut into 5 same-size parts and so on. Answer the following.
 a. After the 5th cut, how many of the smallest pieces of paper are there?
 b. After the nth cut, how many of the smallest pieces are there?
18. Each of the following sequences is labeled either arithmetic or geometric. In each part, find the missing terms.
 a. __, 39, __, __, 69 (arithmetic)
 b. __, 200, __, __, 800 (arithmetic)
 c. __, 5^4, __, __, 5^{10} (geometric)
19. A *Fibonacci-type sequence* is a sequence in which the first two terms are arbitrary and in which every term starting from the third is the sum of the two previous terms. Each of the following is a Fibonacci-type sequence. In each part, find the missing terms.
 a. __, __, 1, 1, __, __, __, __
 b. __, __, __, 10, 13, __, 36, 59
 c. 0, 2, __, __, __, __, __, __
20. Starting with 1 and 1 as the first two terms of the Fibonacci sequence, answer the following.
 a. Add the first three terms. The result is one less than which term?
 b. Add the first four terms. The result is one less than which term?
 c. Add the first five terms. The result is one less than which term?
 d. Write a conjecture regarding the sum of the first n terms of the sequence.
 e. Assuming that your conjecture in part (d) is true, what can you conclude about the sum of the first $n - 2$ terms of the sequence?
21. A new pair of tennis shoes cost \$80. If the price increases each year by 5% of the previous year's price, find the following:
 a. The price after 5 years
 b. The price after n years

Assessment 1-2 B

1. In each of the following, determine a possible pattern and draw the next figure according to that pattern if the sequence continues.
 a. 
 b. 
 c. 
2. Each of the following sequences is either arithmetic or geometric. Identify the sequences and list the next three terms for each.
 a. $2, 6, 10, 14, \ldots$
 b. $0, 13, 26, \ldots$
 c. $4, 16, 64, \ldots$
 d. $2^2, 2^6, 2^{10}, \ldots$
 e. $100 + 4 \cdot 2^{50}, 100 + 6 \cdot 2^{50}, 100 + 8 \cdot 2^{50}, \ldots$

3. Find the 100th term and the nth term for each of the sequences in exercise 2.

4. Use a traditional clock face to determine the next three terms in the following sequence:

$$1, 9, 5, 1, \ldots$$

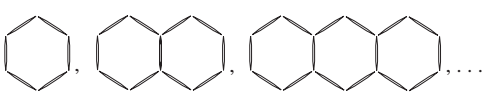
5. Observe the following pattern:

$$1 + 3 = 2^2,$$
$$1 + 3 + 5 = 3^2,$$
$$1 + 3 + 5 + 7 = 4^2$$

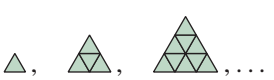
a. Conjecture a generalization based on this pattern.
b. Based on the generalization in (a), find

$$1 + 3 + 5 + 7 + \ldots + 35.$$

6. In the following pattern, one hexagon takes 6 toothpicks to build, two hexagons take 11 toothpicks to build, and so on. How many toothpicks would it take to build
a. 10 hexagons?
b. n hexagons?



7. Each successive figure below is made of small triangles like the first one in the sequence. Conjecture the number of small triangles needed to make
a. the 100th figure?
b. the nth figure?



8. A tank contains 15,360 L of water. At the end of each subsequent day, half of the water is removed and not replaced. How much water is left in the tank after 10 days?

9. The Washington Middle School schedule is an arithmetic sequence. Each period is the same length and includes a 4th period lunch. The first three periods begin at 8:10 A.M., 9:00 A.M., and 9:50 A.M., respectively. At what time does the eighth period begin?

10. The first difference of a sequence is 3, 6, 9, 12, 15, . . . Find the first six terms of the original sequence in each of the following cases:
a. The first term of the original sequence is 3.
b. The sum of the first two terms of the original sequence is 7.
c. The fifth term of the original sequence is 34.

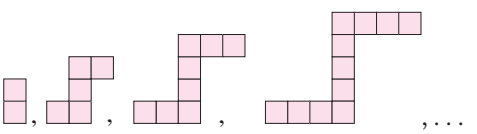
11. List the next three terms to continue a pattern in each of the following. (Finding differences may be helpful.)
a. 3, 8, 15, 24, 35, 48, . . .
b. 1, 7, 18, 37, 67, 111, . . .

12. How many terms are there in each of the following sequences?
a. $1, 3, 3^2, 3^3, \ldots, 3^{99}$
b. $9, 13, 17, 21, 25, \ldots, 353$
c. $38, 39, 40, 41, \ldots, 238$

13. Find the first five terms in sequences with the following nth terms.
a. $5n - 1$
b. $6n - 2$
c. $5n + 1$
d. $n^2 - 1$

14. Find a counterexample for each of the following:
a. If n is a natural number, then $(3 + n)/3 = n$.
b. If n is a natural number, then $(n - 2)^2 = n^2 - 2^2$.

15. Assume the following pattern with terms built of square-tiles figures continues and answer the questions that follow.



a. How many square tiles are there in the seventh figure?
b. How many square tiles are in the nth figure?
c. Is there a figure that has exactly 449 square tiles? If so, which one?

16. Consider the sequences given in the table below. Find the least number, n, such that the nth term of the geometric sequence is greater than the corresponding term in the arithmetic sequence.

Term Number	1	2	3	4	5	6	. . .	n
Arithmetic	200	500	800	1100	1400	1700	. . .	
Geometric	1	3	9	27	81	243	. . .	

17. Female bees are born from fertilized eggs, and male bees are born from unfertilized eggs. This means that a male bee has only a mother, whereas a female bee has a mother and a father. If the ancestry of a male bee is traced 10 generations including the generation of the male bee, how many bees are there in all 10 generations? (*Hint:* The Fibonacci sequence might be helpful.)

18. Each of the following sequences is labeled either arithmetic or geometric. In each part, find the missing terms.
a. __, 49, __, __, 64 (arithmetic)
b. 1, __, __, __, 625 (geometric)
c. __, 3^{10}, __, __, 3^{19} (geometric)
d. a, __, __, __, $5a$ (arithmetic)

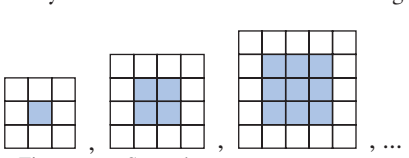
19. Each of the following sequences is a Fibonacci-type sequence (see problem 19 Assessment 1-2A). Find the missing terms.
a. 1, __, __, 7, 11
b. __, 2, __, 4, __
c. __, __, 3, 4, __

20. Starting with 1 and 1 as the first two terms of the Fibonacci sequence, answer the following.
a. Check that $F_1 + F_3 = F_4$
b. Compute $F_1 + F_3 + F_5$. The sum is which term?
c. Compute $F_1 + F_3 + F_5 + F_7$. The sum is which term?
d. Write a conjecture based on the examples in parts (a), (b), and (c).

Mathematical Connections 1-2

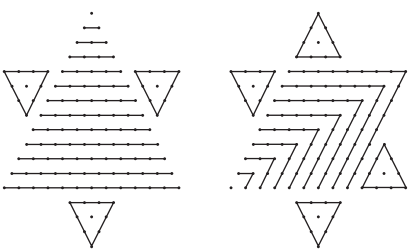
Reasoning

1. a. If a fixed number is added to each term of an arithmetic sequence, is the resulting sequence an arithmetic sequence? Justify the answer.
 b. If each term of an arithmetic sequence is multiplied by a fixed number, will the resulting sequence always be an arithmetic sequence? Justify the answer.
 c. If the corresponding terms of two arithmetic sequences are added, is the resulting sequence arithmetic?
2. A student says she read that Thomas Robert Malthus (1766–1834), a renowned British economist and demographer, claimed that the increase of population will take place, if unchecked, in a geometric sequence, whereas the supply of food will increase in only an arithmetic sequence. This theory implies that population increases faster than food production. The student is wondering why. How do you respond?
3. Abby and Dan are preparing for a GRE (Graduate Record Exam) to take place in 5 months. Abby starts by studying 10 hours the first week and increases her study by 30 minutes per week. Dan starts at 6 hours per week, but increases his time every week by 45 minutes per week. Answer the following.
 a. How many hours did each student study in week 8?
 b. In which week will Dan first catch up with Abby in the number of hours spent studying per week?
4. The *arithmetic average* of two numbers x and y is $\frac{x+y}{2}$. Use *deductive reasoning* to explain why if three numbers a, b, and c form an arithmetic sequence, then b is the arithmetic average of a and c.
5. The numbers x, y, and z are in a Fibonacci-type sequence. If $z = x + y$, use deductive reasoning to find all triples x, y, and z that make an arithmetic sequence as well as consecutive terms in a Fibonacci-type sequence.
6. The figure below shows the first three terms of a sequence of figures containing small square tiles. Some of the tiles are shaded. Notice that the first figure has one shaded tile. The second figure has $2 \cdot 2$, for 2^2, shaded tiles. The third figure has $3 \cdot 3$, or 3^2, shaded tiles. If this pattern of having shaded squares surrounded by white borders continues, answer the following:
 a. How many shaded tiles are there in the nth figure?
 b. How many white tiles are there in the nth figure?



Open-Ended

7. Patterns can be used to count the number of dots on the Chinese checkerboard; two patterns are shown here. Determine several other patterns to count the dots.



$1 + 2 + 3 + \ldots + 13 + 3 \cdot 10$ $\quad$ $1 + 3 + 5 + 7 + \ldots + 17 + 4 \cdot 10$

8. Make up a pattern involving figurate numbers and find the number of dots in the 100th figure. Describe the pattern and how to find the 100th term.
9. A sequence that follows the same pattern as the Fibonacci sequence but in which the first two terms are any numbers is a Fibonacci type sequence. Choose a few such sequences and compare their behavior to the Fibonacci sequence.
10. Use online sources to find two problems involving the Fibonacci sequence appropriate for the 4th or 5th grades. State the problems and solve them, explaining your solutions in a way appropriate for that level.

Cooperative Learning

11. The following pattern is called *Pascal's triangle*. It was named for the mathematician Blaise Pascal (1623–1662).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

 a. Have each person in the group find four different patterns in the triangle and then share them with the rest of the group.
 b. Add the numbers in each horizontal row. Discuss the pattern that occurs.
 c. Use part (b) to find the sum in the 16th row.
 d. What is the sum of the numbers in the nth row?
12. If the following pattern continued indefinitely, the resulting figure would be called the *Sierpinski triangle*, or *Sierpinski gasket*.



In a group, determine each of the following. Discuss different counting strategies.

a. How many black triangles would be in the fifth figure?
b. How many white triangles would be in the fifth figure?
c. How many black triangles are in the *n*th figure?

Connecting Mathematics to the Classroom

13. Joey said that 4, 24, 44, and 64 all have remainder 0 when divided by 4, so all numbers that end in 4 must have 0 remainder when divided by 4. How do you respond?

14. Al and Betty were asked to extend the sequence 2, 4, 8, ... Al said his answer of 2, 4, 8, 16, 32, 64, . . . was the correct one. Betty said Al was wrong and it should be 2, 4, 8, 14, 22, 32, 44, ... What do you tell these students?

15. A student claims the sequence 1, 1, 1, 1, . . . is both arithmetic and geometric and would like to know if there are other such sequences. How do you respond?

16. A student claims that she has found an easy way to find the number of terms in an arithmetic sequence: "You take the difference between the last and first terms and divide by the common difference." How do you respond?

Review Problems

17. In a baseball league consisting of 10 teams, each team plays each of the other teams twice. How may games will be played?

18. How many ways can you make change for 40¢ using only nickels, dimes, and quarters?

19. Tents hold 2, 3, 5, 6, or 12 people. What combinations of tents are possible to sleep 26 people if all tents are fully occupied and only one 12-person tent is used?

National Assessments

National Assessment of Educational Progress (NAEP) Questions

The growing number pattern 3, 4, 6, 9, 13, . . . follows a rule. Explain the rule.

Write a new growing pattern beginning with 21 that follows the same rule. 21, _____, _____, _____, _____, _____

NAEP, Grade 4, 2013

Write the next two numbers in the number pattern.

1 6 4 9 7 12 10 ____ ____

Write the rule that you used to find the two numbers you wrote.

NAEP, Grade 4, 2009

A car can seat c adults. A van can seat 4 more than twice as many adults as the car can. In terms of c, how many adults can the van seat?

A. $c + 8$
B. $c + 12$
C. $2c - 4$
D. $2c + 4$
E. $4c + 2$

NAEP, Grade 8, 2013

Which of the following equations has the same solution as the equation $2x + 6 = 32$?

A. $2x = 38$
B. $x - 3 = 16$
C. $x + 6 = 16$
D. $2(x - 3) = 16$
E. $2(x + 3) = 32$

NAEP, Grade 8, 2009

Hint for Solving the Preliminary Problem

Show that if all the coins taken out were genuine, they would weigh 110 ounces. Also notice that if the first box contained all the fake coins, the weight of all the coins would be 109 ounces.

Chapter 1 Summary

KEY CONCEPTS	DEFINITIONS, DESCRIPTIONS, AND THEOREMS
Section 1-1	
Four-step problem-solving process (p. 2)	• Understanding the problem • Devising a plan • Carrying out the plan • Looking Back
Natural numbers (p. 4)	The numbers 1, 2, 3, 4, 5, . . . The three dots above, called an *ellipsis*, indicate that the pattern of the numbers in the list continues.
Problem-solving strategies (p. 4)	• Look for a pattern • Examine a related problem • Examine a simpler case • Make a table • Identify a subgoal • Make a diagram • Use guess and check • Work backward • Write an equation
Section 1-2	
Conjecture (p. 17)	A statement thought to be true but has not been proven.
Inductive reasoning (p. 17)	Method of making generalizations based on observations and patterns, which may or may not be true.
Deductive reasoning (p. 17)	Method of proving the truth of conjectures through the use of axioms, theorems, undefined terms assumed to be true, and logic.
Counterexample (p. 18)	An example that shows a conjecture is false.
Sequence (p. 19)	An ordered arrangement of numbers, figures, or objects. The individual items of a sequence are the *terms* of the sequence. The *term number* indicates the position of the term in the sequence.
Arithmetic sequence (p. 19)	A sequence in which each successive term from the second term on is obtained from the previous term by adding a fixed number. The fixed number is the *common difference*, or *difference*. An arithmetic sequence involves a *recursive pattern*—each successive term of the sequence is obtained from the previous term(s).
Fibonacci sequence (p. 24)	The sequence in which the first two terms are both 1 and each subsequent term is the sum of the previous two terms: 1, 1, 2, 3, 5, 8, 13, 21, . . . In general, a sequence in which the first two terms are arbitrary and each subsequent term is the sum of the previous two terms in a *Fibonacci-type sequence*.
Geometric sequence (p. 26)	A sequence in which each successive term from the second term on is obtained from the previous term by multiplying by a fixed nonzero number. The fixed number is the *ratio*.
Exponentiation (p. 26)	If n is a natural number, then $a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factor}}$. If $n = 0$ and $a \neq 0$, then $a^0 = 1$.
Figurative numbers (p. 27)	Numbers that can be represented by dots arranged in the shape of geometric figures. Figurative numbers provide examples of sequences that are neither arithmetic nor geometric.
Finding common differences (p. 28)	A technique used to discover patterns in sequences.

Chapter 1 Review

1. If today is Sunday, July 4, and next year is not a leap year, what day of the week will July 4 be on next year?
2. Jackie spent $40 on two items. If she spent $5.90 more on the more expensive item, how much did this item cost? What strategy have you used to solve this problem?
3. A nursery rhyme states:

 A diller, a dollar, a ten o'clock scholar!
 What makes you come so soon?
 You used to come at ten o'clock,
 But now you come at noon.

 Explain whether the rhyme makes sense mathematically.
4. List three more terms that complete a pattern in each of the following; explain your reasoning, and tell whether each sequence is arithmetic or geometric, or neither.
 a. 0, 1, 3, 6, 10, ____, ____, ____,
 b. 52, 47, 42, 37, ____, ____, ____,
 c. 6400, 3200, 1600, 800, ____, ____, ____,
 d. 1, 2, 3, 5, 8, 13, ____, ____, ____,
 e. 2, 5, 8, 11, 14, ____, ____, ____,
 f. 1, 4, 16, 64, ____, ____, ____,
 g. 0, 4, 8, 12, ____, ____, ____,
 h. 1, 8, 27, 64, ____, ____, ____,
5. Find a possible nth term in each of the following.
 a. $5, 8, 11, 14, \ldots$
 b. $3, 9, 27, 81, 243, \ldots$
 c. $2^2 - 1, 2^3 - 1, 2^4 - 1, 2^5 - 1, \ldots$
6. Find the first five terms of the sequences whose nth term is given as follows:
 a. $3n - 2$
 b. $n^2 + n$
 c. $4n - 1$
7. Find the following sums:
 a. $2 + 4 + 6 + 8 + 10 + \ldots + 200$
 b. $51 + 52 + 53 + 54 + \ldots + 151$
8. Produce a counterexample, if possible, to disprove each of the following. If a statement is true, justify it.
 a. If two odd numbers are added, then the sum is odd.
 b. If a number is odd, then it ends in a 1 or a 3.
 c. If two even numbers are added, then the sum is even.
9. Complete the following magic square; that is, complete the square so that the sum in each row, column, and diagonal is the same.

16	3	2	13
	10		
9		7	12
4		14	

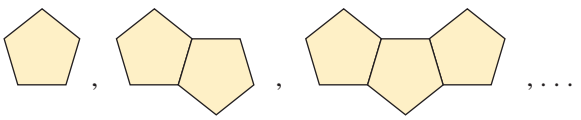
10. How many people can be seated at 12 square tables lined up end to end if each table individually holds four persons?
11. Solve the following equations:
 a. $\square + 2^{60} = 2^{61}$ b. $\square^2 = 625$
12. If fence posts are to be placed in a row 5 m apart, how many posts are needed for 100 m of fence?
13. If a complete rotation of a car tire moves a car forward 6 ft, how many rotations of the tire occur before the tire goes off its 50,000 mi warranty?
14. The members of Mrs. Grant's class are standing in a circle; they are evenly spaced and are numbered in order. The student with number 7 is standing directly across from the student with number 17. How many students are in the class?
15. A carpenter has three large boxes. Inside each large box are two medium-sized boxes. Inside each medium-sized box are five small boxes. How many boxes are there altogether?
16. Use differences to find the next term in the following sequence:

 $5, 15, 37, 77, 141, ____$
17. An ant farm can hold 100,000 ants. If the farm held 1500 ants on the first day, 3000 ants on the second day, 6000 ants on the third day, and so on forming a geometric sequence, in how many days will the farm be full?
18. Toma's team entered a mathematics contest where teams of students compete by answering questions that are worth either 3 points or 5 points. No partial credit was given. Toma's team scored 44 points on 12 questions. How many 5-point questions did the team answer correctly?
19. Three pieces of wood are needed for a project. They are to be cut from a 90-cm-long piece of wood. The longest piece is to be 3 times as long as the middle-sized piece and the shortest piece is to be 10 cm shorter than the middle-sized piece. How long are the pieces?
20. How many four-digit numbers have the same digits as 1993?
21. If n and m are natural numbers and $n > m$, how many terms are in the arithmetic sequence $m, m + 1, m + 2, \ldots, n$? (Your answer should be in terms of n and m.) What strategy (or strategies) have you used to answer the question? Why?
22. We have two containers, one of which holds 7 cups and the other holds 4 cups. How can we measure exactly 5 cups of water, if we have an unlimited amount of water with which to start?
23. The following geometric arrays suggest a sequence of numbers: $2, 6, 12, 20, \ldots$



 a. Find the next three terms.
 b. Find the 100th term.
 c. Find the nth term.

24. Each side of each pentagon below is 1 unit long.



a. Draw a possible next figure in the sequence.
b. What is the perimeter (distance around) of each of the first four figures?
c. What is the perimeter of the 100th figure?
d. What is the perimeter of the nth figure?

25. a. If every second term in an arithmetic sequence is circled, do the circled terms always constitute an arithmetic sequence? Justify your answer.
b. Answer the question in part (a) again, but replace "arithmetic" by "geometric."

26. If for every natural number n, the sum of the first n terms of a certain sequence is $n^2 - n$, find the 4th term. Justify your answer.

27. Each of the following is a Fibonacci-type sequence. Find the missing terms. In part (c), your answer should be in terms of a. and b.
a. 13, ___, ___, 27
b. 137, ___, ___, 163
c. b, ___, ___, a

Chapter 2

Introduction to Logic and Sets

2-1 Reasoning and Logic: An Introduction

2-2 Describing Sets

2-3 Other Set Operations and Their Properties



Preliminary Problem

Maria was given the following set of data about 37 college women and invitations to join social organizations: 20 received invitations for Alpha Pi; 21 for Beta Zeta; 12 for Gamma Iota; 10 for Alpha Pi and Beta Zeta; 5 for Beta Zeta and Gamma Iota; 9 for Alpha Pi and Gamma Iota; and 5 received no invitations. She was asked to determine how many women were invited to join all three organizations and how many were invited to Alpha Pi and Beta Zeta but not Gamma Iota. Can you determine the numbers she needs?

If needed, see Hint on page 79.

In the *Common Core Standards for Mathematical Practice*, we find the following:

Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen [to] or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. (p. 7)

In this section, tools needed to evaluate arguments are introduced.

2-1 Reasoning and Logic: An Introduction

2-1 Objectives

Students will be able to understand and explain

- Quantifiers and their effects on statements.
- Different forms of statements.
- How to determine if two statements are logically equivalent.
- How to develop logical arguments.
- How to determine whether an argument is valid.

Logic is a tool used in mathematical thinking, reasoning, and sense making. We present some basics of logic in this section and use them in work with set operation in the next section. In logic, a **statement** *is a sentence that is either true or false, but not both.* The following expressions are not statements because their truth values cannot be determined:

1. She has blue eyes.
2. $x + 7 = 18$
3. $2y + 7 > 1$
4. $2 + 3$
5. How are you?
6. Look out!
7. Lincoln was the best president.

Expressions (**1**), (**2**), and (**3**) become statements if, for (**1**), "she" is identified, and for (**2**) and (**3**), values are assigned to x and y, respectively. However, an expression involving he or she or x or y may already be a statement. For example, "If he is over 210 cm tall, then he is over 2 m tall" and "$2(x + y) = 2x + 2y$" are both statements because they are true no matter who *he* is or what the numerical values of x and y are.

Negation and Quantifiers

From a given statement p it is possible to create the **negation** of p, denoted by $\sim p$, which means not p. *If a statement is true, its negation is false; if the statement is false, its negation is true.* Consider the statement "Snow is falling." The negation of this statement may be stated simply as "Snow is not falling."

Example 1

Negate each of the following statements.

a. $2 + 3 = 5$
b. A hexagon has six sides.

Solution

a. $2 + 3 \neq 5$
b. A hexagon does not have six sides.

At a given time, sentences such as "The shirt I have on is blue" and "The shirt I have on is green" are statements. However, they are not negations of each other. A statement and its negation must have opposite truth values in all possible cases. If the shirt I have on is actually red,

then both of the statements are false and, hence, cannot be negations of each other. However, at a given time the statements "The shirt I have on is blue" and "The shirt I have on is not blue" are negations of each other because they have opposite truth values no matter what color the shirt really is.

Some statements involve **quantifiers** and are more complicated to negate. Quantifiers include words such as *all*, *some*, *every*, and *there exists*.

- The quantifiers *all*, *every*, and *no* refer to each and every element and are **universal quantifiers.**
- The quantifiers *some* and *there exists at least one* refer to one or more, or possibly all, of the elements and are **existential quantifiers.**
- *All*, *every*, and *each* have the same mathematical meaning. Similarly, *some* and *there exists at least one* have the same meaning.

Assume the following is true: "Some professors at Paxson University have blue eyes." This means that at least one professor at Paxson University has blue eyes. It does not rule out the possibilities that *all* the Paxson professors have blue eyes or that some of the Paxson professors do not have blue eyes. Because the negation of a true statement is false, neither "Some professors at Paxson University do not have blue eyes" nor "All professors at Paxson have blue eyes" are negations of the original statement. One possible negation of the original statement is "No professors at Paxson University have blue eyes." To discover if one statement is a negation of another, we use arguments to determine whether they have opposite truth values in all possible cases.

Below we give general forms of quantified statements with their corresponding negations.

Statement	**Negation**
Some a's are b's.	No a is b.
Some a's are not b's.	All a's are b's.
All a's are b's.	Some a's are not b's.
No a is b.	Some a's are b's.

Example 2

Negate each of the following regardless of its truth value:

a. All students like hamburgers.
b. Some people like mathematics.
c. There exists a natural number n such that $3n = 6$.
d. For all natural numbers n, $3n = 3n$.

Solution

a. Some students do not like hamburgers.
b. No people like mathematics.
c. For all natural numbers n, $3n \neq 6$.
d. There exists a natural number n such that $3n \neq 3n$.

Truth Tables and Compound Statements

To investigate the truth of statements, consider the following puzzle by a foremost writer of logic puzzles, Raymond Smullyan. One of Smullyan's books, *The Lady or the Tiger?*, has a puzzle about a prisoner who must make a choice between two rooms. Each room has a sign on the door and the prisoner knows that exactly *one* of the signs is true. The signs are shown in Figure 1.

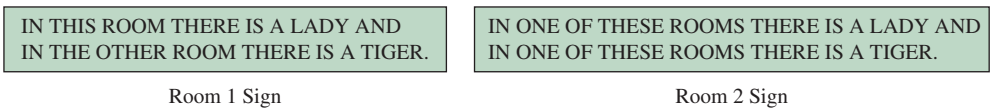


Figure 1

Table 1

Statement p	Negation $\sim p$
T	F
F	T

Table 2

p	q	Conjunction $p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 3

p	q	Disjunction $p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

With the information on the signs in Figure 1, the prisoner can choose the correct room. Consider that if the Room 1 sign is true, then the Room 2 sign must be true. Because this cannot happen, the Room 2 sign must be true, making the Room 1 sign false. Because the Room 1 sign is false, the lady can't be in Room 1 and must be in Room 2.

A symbolic system can help in the study of logic. **Truth tables** are often used to show all possible true-false patterns for statements. Table 1 summarizes the truth table for p and $\sim p$.

From two given statements, it is possible to create a new, *compound statement* by using a connective such as *and*. A **compound statement** may be formed by combining two or more statements. For example, "Snow is falling" and "The ski run is open" together with *and* give "Snow is falling and the ski run is open." Other compound statements can be obtained by using the connective *or*. For example, "Snow is falling or the ski run is open." In logic the symbols $\wedge$ and $\vee$ are used to represent the connectives *and* and *or*, respectively. For example, if p represents "Snow is falling" and q represents "The ski run is open," then "Snow is falling and the ski run is open" is denoted by $p \wedge q$. Similarly, "Snow is falling or the ski run is open" is denoted by $p \vee q$.

The truth value of any compound statement, such as $p \wedge q$, is defined using the truth value of each of the simple statements. Because each of the statements p and q may be either true or false, there are four distinct possibilities in the truth table shown in Table 2. The compound statement $p \wedge q$ is the **conjunction** of p and q and is defined to be true if, and only if, both p and q are true. Otherwise, it is false.

The compound statement $p \vee q$ is the **disjunction** p *or* q. In everyday language, *or* is not always interpreted in the same way. In logic, we use an *inclusive or*. The statement "I will go to a movie or I will read a book" means I will either go to a movie, or read a book, or do both. Hence, in logic, p or q, symbolized $p \vee q$, is defined to be false if both p and q are false and true in all other cases. This is summarized in Table 3.

Example 3

Let p: $2 + 3 = 5$, q: $2 \cdot 3 = 6$, and r: $5 + 3 = 9$.

Classify each of the following statements as true or false.

a. $p \wedge q$
b. $q \vee r$
c. $\sim p \vee r$
d. $\sim p \wedge \sim q$
e. $\sim (p \wedge q)$
f. $(p \wedge q) \vee \sim r$

Solution

a. p is true and q is true, so $p \wedge q$ is true.
b. q is true and r is false, so $q \vee r$ is true.
c. $\sim p$ is false and r is false, so $\sim p \vee r$ is false.
d. $\sim p$ is false and $\sim q$ is false, so $\sim p \wedge \sim q$ is false.
e. $p \wedge q$ is true so $\sim (p \wedge q)$ is false.
f. $p \wedge q$ is true and $\sim r$ is true, so $(p \wedge q) \vee \sim r$ is true.

Truth tables are used not only to summarize the truth values of compound statements but also to determine if two statements are *logically equivalent*. *Two statements are* ***logically equivalent*** *if, and only if, they have the same truth values in every possible situation.* If p and q are logically equivalent, we write $p \equiv q$.

Example 4

Show that $\sim(p \wedge q) \equiv \sim p \vee \sim q$ [First of De Morgan's Laws for logic].

Solution Two compound statements are logically equivalent if they have the same truth values. Truth tables for these statements are given in Table 4.

Table 4

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Because the two statements have the same truth values, we know that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Example 4 shows that $\sim(p \wedge q) \equiv \sim p \vee \sim q$. In the same way, we can show that $\sim(p \vee q) \equiv \sim p \wedge \sim q$. These equivalencies are **De Morgan's Laws** for logic.

Theorem 2-1: De Morgan's Laws for Logic

a. $\sim(p \wedge q) \equiv \sim p \vee \sim q$
b. $\sim(p \vee q) \equiv \sim p \wedge \sim q$

NOW TRY THIS 1

a. Use truth tables to confirm the second De Morgan's Law for logic.
b. Write a sentence showing the use of De Morgan's second law in everyday language.

Conditionals and Biconditionals

Statements expressed in the form "if p, then q" are **conditionals**, or **implications**, and are denoted by $p \rightarrow q$. Such statements also can be read "p implies q." The "if" part of a conditional is the **hypothesis** of the implication and the "then" part is the **conclusion**. Many types of statements can be put in "if-then" form. An example follows:

Statement: All equilateral triangles have acute angles.

If-then form: If a triangle is equilateral, then it has acute angles.

Hypothesis Conclusion

Historical Note



George Boole (1815–1864), born in Lincoln, England, is called "the father of logic." As a professor at Queens College in Ireland, he used symbols to represent concepts and developed a system of algebraic manipulations to accompany the symbols. His work, a marriage of logic and mathematics, is known as Boolean algebra.

An implication may also be thought of as a promise. Suppose Betty makes the promise "If I get a raise, then I will take you to dinner." If Betty keeps her promise, the implication is true; if Betty breaks her promise, the implication is false. Consider the four possibilities in Table 5.

Table 5

Case	p	q	Translation of Symbols	Result
(1)	T	T	Betty gets the raise; she takes you to dinner.	Promise Kept
(2)	T	F	Betty gets the raise; she does not take you to dinner.	Promise Broken
(3)	F	T	Betty does not get the raise; she takes you to dinner.	Promise Kept
(4)	F	F	Betty does not get the raise; she does not take you to dinner.	Promise Kept

Table 6

p	q	Implication $q \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The only case in which Betty breaks her promise is when she gets her raise and fails to take you to dinner, case (2). If she does not get the raise, she can either take you to dinner or not without breaking her promise. The definition of implication is summarized in Table 6. Observe that the only case for which the implication is false is when p is true and q is false.

An implication can be worded in several equivalent ways, as follows:

1. If the sun is shining, then the swimming pool is open. (If p, then q.)
2. If the sun is shining, the swimming pool is open. (If p, q.)
3. The swimming pool is open if the sun is shining. (q if p.)
4. The sun is shining implies the swimming pool is open. (p implies q.)
5. The sun is shining only if the pool is open. (p only if q.)
6. The sun's shining is a sufficient condition for the swimming pool to be open. (p is a sufficient condition for q.)
7. The swimming pool's being open is a necessary condition for the sun to be shining. (q is a necessary condition for p.)

A statement in the form $p \rightarrow q$ has three related implication statements, as follows:

Statement:	If p, then q.	$p \rightarrow q$
Converse:	If q, then p.	$q \rightarrow p$
Inverse:	If not p, then not q.	$\sim p \rightarrow \sim q$
Contrapositive:	If not q, then not p.	$\sim q \rightarrow \sim p$

Example 5

Write the converse, the inverse, and the contrapositive for the following statement.

If I am in San Francisco, then I am in California.

Solution

Converse:	If I am in California, then I am in San Francisco.
Inverse:	If I am not in San Francisco, then I am not in California.
Contrapositive:	If I am not in California, then I am not in San Francisco.

Example 5 illustrates that if an implication is true, its converse and inverse are not necessarily true. However, the contrapositive is true. We check these observations on the following true statement: *If a number is a natural number, the number is not 0.* The natural numbers are 1, 2, 3, 4, 5, 6, ... We check the truth of the converse, inverse, and contrapositive.

Converse: *If a number is not 0, then it is a natural number*. This is false; for example, $^{-}6$ is not 0 and it is not a natural number.

Inverse: *If a number is not a natural number, then it is 0*. This is false; for example, $^{-}6$ is not a natural number and it also is not 0.

Contrapositive: *If a number is 0, then it is not a natural number*. This is true because the natural numbers are 1, 2, 3, 4, 5, 6

The contrapositive of the last statement is the original statement. Hence, the preceding discussion suggests that a statement and its contrapositive are logically equivalent. It follows that a statement and its contrapositive cannot have opposite truth values. We summarize this in the following theorem.

Theorem 2-2: Equivalence of a statement and its contrapositive

The implication $p \rightarrow q$ and its contrapositive $\sim q \rightarrow \sim p$ are logically equivalent; that is, $p \rightarrow q \equiv \sim q \rightarrow \sim p$.

Example 6

Use truth tables to prove that $p \rightarrow q \equiv \sim q \rightarrow \sim p$.

Solution Truth tables for these statements are given in Table 7.

Table 7

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Because the two statements have the same truth values, $p \rightarrow q \equiv \sim q \rightarrow \sim p$.

Connecting a statement and its converse with the connective *and* gives $(p \rightarrow q) \wedge (q \rightarrow p)$. This compound statement can be written as $p \leftrightarrow q$ and usually is read "p if, and only if, q." The statement "p if, and only if, q" is a **biconditional**. *Now Try This 2* considers when a biconditional is true.

NOW TRY THIS 2

Build a truth table to determine when a biconditional is true.

Valid Reasoning

In problem solving, the reasoning or argument is said to be **valid** if the conclusion follows unavoidably from the hypotheses. The basis of this determination is a consideration of the possibility of having a true hypothesis and a false conclusion (see Table 6) which yields a false statement or an invalid argument. Consider the following examples:

Hypotheses: All dogs are animals.
Goofy is a dog.

Conclusion: Goofy is an animal.

The statement "All dogs are animals" can be pictured with the **Euler diagram** in Figure 2(a).

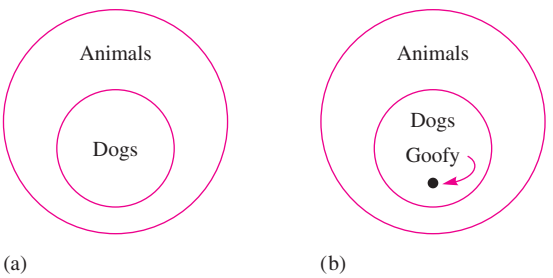


Figure 2

The information "Goofy is a dog" implies that Goofy also belongs to the circle containing dogs, as pictured in Figure 2(b). Goofy also belongs to the circle containing animals. Thus, the reasoning is valid because it is impossible to draw a picture that satisfies the hypotheses and contradicts the conclusion.

Consider the following argument.

Hypotheses:	All elementary schoolteachers are mathematically literate. Some mathematically literate people are not children.
Conclusion:	No elementary schoolteacher is a child.

Let E represent elementary schoolteachers, M represent mathematically literate people, and C represent children. Then the statement "All elementary schoolteachers are mathematically literate" can be pictured as in Figure 3(a). The statement "Some mathematically literate people are not children" can be pictured in several ways. Three of these are illustrated in Figure 3(b) through (d). The conclusion appears to be true with the drawings in Figure 3(b) and (c).

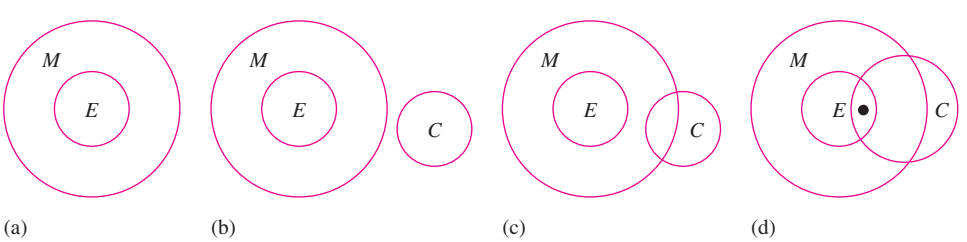


Figure 3

However, according to Figure 3(d), it is possible that some elementary schoolteachers are children, as noted by the dot placed in the figure and yet the given statements are satisfied. Therefore, the conclusion that "No elementary schoolteacher is a child" does not follow from the given hypotheses. Hence, the reasoning is not valid.

If even one picture can be drawn to satisfy the hypotheses of an argument and contradict the conclusion, the argument is not valid. However, to show that an argument is valid, all possible pictures must show that there are no contradictions. *There must be no way to satisfy the hypotheses and contradict the conclusion if the argument is valid.*

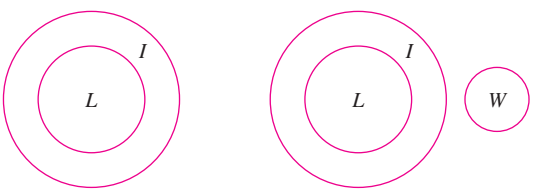
Example 7

Determine whether the following argument is valid:

Hypotheses: In Washington, D.C., all lobbyists have influence.
No one in Washington, D.C., over 6 ft tall has influence.

Conclusion: Persons over 6 ft tall are not lobbyists in Washington, D.C.

Solution If L represents the lobbyists in Washington, D.C., and I the people who have influence, the first hypothesis is pictured as shown in Figure 4(a). If W represents the people in Washington, D.C., over 6 ft tall, the second hypothesis is pictured in Figure 4(b). Because people over 6 ft tall are outside the circle representing those who have influence and lobbyists are in the circle I, the conclusion is valid and no person over 6 ft tall is a lobbyist in Washington, D.C.



(a) (b)

Figure 4

A different method for determining whether an argument is valid uses **direct reasoning** and a form of argument called the **law of detachment** (or ***modus ponens***). For example, consider the following argument.

Hypotheses: If the sun is shining, then we shall go to the beach.
The sun is shining.

Conclusion: We shall go to the beach.

In general, the law of detachment (or *modus ponens*) is stated as follows.

Theorem 2-3: Law of Detachment (*Modus Ponens*)

If the statement "if p, then q" is true, and p is true, then q must be true.

Example 8

Show that $[(p \rightarrow q) \wedge p] \rightarrow q$ is always true.

Solution A truth table for this implication is given in Table 8.

Table 8

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Table 8 is a truth table proof of the law of detachment (*modus ponens*). The statement $[(p \rightarrow q) \wedge p] \rightarrow q$ is a **tautology**, that is, a statement that is true all the time.

Example 9

Determine whether the following argument is valid.

Hypotheses: If x is a natural number greater than 2, then $x^2 > 4$.
$x > 2$

Conclusion: $x^2 > 4$

Solution Using the law of detachment, *modus ponens*, we see that the conclusion is valid.

A different type of reasoning, **indirect reasoning**, uses a form of argument called the **law of denying the consequent** (or ***modus tollens***). For example, consider the following argument.

Hypotheses: If a figure is a square, then it is a rectangle.
The figure is not a rectangle.

Conclusion: The figure cannot be a square.

In general, the law of denying the consequent (or *modus tollens*) is stated as follows:

Theorem 2-4: Law of Denying the Consequent (*Modus Tollens*)

With a conditional accepted as true but having a false conclusion, the hypothesis must be false. Symbolically if $p \rightarrow q$ is true and q is not true, then p is not true.

The *Law of Denying the Consequent* follows from the fact that an implication and its contrapositive are logically equivalent.

Example 10

Determine conclusions for each of the following pairs of true statements:

a. If a person lives in Boston, then the person lives in Massachusetts. Jessica does not live in Massachusetts.
b. If $x = 3$, then $2x \neq 7$. Suppose $2x = 7$.

Solution
a. Jessica does not live in Boston (*modus tollens*).
b. $x \neq 3$ (*modus tollens*)

Finally we consider the **chain rule (*transitivity*)**. Consider the following argument.

Hypotheses: If I save, I will retire early.
If I retire early, I will play golf.

Conclusion: If I save, I will play golf.

In general, the chain rule is stated as follows and proved in the Assessment 2-1A.

Theorem 2-5: Chain Rule (Transitivity)

If "if p, then q" and "if q, then r" are true, then "if p, then r" is true. Symbolically, $[(p \to q) \wedge (q \to r)] \to (p \to r)$.

Example 11

Determine valid conclusions for the following true statements.

a. If a triangle is equilateral, then it is isosceles. If a triangle is isosceles, it has at least two congruent sides.
b. If a number is a whole number, then it is an integer. If a number is an integer, then it is a rational number. If a number is a rational number, then it is a real number.

Solution

a. If a triangle is equilateral, then it has at least two congruent sides.
b. If a number is a whole number, then it is a real number.

People often draw invalid conclusions based on advertising or other information. Assume, for example, the statement "Healthy people eat Super-Bran cereal" is true. Are the following conclusions valid?

If a person eats Super-Bran cereal, then the person is healthy.

If a person is not healthy, the person does not eat Super-Bran cereal.

If the original statement is denoted by $p \to q$, where p is "a person is healthy" and q is "a person eats Super-Bran cereal," then the first conclusion is the converse of $p \to q$ (that is, $q \to p$) and the second conclusion is the inverse of $p \to q$ (that is, $\sim p \to \sim q$). Hence, neither is valid.

Assessment 2-1A

1. Determine which of the following are statements and then classify each statement as true or false.
a. $2 + 4 = 8$
b. Los Angeles is a state in the United States.
c. What time is it?
d. $3 \cdot 2 = 6$

2. Use quantifiers to make each of the following true, where n is a natural number.
a. $n + 8 = 11$ **b.** $n^2 = 4$
c. $n + 3 = 3 + n$ **d.** $5n + 4n = 9n$

3. Use quantifiers to make each equation in exercise 2 false.

4. Write the negation of each of the following statements.
a. This book has 500 pages.
b. $3 \cdot 5 = 15$
c. All dogs have four legs.
d. Some rectangles are squares.
e. Not all rectangles are squares.
f. No dogs have fleas.

5. Identify the following as true or false.
a. For some natural numbers n, $n < 6$ and $n > 3$.
b. For all natural numbers n, $n > 0$ or $n < 5$.

6. Complete each of the following truth tables.

a.

p	$\sim p$	$\sim(\sim p)$
T		
F		

b.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T			
F			

c. Based on part (a), is p logically equivalent to $\sim(\sim p)$?
d. Based on part (b), is $p \vee \sim p$ logically equivalent to $p \wedge \sim p$?

7. a. Complete the following truth table.

p	q	$p \to q$	$\sim p$	$\sim p \vee q$

b. Use columns 3 and 5 to obtain two logically equivalent statement.

c. Use your answer to part (b) to write an implication that is logically equivalent to the following: $2 + 3 \neq 5$ or $4 + 6 = 10$.

8. If q stands for "This course is easy" and r stands for "Lazy students do not study," write each of the following in symbolic form.

a. This course is easy and lazy students do not study.

b. Lazy students do not study or this course is not easy.

c. It is false that both this course is easy and lazy students do not study.

d. This course is not easy.

9. Decide on the truth value of each of the following statements.

a. $2 + 3 = 5$ and $4 + 7 = 10$.

b. The president of the United States in 2013 was not Barack Obama.

c. With every seat filled, the Supreme Court of the United States has 12 justices.

d. If a triangle has three sides of the same length, then the triangle has two sides of the same length.

e. If a triangle has two sides of the same length, then the triangle has three sides of the same length.

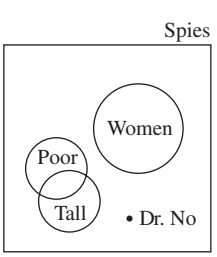
10. Write the negation of each statement in exercise 9.

11. For each of the following, is the pair of statements logically equivalent?

a. $\sim(p \vee q)$ and $\sim p \vee \sim q$

b. $\sim(p \wedge q)$ and $\sim p \wedge \sim q$

12. Describe Dr. No as completely as possible.



13. Write each of the following in symbolic form if p is the statement "It is raining" and q is the statement "The grass is wet."

a. If it is raining, then the grass is wet.

b. If it is not raining, then the grass is wet.

c. If it is raining, then the grass is not wet.

d. The grass is wet if it is raining.

e. The grass is not wet implies that it is not raining.

f. The grass is wet if, and only if, it is raining.

14. For each of the following implications, write the converse, inverse, and contrapositive.

a. If a triangle is scalene, then it has no two sides of the same length.

b. If an angle is not acute, then it is a right angle.

c. If Mary is a U.S. citizen, then she is not a citizen of Cuba.

d. If a number is a whole number, then it is not a natural number.

15. Complete the truth table below to determine whether the negation of $p \wedge \sim q$ is the negation of $p \to q$.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$p \to q$	$\sim(p \to q)$

16. Write a statement logically equivalent to the statement "If a number is a multiple of 8, then it is a multiple of 4."

17. Investigate the validity of each of the following arguments.

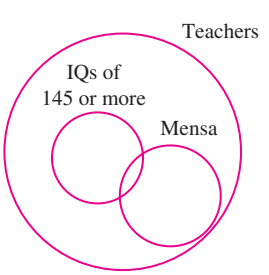
a. All squares are quadrilaterals.
All quadrilaterals are polygons.
Conclusion: All squares are polygons.

b. If a student is a freshman, then the student takes mathematics.
Jane is a sophomore.
Conclusion: Jane does not take mathematics.

18. Use the truth table below to prove the chain rule (transitivity).

p	q	r	$p \to q$	$q \to r$	$(p \to r)$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$

19. Use the Euler diagram pictured to investigate the validity of the argument that "Some teachers with IQs of 145 are in Mensa."



20. For each of the following, form a conclusion that follows logically from the given statements.
- **a.** Some students in Beta Club are in Integrated Mathematics I.
 All students who are in Integrated Mathematics I make A's.
- **b.** If I study for the final, then I will pass the final.
 If I pass the final, then I will pass the course.
 If I pass the course, then I will look for a teaching job.
- **c.** Every equilateral triangle is isosceles.
 There exist triangles that are equilateral.

21. Write the following in if-then form.
- **a.** Every figure that is a square is a rectangle.
- **b.** All integers are rational numbers.
- **c.** Polygons with exactly three sides are triangles.

22. Use De Morgan's Laws to write a negation of each of the following.
- **a.** $3 \cdot 2 = 6$ and $1 + 1 \neq 3$.
- **b.** You can pay me now or you can pay me later.

Assessment 2-1B

1. Determine which of the following are statements and then classify each statement as true or false:
- **a.** Shut the window.
- **b.** He is in town.
- **c.** $2 \cdot 2 = 2 + 2$
- **d.** $2 + 3 = 8$
- **e.** Stay put!

2. Use quantifiers to make each of the following true, where n is a natural number:
- **a.** $n + 0 = n$
- **b.** $n + 1 = n + 2$
- **c.** $3(n + 2) = 12$
- **d.** $n^3 = 8$

3. Use quantifiers to make each equation in exercise 2 false.

4. Write the negation of each of the following statements.
- **a.** $6 < 8$
- **b.** Some cats do not have nine lives.
- **c.** All squares are rectangles.
- **d.** Not all numbers are positive.
- **e.** Some people have blond hair.

5. Identify the following as true or false.
- **a.** For some natural numbers n, $n > 5$ and $n > 2$.
- **b.** For all natural numbers n, $n > 5$ or $n < 5$.

6.
- **a.** If you know that p is true, what can you conclude about the truth value of $p \vee q$, even if you don't know the truth value of q?
- **b.** If you know that p is false, what can you conclude about the truth value of $p \rightarrow q$?

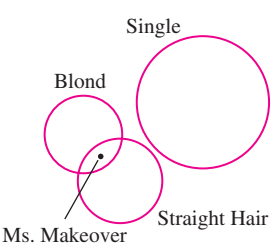
7. If q stands for "You said goodbye" and r stands for "I said hello," write each of the following in symbolic form.
- **a.** You said goodbye and I said hello.
- **b.** You said goodbye and I did not say hello.
- **c.** I did not say hello or you did not say goodbye.
- **d.** It is false that both you said goodbye and I said hello.

8. Decide on the truth value of each of the following statements.
- **a.** $4 + 6 = 10$ or $2 + 3 = 5$.
- **b.** A National Football League team can have more than 11 players on the field while a game is in progress.
- **c.** The first president of the United States was George Washington.
- **d.** If a quadrilateral has three sides of the same length, the quadrilateral has four sides of the same length.
- **e.** If a rectangle has four sides of the same length, then it has three sides of the same length.

9. Write the negation of each statement in exercise 8.

10. For each of the following, is the pair of statements logically equivalent?
- **a.** $\sim(p \vee q)$ and $\sim p \wedge \sim q$
- **b.** $\sim(p \wedge q)$ and $\sim p \vee \sim q$

11. Use the Euler diagram to describe Ms. Makeover as completely as possible.



12. Write each of the following in symbolic form if p is the statement "You build it" and q is the statement "They will come."
- **a.** If you build it, they will come.
- **b.** If you do not build it, then they will come.
- **c.** If you build it, they will not come.
- **d.** They will come if you build it.
- **e.** If you do not build it, then they will not come.
- **f.** If they will not come, then you do not build it.

13. For each of the following implications, state the converse, inverse, and contrapositive.
- **a.** If $x = 3$, then $x^2 = 9$.
- **b.** If it snows, then classes are canceled.

14. Iris makes the true statement "If it rains, then I am going to the movies." Does it follow logically that "if it does not rain, then Iris does not go to the movies?"

15. Investigate the validity of each of the following arguments.
- **a.** All women are mortal.
 Hypatia was a woman.
 Conclusion: Hypatia was mortal.

b. All of the original Harry Potter books were written by J. K. Rowling.
Dirty Harry was not written by J. K. Rowling.
Conclusion: *Dirty Harry* is not an original Harry Potter book.

c. Some whole numbers are natural numbers.
Seven is a whole number.
Conclusion: Seven is a natural number.

16. For each of the following, form a conclusion that follows logically from the given statements.

a. All students in Integrated Mathematics I are in Kappa Mu Epsilon.
Helen is in Integrated Mathematics I.

b. All engineers need mathematics.
Ron does not need mathematics.

c. All bicycles have tires.
All tires use rubber.

17. Write each of the following in if-then form.

a. All natural numbers are real numbers.

b. Every circle is a closed figure.

18. Use De Morgan's Laws to write a negation of each of the following.

a. $3 + 5 \neq 9$ and $3 \cdot 5 = 15$.

b. I am going or she is going.

19. Use the chain rule to form a logical conclusion based on the following statements.
A square is a rectangle. A rectangle is a parallelogram.

Mathematical Connections 2-1

Reasoning

1. Explain why commands and questions are not statements.

2. Explain how to write the negation of a quantified statement in the form "Some A's are B's."

3. **a.** Describe under what conditions a disjunction is true.

b. Describe under what conditions an implication is true.

4. Consider the nursery rhyme.

For want of a nail, the shoe was lost.
For want of a shoe, the horse was lost.
For want of a horse, the rider was lost.
For want of a rider, the battle was lost.
For want of a battle, the war was lost.
Therefore, for want of a nail, the war was lost.

a. Write each line as an *if-then* statement.

b. Does the conclusion (the *therefore* statement) follow logically. Why?

5. In an e-mail address line, a comma or a semicolon is used to separate addresses. From an e-mail sender's standpoint, explain the logical meaning of the punctuation mark.

6. If a statement and its converse are both true, which of the other forms, inverse and contrapositive, are also true, if any?

Open-Ended

7. Table 5 shows a situation used to motivate the definition of the truth values for the implication $p \rightarrow q$. Describe a similar situation that could be used to motivate this same implication.

8. A student in a logic class suggested that one could take the negation of "both sides" of a tautology and construct another tautology. Is this correct? Justify your answer.

Cooperative Learning

9. Discuss the paradox arising from the following set of statements.
(a) This textbook is 2000 pages long.
(b) The author of this textbook is Dante.
(c) The statements (a), (b), and (c) are all false.

10. With classmates divided into small groups, each group should write a series of statements and give them to another group to have the second group write any possible conclusions. The material with conclusions should then be returned to the original writing group for evaluation of the conclusions. The original statements, possible conclusions, and evaluations should all be shared with the class.

Connecting Mathematics to the Classroom

11. Write an explanation for a seventh grader showing how you could explain that even with a false hypothesis, an argument can still be valid.

12. Analyze the following argument that was presented by a sixth-grade student.
Joe said, "Mr. Johnson said that some of us could go to recess at 10:00 A.M., and all got to go but me. He lied."
Describe circumstances in the situation where Mr. Johnson would have lied.

2-2 Describing Sets

2-2 Objectives

Students will be able to understand and explain

- Set language and structure as applied to elementary mathematics.
- Connections between finite sets and whole numbers.
- Uses of one-to-one correspondence.

CCSS **In the *Common Core Standards for Mathematical Practice*, we find the following:**

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for symbols involved. (p. 6)

In this chapter, the groundwork is laid for an understanding of the structure of sets and the logic needed to reason abstractly. This entails both making sense of quantities and their relationships in problem situations and understanding how one begins to "use stated assumptions, definitions, and previously established results in constructing arguments." (p. 6)

In the years from 1871 through 1884, Georg Cantor created *set theory*, which had a profound effect on research and mathematics teaching. Sets, and relations between sets, form a basis for teaching children the *concept* of the set of whole numbers, $\{0, 1, 2, 3, \ldots\}$, and the concept of "less than" as well as addition, subtraction, and multiplication of whole numbers. Understanding whole numbers and operations on whole numbers can be enhanced by the notion of sets. In this section we introduce set notation and relations between sets.

The Language of Sets

A **set** is understood to be any collection of objects. Individual objects in a set are **elements**, or **members**, of the set. For example, each letter is an element of the set of letters in the English language. Capital letters are generally used to name sets. The elements of the set are listed inside a pair of braces, { }. The set A of lowercase letters of the English alphabet can be written in set notation as follows:

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

The order in which the elements are written makes no difference, and *each element is listed only once*. Thus, $\{b, o, k\}$ and $\{k, o, b\}$ are considered to be the same set.

We show that an element belongs to a set by using the symbol $\in$. Thus, if $A = \{a, b, c\}$, then $b \in A$. If an element does not belong to a set, we use the symbol $\notin$. For example, $3 \notin A$. In mathematics, the same letter, one lowercase and the other uppercase, cannot be freely interchanged. For example, we have $b \in A$ but $B \notin A$.

A set must be **well defined**; that is, if we are given a set and some particular object, then we must be able to tell whether the object does or does not belong to the set. For example, the set of all citizens of Pasadena, California, who ate rice on January 1, 2014, is well defined. We

Historical Note



Georg Cantor (1845–1918) obtained his doctorate in mathematics in Berlin at age 22. Most of his academic career was spent at the University of Halle. His hope of becoming a professor at the University of Berlin did not materialize because his work gained little recognition during his lifetime. After his death Cantor's work in set theory was praised as an astonishing product of mathematical thought, one of the most beautiful realizations of human activity. ●

personally may not know if a particular resident of Pasadena ate rice or not, but that resident either belongs or does not belong to the set. On the other hand, the set of all tall people is not well defined because there is no clear meaning of "tall."

We may use sets to define mathematical terms. For example from Chapter 1, we have the set of natural numbers in $N = \{0, 1, 2, 3, \ldots\}$. Two common methods of describing sets are the **listing**, or **roster method**, and **set-builder notation**, as seen in the examples:

Listing, or roster method: $C = \{1, 2, 3, 4\}$
Set-builder notation: $C = \{x \mid x \in N, x < 5\}$
The latter notation is read as follows:

C	$=$	$\{$	x	$\mid$	$x \in N,$	$x < 5\}$
Set C	is equal to	the set of	all elements x	such that	x is an element of the natural numbers, and	x is less than 5

In the set-builder notation $C = \{x \mid x \in N \text{ and } x < 5\}$; the comma is often a place holder for "and." With this notation both conditions, $x \in N$ and $x < 5$, must be true.

In set-builder notation any lowercase letter can be used to represent a general element. Set-builder notation is useful when the individual elements of a set are not known or they are too numerous to list. For example, the set of decimals between 0 and 1 can be written as

$$D = \{x \mid x \text{ is a decimal between 0 and 1}\}.$$

It would be impossible to list all the elements of D. Hence the set-builder notation is indispensable here.

Example 12

Use set-builder notation to write the following sets whose elements are terms of arithmetic sequences.

a. $\{2, 4, 6, 8, 10, \ldots\}$ **b.** $\{1, 3, 5, 7, \ldots\}$

Solution

a. $\{x \mid x \text{ is an even natural number}\}$. Because every even natural number can be written as 2 times some natural number, this set can also be written as $\{x \mid x = 2n, n \in N\}$, or $\{2n \mid n \in N\}$.

b. $\{x \mid x \text{ is an odd natural number}\}$. Because every odd natural number can be written as some even number minus 1, this set can also be written as $\{x \mid x = 2n - 1, n \in N\}$, or $\{2n - 1 \mid n \in N\}$.

In Example 12, we note that the elements in the sets of parts (a) and (b) constitute sequences where elements are ordered.

Example 13

Each of the following sets is described in set-builder notation. Write each of the sets by listing its elements.

a. $C = \{2k + 1 \mid k = 3, 4, 5\}$
b. $D = \{x \mid x \text{ is an even natural number less than 8}\}$

Solution

a. We substitute 3, 4, and 5 for k in $2k + 1$ and obtain the corresponding values shown in Table 9. Thus, $C = \{7, 9, 11\}$.

Table 9

k	$2k + 1$
3	$2 \cdot 3 + 1 = 7$
4	$2 \cdot 4 + 1 = 9$
5	$2 \cdot 5 + 1 = 11$

b. $D = \{2, 4, 6\}$

As noted earlier, the order in which the elements are listed does not matter. If sets A and B are **equal**, written $A = B$, then every element of A is an element of B, and every element of B is an element of A. If A does not equal B, we write $A \neq B$.

Definition of Equal Sets

Two sets are **equal** if, and only if, they contain exactly the same elements.

One-to-One Correspondence

One of the most useful concepts in set theory is a **one-to-one correspondence** between two sets. For example, consider the set of people $P = \{\text{Tomas, Dick, Mari}\}$ and the set of swimming lanes $S = \{1, 2, 3\}$. Suppose each person in P is to swim in a lane numbered 1, 2, or 3 so that no two people swim in the same lane. Such a person-lane pairing is a one-to-one correspondence. One way to exhibit a one-to-one correspondence is shown in Figure 5 with arrows connecting corresponding elements.

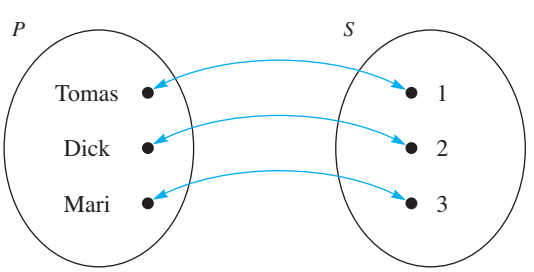


Figure 5

Other possible one-to-one correspondences exist between the sets P and S. All six possible one-to-one correspondences between sets P and S can be listed as follows:

1. Tomas $\leftrightarrow$ 1
Dick $\leftrightarrow$ 2
Mari $\leftrightarrow$ 3

2. Tomas $\leftrightarrow$ 1
Dick $\leftrightarrow$ 3
Mari $\leftrightarrow$ 2

3. Tomas $\leftrightarrow$ 2
Dick $\leftrightarrow$ 1
Mari $\leftrightarrow$ 3

4. Tomas $\leftrightarrow$ 2
Dick $\leftrightarrow$ 3
Mari $\leftrightarrow$ 1

5. Tomas $\leftrightarrow$ 3
Dick $\leftrightarrow$ 1
Mari $\leftrightarrow$ 2

6. Tomas $\leftrightarrow$ 3
Dick $\leftrightarrow$ 2
Mari $\leftrightarrow$ 1

Notice that each listing 1–6 represents a single one-to-one correspondence between the sets P and S. A complete set of one-to-one correspondences between sets P and S can also be listed using a table as in Table 10.

Table 10

	Lanes		
Pairings	**1**	**2**	**3**
1.	Tomas	Dick	Mari
2.	Tomas	Mari	Dick
3.	Dick	Tomas	Mari
4.	Dick	Mari	Tomas
5.	Mari	Tomas	Dick
6.	Mari	Dick	Tomas

The general definition of one-to-one correspondence between two sets follows.

Definition of One-to-One Correspondence

If the elements of sets P and S can be paired so that for each element of P there is exactly one element of S and for each element of S there is exactly one element of P, then the two sets P and S are in **one-to-one correspondence**.

NOW TRY THIS 3

Consider a set of four people $\{A, B, C, D\}$ and a set of four swimming lanes $\{1, 2, 3, 4\}$.

a. Exhibit all the one-to-one correspondences between the two sets.
b. How many such one-to-one correspondences are there?
c. Find the number of one-to-one correspondences between two sets with five elements each and explain your reasoning.

Figure 6 shows all possible one-to-one correspondences from Table 10. To read the tree diagram and see the one-to-one correspondence, we follow each branch. The person occupying a specific lane in a correspondence is listed below the lane number. For example, the top branch gives the pairing (Tomas, 1), (Dick, 2), and (Mari, 3).

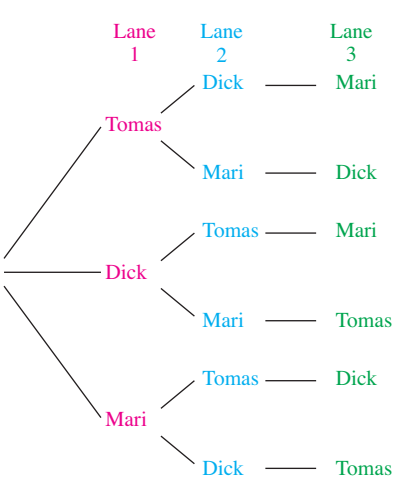


Figure 6

Observe in Figure 6 when assigning a swimmer to lane 1, we have a choice of three people: Tomas, Dick, or Mari. If we put Tomas in lane 1, then he cannot be in lane 2, and hence the second lane must be occupied by either Dick or Mari. In the same way, we see that if Dick is in lane 1, then there are two choices for lane 2: Tomas or Mari. Similarly, if Mari is in lane 1, then again there are two choices for the second lane: Tomas or Dick. Thus, for each of the three ways we can fill the first lane, there are two subsequent ways to fill the second lane, and hence there are $2 + 2 + 2$, or $3 \cdot 2$, or 6 ways to arrange the swimmers in the first two lanes. For each arrangement of the swimmers in the first two lanes, there remains only one possible swimmer to fill the third lane. For example, if Mari fills the first lane and Dick fills the second, then Tomas must be in the third. Thus, the total number of arrangements for the three swimmers is equal to $3 \cdot 2 \cdot 1$, or 6.

Similar reasoning can be used to find how many ice-cream arrangements are possible on a two-scoop cone if 10 flavors are offered. If we count chocolate and vanilla (chocolate on bottom and vanilla on top) different from vanilla and chocolate (vanilla on bottom and chocolate on top) and allow two scoops to be of the same flavor, we proceed as follows. There are 10 choices for the first scoop, and for each of these 10 choices there are 10 subsequent choices for the second scoop. Thus, the total number of arrangements is $10 \cdot 10$, or 100.

The counting argument used to find the number of possible one-to-one correspondences between the set of swimmers and the set of lanes and the previous problem about ice-cream-scoop arrangements are examples of the Fundamental Counting Principle.

Theorem 2-6: Fundamental Counting Principle

If event M can occur in m ways and, after it has occurred, event N can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.

NOW TRY THIS 4

Write a mathematical expression showing how many one-to-one correspondences there are between two sets with 17 elements each.

Equivalent Sets

Closely associated with one-to-one correspondences is the concept of **equivalent sets**. For example, suppose a room contains 20 chairs with one student is sitting in each chair and no one standing. There is a one-to-one correspondence between the set of chairs and the set of students in the room. In this case, the set of chairs and the set of students are not equal sets because the elements are different, but they are considered equivalent sets.

Definition of Equivalent Sets

Two sets A and B are **equivalent**, written $A \sim B$, if, and only if, there exists a one-to-one correspondence between the sets.

The term *equivalent* should not be confused with *equal*. The difference is seen in Example 14.

Example 14

Let

$$A = \{p, q, r, s\}, \quad B = \{a, b, c\}, \quad C = \{x, y, z\}, \quad \text{and} \quad D = \{b, a, c\}.$$

Compare the sets, using the terms *equal* and *equivalent*.

Solution

Each set is both equivalent to and equal to itself.
Sets A and B are not equivalent $(A \not\sim B)$ and not equal $(A \neq B)$.
Sets A and C are not equivalent $(A \not\sim C)$ and not equal $(A \neq C)$.
Sets A and D are not equivalent $(A \not\sim D)$ and not equal $(A \neq D)$.
Sets B and C are equivalent $(B \sim C)$ but not equal $(B \neq C)$.
Sets B and D are equivalent $(B \sim D)$ and equal $(B = D)$.
Sets C and D are equivalent $(C \sim D)$ but not equal $(C \neq D)$.

Cardinal Numbers

Without knowing how to count, a child might tell that there are as many fingers on the left hand as on the right hand by matching the fingers on one hand with the fingers on the other hand, as in Figure 7. Naturally placing the fingers so that the left thumb touches the right thumb, the left index finger touches the right index finger, and so on, exhibits a one-to-one correspondence between the fingers of the two hands. Similarly, without counting, children realize that if every student in a class sits in a chair and no chairs are empty, there are as many chairs as students, and vice versa.

A one-to-one correspondence between sets helps explain the concept of a number. Consider the five sets $\{a, b\}$, $\{p, q\}$, $\{x, y\}$, $\{b, a\}$, and $\{*, \#\}$; the sets are equivalent to one another and share the property of "twoness"; that is, these sets have the same *cardinal number*, namely, 2. The **cardinal number** of a set S, denoted $n(S)$, indicates the number of elements in the set S. If $S = \{a, b\}$, the cardinal number of S is 2, and we write $n(S) = 2$. *If two sets, A and B, are equivalent, then A and B have the same cardinal number; that is, $n(A) = n(B)$.*

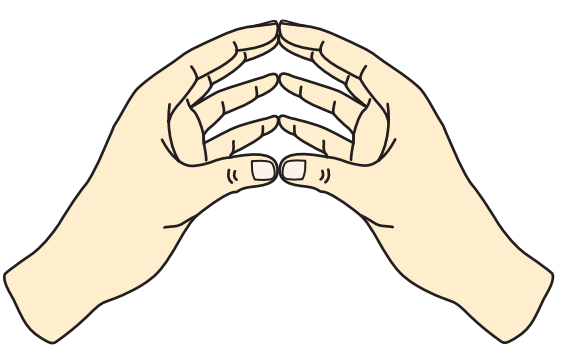


Figure 7

The Empty Set

A set that contains no elements has cardinal number 0 and is an **empty**, or **null, set**. The empty set is designated by the symbol $\varnothing$ or $\{\ \}$. Two examples of empty sets follow.

$$C = \{x \mid x \text{ was a state of the United States before 1200 CE}\}$$
$$D = \{x \mid x \text{ is a natural number such that } x^2 = 17\}$$

The empty set is often incorrectly written as $\{\varnothing\}$. This set is not empty because it contains one element, namely $\varnothing$. Likewise, $\{0\}$ does not represent the empty set. Why?

A set is a **finite set** if the cardinal number of the set is zero or a natural number. The set N of natural numbers is an **infinite set**; it is not finite. The set W, containing all the natural numbers and 0, is the set of **whole numbers**: $W = \{0, 1, 2, 3, \ldots\}$. W is an infinite set.

NOW TRY THIS 5

Explain why there can be no greatest natural number; that is, explain why the set of natural numbers is not a finite set.

More About Sets

The **universal set**, or the **universe**, denoted U, is the set that contains all elements being considered in a given discussion. Suppose $U = \{x \mid x \text{ is a person living in California}\}$ and $F = \{x \mid x \text{ is a female living in California}\}$. The universal set, U, and set F can be represented by a diagram, as in Figure 8(a). The universal set is represented by a large rectangle, and F is indicated by the circle inside the rectangle, as shown in Figure 8(a). This figure is an example of a **Venn diagram**, named after the British logician and philosopher John Venn (1834–1923). The set of elements in the universe that are not in F, denoted by $\overline{F}$, is the set of males living in California and is the **complement** of F. It is represented by the shaded region in Figure 8(b).

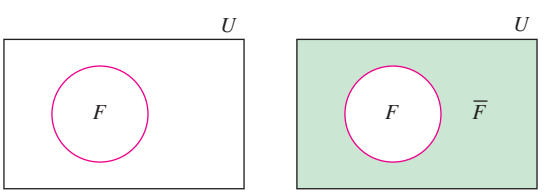


Figure 8

Definition of Set Complement

The **complement** of a set A, written $\overline{A}$, is the set of all elements in a given universal set U that are not in A; that is, $\overline{A} = \{x \mid x \in U \text{ and } x \notin A\}$.

Example 15

a. If $U = \{a, b, c, d\}$ and $B = \{c, d\}$, find $\overline{B}$, $\overline{U}$, and $\overline{\varnothing}$.
b. If $U = \{x \mid x \text{ is an animal in the zoo}\}$ and $S = \{x \mid x \text{ is a snake in the zoo}\}$, describe $\overline{S}$.
c. If $U = N$, $E = \{2, 4, 6, 8, \ldots\}$, and $O = \{1, 3, 5, 7, \ldots\}$, find $\overline{E}$, and $\overline{O}$.

Solution

a. $\overline{B} = \{a, b\}$, $\overline{U} = \varnothing$, and $\overline{\varnothing} = U$
b. $\overline{S} = \{x \mid x \text{ is an animal in the zoo and } x \text{ is not a snake}\}$
c. $\overline{E} = O$ and $\overline{O} = E$

Subsets

Consider the sets $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6\}$. All the elements of B are contained in A and we say that B is a **subset** of A. We write $B \subseteq A$. In general, we have the following:

Definition of Subset

B is a **subset** of A, written $B \subseteq A$, if, and only if, every element of B is an element of A.

This definition allows B to be equal to A. The definition is written with the phrase "if, and only if," which means "if B is a subset of A, then every element of B is an element of A, and if every element of B is an element of A, then B is a subset of A." An alternate definition follows.

Alternate Definition of Set Equality

Two sets A and B are equal if, and only if, $A \subseteq B$ and $B \subseteq A$.

When a set A is not a subset of another set B, we write $A \not\subseteq B$. To show that $A \not\subseteq B$, we must find at least one element of A that is not in B. If $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$, then A is not a subset of B because 5 is an element of A but not of B. Likewise, $B \not\subseteq A$ because 2 belongs to B but not to A.

The Empty Set as a Subset of Every Set

It is not obvious how the empty set fits the definition of a subset because no elements in the empty set are elements of another set. To investigate this problem, we use the strategies of *indirect reasoning* and *looking at a special case*.

For the set $\{1, 2\}$, either $\varnothing \subseteq \{1, 2\}$ or $\varnothing \not\subseteq \{1, 2\}$. Suppose $\varnothing \not\subseteq \{1, 2\}$. Then there must be some element in $\varnothing$ that is not in $\{1, 2\}$. Because the empty set has no elements, there cannot be an element in the empty set that is not in $\{1, 2\}$. Consequently, $\varnothing \not\subseteq \{1, 2\}$ is false. Therefore, the only other possibility, $\varnothing \subseteq \{1, 2\}$, is true. The same reasoning can be applied in the case of the empty set and any other set. Therefore, *the empty set is a subset of every set*.

Proper Subsets

For sets $A = \{a, b, c\}$ and $B = \{c, b, a\}$, we have $B \subseteq A$, $A \subseteq B$, and $B = A$. If C is a subset of A and C is not equal to A, then C is a **proper subset** of A, written $C \subset A$. This means that every element of C is contained in A and there is at least one element of A that is not in C. For example, if $X = \{1, 2\}$ and $Y = \{1, 2, 3\}$ then $X \subset Y$.

Definition of Proper Subset

For all sets A and B, B is a **proper subset** of A, written $B \subset A$, if, and only if, $B \subseteq A$ and $B \neq A$; that is, every element of B is an element of A, and there is at least one element of A that is not an element of B.

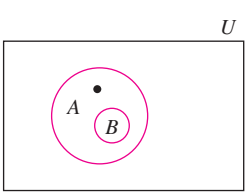


Figure 9

To indicate a proper subset, sometimes a Venn diagram like the one shown in Figure 9 is used, showing a dot (an element) in A that is not in B. The empty set is a proper subset of every non-empty set. (Why?)

Example 16

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3\}$, and $P = (x | x = 2^n - 1, n \in N\}$.

a. Identify all the subset relationships that occur among these sets.
b. Identify all the proper subset relationships that occur among these sets.
c. If $C = \{2k | k \in N\}$ and $D = \{4k | k \in N\}$, show that one of the sets is a subset of the other.

Solution

a. Because $2^1 - 1 = 1$, $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^4 - 1 = 15$, $2^5 - 1 = 31$, and so on, $P = \{1, 3, 7, 15, 31, \ldots\}$. Thus, $B \subseteq P$. Also $B \subseteq A$, $A \subseteq A$, $B \subseteq B$ and $P \subseteq P$.
b. $B \subset A$ and $B \subset P$.
c. $C = \{2 \cdot 1, 2 \cdot 2, 2 \cdot 3, 2 \cdot 4, \ldots\} = \{2, 4, 6, 8, \ldots\}$ and $D = \{4 \cdot 1, 4 \cdot 2, 4 \cdot 3, 4 \cdot 4, \ldots\} = \{4, 8, 12, 16, \ldots\}$. Each element in D appears in C. This is true because $4k = 2(2k)$. Therefore, every element of D is an element of C and $D \subseteq C$.

Subsets and elements of sets are often confused. We say that $2 \in \{1, 2, 3\}$. But because 2 is not a set, we cannot substitute the symbol $\subseteq$ for $\in$. However, $\{2\} \subseteq \{1, 2, 3\}$ and $\{2\} \subset \{1, 2, 3\}$. Notice that $\{2\} \notin \{1, 2, 3\}$ and $2 \not\subseteq \{1, 2, 3\}$.

Inequalities: An Application of Set Concepts

The notion of proper subset and the concept of one-to-one correspondence can be used to define the concept of "less than" among natural numbers. The set $\{a, b, c\}$ has fewer elements than the set $\{w, x, y, z\}$ because when we try to pair the elements of the two sets, as in

$$\begin{matrix} \{a, b, c\} \\ \updownarrow\ \updownarrow\ \updownarrow \\ \{x, y, z, w\} \end{matrix}$$

we see that there is at least one element of the second set that is not paired with an element of the first set. The set $\{a, b, c\}$ is equivalent to $\{x, y, z\}$, a proper subset of the set $\{x, y, z, w\}$. The generalization leads to a definition of **less than**.

Definition of Less Than and Greater Than

If A and B are finite sets, A has fewer elements than B if A is equivalent to a proper subset of B.

We say that $n(A)$ is **less than** $n(B)$ and write $n(A) < n(B)$. We say that b is **greater than** a, written $b > a$, if, and only if, $a < b$.

If A and B are finite sets and $A \subset B$, then A has fewer elements than B and it is not possible to find a one-to-one correspondence between the sets. Consequently, A and B are not equivalent. However, when both sets are infinite and $A \subset B$, the sets could be equivalent. For example, consider the set N of natural numbers and the set E of even natural numbers. $E \subset N$, but it is still possible to find a one-to-one correspondence between the sets. To do so, we correspond each number in set N to a number in set E that is twice as great; that is, $n \in N$ corresponds to $2n \in E$, as shown next.

$$N = \{1, 2, 3, 4, 5, \ldots, n, \ldots\}$$
$$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\quad \searrow$$
$$E = \{2, 4, 6, 8, 10, \ldots, 2n, \ldots\}$$

Notice that in the correspondence, every element of N corresponds to a unique element in E and, conversely, every element of E corresponds to a unique element in N. For example, 11 in N corresponds to $2 \cdot 11$, or 22, in E. And 100 in E corresponds to $100 \div 2$, or 50, in N. Thus, $N \sim E$; that is, N and E are equivalent.

Students sometimes have difficulty with infinite sets and especially with their cardinal numbers, called **transfinite numbers**. As shown, E is a proper subset of N, but because they can be placed in a one-to-one correspondence, they are equivalent and have the same cardinal number. Georg Cantor was the first to introduce the concept of a transfinite number. He used the first letter of the Hebrew alphabet $\aleph_0$ (aleph-null) for the cardinal number of the set N.

Problem Solving Passing a Senate Measure

A committee of senators consists of Abel, Baro, Carni, and Davis. Suppose each member of the committee has one vote and a simple majority is needed to either pass or reject any measure. A measure that is neither passed nor rejected is considered to be blocked and will be voted on again. Determine the number of ways a measure could be passed or rejected and the number of ways a measure could be blocked.

Understanding the Problem We are asked to determine how many ways the committee of four could pass or reject a measure and how many ways the committee of four could block a measure. To pass or reject a measure requires a winning coalition, that is, a group of senators who can pass or reject the measure, regardless of what the others do. To block a proposal, there must be a blocking coalition, that is, a group who can prevent any measure from passing but who cannot reject the measure.

Devising a Plan To solve the problem, we can *make a list* of subsets of the set of senators. Any subset of the set of senators with three or four members will form a winning coalition. Any subset of the set of senators with exactly two members will form a blocking coalition.

Carrying Out the Plan We list all subsets of the set $S = \{\text{Abel, Baro, Carni, Davis}\}$. For ease, we identify the members as follows: A—Abel, B—Baro, C—Carni, D—Davis.

$\varnothing$	$\{A\}$	$\{A, B\}$	$\{A, B, C\}$	$\{A, B, C, D\}$
	$\{B\}$	$\{A, C\}$	$\{A, B, D\}$	
	$\{C\}$	$\{A, D\}$	$\{A, C, D\}$	
	$\{D\}$	$\{B, C\}$	$\{B, C, D\}$	
		$\{B, D\}$		
		$\{C, D\}$		

There are five subsets with at least three members that can form a winning coalition to pass or reject a measure and six subsets with exactly two members that can block a measure.

Looking Back Other questions that might be considered include:

1. How many minimal winning coalitions are there? In other words, how many subsets are there of which no proper subset could pass a measure?
2. Devise a method to solve this problem without listing all subsets.
3. In "Carrying Out the Plan," 16 subsets of $\{A, B, C, D\}$ are listed. Use that result to systematically list all the subsets of a committee of five senators. Can you find the number of subsets of the 5-member committee without actually counting the subsets?

Number of Subsets of a Finite Set

How many subsets can be made from a set containing n elements? To obtain a general formula, we use the strategy of *examine a simpler case* first.

1. If $P = \{a\}$, then P has two subsets, $\varnothing$ and $\{a\}$.
2. If $Q = \{a, b\}$, then Q has four subsets, $\varnothing$, $\{a\}$, $\{b\}$, and $\{a, b\}$.
3. If $R = \{a, b, c\}$, then R has eight subsets, $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{a, b, c\}$.

We could list all the subsets of a given set by using a tree diagram. For example, tree diagrams for the subsets of $Q = \{a, b\}$ and $R = \{a, b, c\}$ are given in Figure 10(a) and (b) respectively.

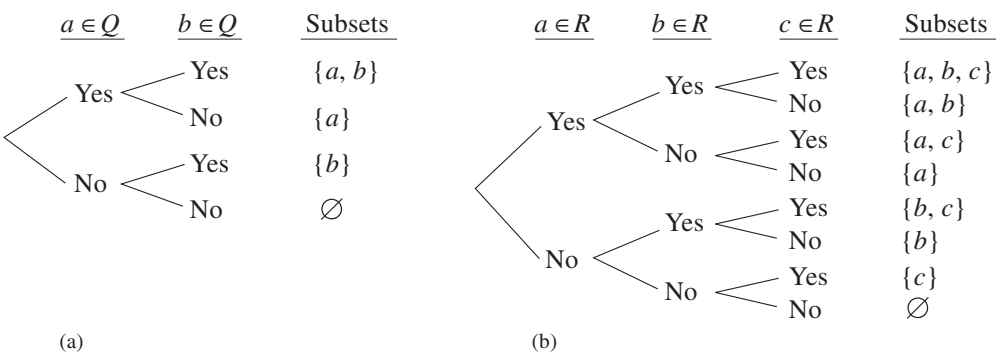


Figure 10

Using the information from these cases, we *make a table* and *search for a pattern*, as in Table 11.

Table 11

Number of Elements	Number of Subsets
1	2, or 2^1
2	4, or 2^2
3	8, or 2^3
·	·
·	·
·	·

Table 11 suggests that for four elements, there might be 2^4, or 16, subsets. Is this correct? If $S = \{a, b, c, d\}$, then all the subsets of $R = \{a, b, c\}$ are also subsets of S. Eight new subsets are also formed by including the element d in each of the eight subsets of R. The eight new subsets are $\{d\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, and $\{a, b, c, d\}$. Thus, there are twice as many subsets of set S (with four elements) as there are of set R (with three elements). Consequently, there are $2 \cdot 8$, or 2^4, subsets of a set with four elements. Because including one more element in a finite set doubles the number of possible subsets of the new set, a set with five elements will have $2 \cdot 2^4$, or 2^5, subsets, and so on. In each case, the number of elements and the power of 2 used to obtain the number of subsets are equal. *Therefore, if there are n elements in a set, 2^n subsets can be formed.* If we apply this formula to the empty set; that is, when $n = 0$, we have $2^0 = 1$. The pattern is meaningful because the empty set has only one subset—itself.

NOW TRY THIS 6

a. How many proper subsets does a set with five elements have?
b. Write a formula to show how many proper subsets a set with n elements has. Explain your reasoning.

Assessment 2-2A

1. Write the following sets using the listing (roster) method or using set-builder notation.
a. The set of letters in the word *assessment*
b. The set of natural numbers greater than 20

2. Rewrite the following using mathematical symbols.
a. P is equal to the set containing p, q, r, and s.
b. The set consisting of the elements 1 and 2 is a proper subset of $\{1, 2, 3\}$.
c. The set consisting of the elements 0 and 1 is not a subset of $\{1, 2, 3\}$.

3. Which of the following pairs of sets can be placed in one-to-one correspondence?
a. $\{1, 2, 3, 4, 5\}$ and $\{m, n, o, p, q\}$
b. $\{a, b, c, d, e, f, \ldots, m\}$ and $\{1, 2, 3, 4, 5, 6, \ldots, 13\}$
c. $\{x|x$ is a letter in the word *mathematics*$\}$ and $\{1, 2, 3, 4, \ldots, 11\}$

4. How many one-to-one correspondences are there between two sets with
a. 6 elements each?
b. n elements each? Explain your reasoning.

5. How many one-to-one correspondences are there between the sets $\{x, y, z, u, v\}$ and $\{1, 2, 3, 4, 5\}$ if in each correspondence
a. x must correspond to 5?
b. x must correspond to 5 and y to 1?
c. x, y, and z must correspond to odd numbers?

6. Which of the following represent equal sets?
$A = \{a, b, c, d\}$ $B = \{x, y, z, w\}$
$C = \{c, d, a, b\}$ $D = \{x|1 \leq x \leq 4 \text{ and } x \in N\}$
$E = \varnothing$ $F = \{\varnothing\}$
$G = \{0\}$ $H = \{\ \}$
$I = \{x|x = 2n + 1 \text{ and } n \in W\}$
$L = \{x|x = 2n - 1 \text{ and } n \in N\}$

7. Find the cardinal number of each of the following sets. Assume the pattern of elements continues in each part in the order given.
a. $\{201, 202, 203, \ldots, 1100\}$
b. $\{1, 3, 5, \ldots, 101\}$
c. $\{1, 2, 4, 8, 16, \ldots, 1024\}$
d. $\{x|x = k^3, k = 1, 2, 3, \ldots, 100\}$

8. If U is the set of all college students and A is the set of all college students with a straight-A average, describe $\overline{A}$.

9. Suppose B is a proper subset of C.
a. If $n(C) = 8$, what is the maximum number of elements in B?
b. If $n(B) = 8$, what is the maximum number of elements in C?

10. Suppose C is a subset of D and D is a subset of C.
a. If $n(C) = 5$, find $n(D)$.
b. What other relationship exists between sets C and D?

11. If $A = \{a, b, c, d, e\}$,
a. how many subsets does A have?
b. how many proper subsets does A have?
c. how many subsets does A have that include the elements a and e?

12. If a set has 127 proper subsets, how many elements are in the set?

13. Identify all the possible proper subset relationships that occur among the following sets.
$A = \{3n|n \in N\}, B = \{6n|n \in N\}$,
$C = \{12n|n \in N\}$.

14. Indicate which symbol, $\in$ or $\notin$, makes each of the following statements true.
a. 0_____$\varnothing$
b. 1024_____$\{x|x = 2^n \text{ and } n \in N\}$
c. 3002_____$\{x|x = 3n - 1 \text{ and } n \in N\}$
d. x_____$\{x|x = 2^n \text{ and } n \in N\}$

15. Indicate which symbol, $\subseteq$ or $\not\subseteq$, makes each part of problem 14 true.

16. Answer each of the following. If your answer is *no*, tell why.
a. If $A = B$, can we always conclude that $A \subseteq B$?
b. If $A \subseteq B$, can we always conclude that $A \subset B$?
c. If $A \subset B$, can we always conclude that $A \subseteq B$?
d. If $A \subseteq B$, can we always conclude that $A = B$?

17. Use the definition of *less than* to show each of the following.
a. $3 < 100$
b. $0 < 3$

18. On a certain senate committee there are seven senators: Abel, Brooke, Cox, Dean, Eggers, Funk, and Gage. Three of these members are to be appointed to a subcommittee. How many possible subcommittees are there?

19. Name two infinite sets that are equivalent but not equal.

20. Write an argument to show that the set of even natural numbers and the set of odd natural numbers should have the same cardinal number.

21. Draw a Venn diagram showing the relationship between the set of natural numbers and the set of whole numbers.

22. Draw a Venn diagram depicting the Adamsville Beta Club, the officers of the Beta Club, and a member Sanna who is not an officer of the Beta Club.

23. If the set of officers of the Adamsville Beta Club is equivalent to the set of members of the Adamsville Beta Club, what can you infer?

Assessment 2-2B

1. Write the following sets using the listing (roster) method or set-builder notation.
 a. The set of letters in the word *algebra*
 b. The set of natural numbers less than 10

2. Rewrite the following using mathematical symbols.
 a. Q is equal to the set whose elements are q, r, and s.
 b. The set containing 1 and 3 is equal to the set containing 3 and 1.
 c. The set containing 1 and 3 only is not a proper subset of $\{1, 4, 6\}$

3. Which of the following pairs of sets can be placed in a one-to-one correspondence?
 a. $\{1, 2, 3, 4\}$ and $\{w, c, y, z\}$
 b. $\{1, 2, 3, \ldots, 25\}$ and $\{a, b, c, d, \ldots, x, y\}$
 c. $\{x | x$ is a letter in the word *geometry*$\}$ and $\{1, 2, 3, 4, 5, 6, 7, 8\}$

4. How many one-to-one correspondences exist between two sets with
 a. 8 elements each?
 b. $n - 1$ elements each? Explain your reasoning.

5. How many one-to-one correspondences are there between the sets $\{a, b, c, d\}$ and $\{1, 2, 3, 4\}$ if in each correspondence
 a. b must correspond to 3?
 b. b must correspond to 3 and d to 4?
 c. a and c must correspond to even numbers?

6. Which of the following represent equal sets?
 $A = \{a, b, c\}$ $\quad B = \{x, y\}$
 $C = \{c, a, b\}$ $\quad D = \{x | 1 \leq x \leq 3$ and $x \in N\}$
 $I = \{x | x = 2n$ and $n \in W\}$,
 $K = \{2, 4, 6, 8, 10, 12, \ldots\}$
 $L = \{x | x = 2n - 1$ and $n \in N\}$

7. Find the cardinal number of each of the following sets. Assume the pattern continues in each part.
 a. $\{19, 20, 21, \ldots, 99\}$
 b. $\{2, 4, 6, 8, \ldots, 1002\}$
 c. $\{x^2 | x = 1, 3, 5, 7, \ldots, 99\}$
 d. $\{x | x = x + 1$ and $x \in N\}$

8. If U is the set of all women and G is the set of alumnae of Georgia State University, describe $\overline{G}$.

9. Suppose $A \subseteq B$.
 a. Explain why $n(A)$ could equal 0.
 b. Is it possible for set B to be the empty set? If so, give an example of sets A and B satisfying this. If not, explain why not.

10. If two sets are subsets of each other, what other relationships must they have?

11. If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
 a. how many subsets does A have?
 b. how many proper subsets does A have?

12. If a set has 16 subsets, how many elements are in the set?

13. Identify all possible proper subset relationships that occur among the following sets.
 $A = \{3n + 1 | n \in N\}$, $\quad B = \{6n + 1 | n \in N\}$,
 $C = \{12n + 1 | n \in N\}$

14. Indicate which symbol, $\in$ or $\notin$, makes each of the following statements true.
 a. $\varnothing$_____$\{\varnothing\}$
 b. 1022_____$\{s | s = 2^n - 2$ and $n \in N\}$
 c. 3004_____$\{x | x = 3n + 1$ and $n \in N\}$
 d. 17____ N

15. Answer each of the following. If your answer is *no*, tell why.
 a. If $A \subseteq B$, can we always conclude that $A = B$?
 b. If $A \subset B$, can we conclude that $A = B$?
 c. If A and B can be placed in a one-to-one correspondence, must $A = B$?
 d. If A and B can be placed in a one-to-one correspondence, must $A \subseteq B$?

16. Use the definition of *less than* to show each of the following.
 a. $0 < 2$
 b. $99 < 100$

17. How many ways are there to stack an ice-cream cone with 4 scoops if the choices are
 a. vanilla, chocolate, rhubarb, and strawberry and each scoop must be different?
 b. vanilla, chocolate, rhubarb, and strawberry and there are no restrictions on different scoops?

18. If two sets have 720 one-to-one correspondences pairing them, how many elements are in each set?

19. Write an argument showing that the set of whole numbers can be placed in a one-to-one correspondence with the set of natural numbers.

20. Draw a Venn diagram showing the members of Congress of the United States, the members of the House of Representatives, and the members of the Senate.

21. In a state swim meet, if the set of swimmers in the 100-m butterfly race is equal to the set of swimmers from the Maryville Swim Team, what can you infer?

22. In some states of the United States, to vote in a primary election, one has to declare a party affiliation of Republican or Democrat. Draw a Venn diagram depicting this information for such a state.

23. If set A and set B are such that set B has one more element than set A, describe a possible relationship between the two sets using some of the following words: *subset*, *proper subset*, *equivalent*, and *equal*.

24. Find an infinite set A such that
 a. $\overline{A}$ is finite.
 b. $\overline{A}$ is infinite.

Mathematical Connections 2-2

Reasoning

1. Explain the difference between a well-defined set and one that is not. Give examples.
2. Which of the following sets are not well defined? Explain.
 - **a.** The set of wealthy schoolteachers
 - **b.** The set of great books
 - **c.** The set of natural numbers greater than 100
 - **d.** The set of subsets of $\{1, 2, 3, 4, 5, 6\}$
 - **e.** The set $\{x \mid x \neq x \text{ and } x \in N\}$
3. Explain why the empty set cannot be put in a one-to-one correspondence with any non-empty set.
4. Explain why $\{\varnothing\}$ has $\varnothing$ as an element and also as a subset.
5. For sets A and B, tell how you would show that $A \not\subseteq B$.
6. Explain why the word "finite" was used in the definition of *less than*.
7. Define *less than or equal to* in a way similar to the definition of *less than*.

Open-Ended

8. **a.** Give three examples of sets A and B and a universal set U such that $A \subset B$; find $\overline{A}$ and $\overline{B}$.
 b. Based on your observations, conjecture a relationship between $\overline{B}$ and $\overline{A}$.
 c. Demonstrate your conjecture in (b) using a Venn diagram.
9. Draw a Venn diagram depicting $\overline{A}$. Shade $\overline{\overline{A}}$. What is the simplest symbol to use for the area you shaded? Why?
10. A set has elements in an arithmetic sequence $\{3, 5, 7, 9, \ldots, 1009\}$. Show how a one-to-one correspondence might be used to find the number of elements in this set. Show the correspondence and any patterns used to find the cardinal number of the sequence.
11. A set of 100 people on a tour includes only men and women. If there is at least one person of each sex on the tour, draw a Venn diagram showing all the information you can deduce and write all number sentences and set relationships you can find.

Cooperative Learning

12. Choose 4 people from your class and have them stand in a row in front of the class. Give each of the people one of the numbers on a sheet of paper: 1, 2, 3, or 4. As a class, gather and re-distribute the sheets of paper as many times as you can to exhibit all the possible arrangements of the symbols 1, 2, 3, and 4. How many arrangements are possible?

Connecting Mathematics to the Classroom

13. A student asks if A is a subset of B and B is a subset of C, do we know that A is a subset of C? How do you respond?
14. A student states that given any two sets A and B, either A is a subset of B or B is a subset of A. What assumptions might the student be making?
15. A third grader asked his teacher whether infinite sets could be compared. How might you respond?

Review Problems

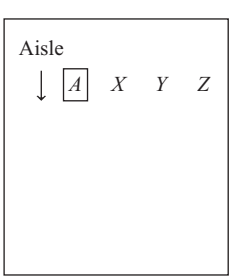
16. If p is false and q is true, find the truth values for each of the following.
 - **a.** $p \wedge q$
 - **b.** $\sim q$
 - **c.** $\sim p \wedge q$
 - **d.** $\sim (p \wedge q)$
 - **e.** $\sim q \wedge \sim p$
17. Find the truth value for each statement in exercise 16 if p is true and q is false.
18. Complete the following truth table.

p	q	$\sim q$	$p \vee \sim q$
T	T		
T	F		
F	T		
F	F		

National Assessments

National Assessment of Educational Progress (NAEP) Question

Four people—*A*, *X*, *Y*, and *Z*—go to a movie and sit in adjacent seats. If *A* sits in the aisle seat, list all possible arrangements of the other three people. One of the arrangements is shown below.



NAEP, Grade 12, 1996

2-3 Other Set Operations and Their Properties

2-3 Objectives

Students will be able to understand and explain

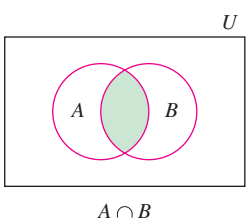
- Relations between set operations and logic connectives.
- Properties of set and logic operations.
- Venn diagrams to sort and reason with data.

In the grade 1 *Common Core Standards*, we find that students should "organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another." (p. 16)

In this section we examine different operations and tools to aid in this type of work. What is presented here should give tools to future teachers to use with organizing, representing, and interpreting small amounts of data. One operation, finding the complement of a set, used in this regard was seen in Section 2-2.

Finding the complement of a set is an operation that acts on only one set at a time. In this section, we consider binary operations that act on two sets at a time and use such logic terms as *and* and *or*.

Set Intersection



$A \cap B$

Figure 11

Suppose that during the fall quarter, a college wants to mail a survey to all its students who are enrolled in both art and biology classes. To do this, the school officials must identify those students who are taking both classes. If A and B are the set of students taking art courses and the set of students taking biology courses, respectively, during the fall quarter, then the desired set of students includes those common to A *and* B, or the **intersection** of A and B. The intersection of sets A and B is the shaded region in Figure 11, which depicts the possibility of A and B containing common elements. The intersection might contain no elements.

Definition of Set Intersection

The **intersection** of two sets A and B, written $A \cap B$, is the set of all elements common to both A and B; $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

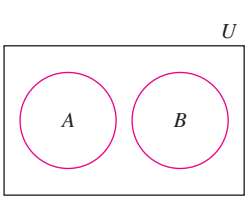


Figure 12

For all sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. The key word in the definition of *intersection* is *and* (see Section 2-1). In everyday language, as in mathematics, *and* implies that both conditions must be met. In the example, the desired set is the set of those students enrolled in both art and biology.

If sets such as A and B have no elements in common, they are *disjoint sets*. In other words, two sets A and B are **disjoint** if, and only if, $A \cap B = \varnothing$. For example, if there are no students that are taking both art (A) and biology (B), then the sets are disjoint. The Venn diagram for this situation could be drawn as in Figure 12.

Example 17

Find $A \cap B$ in each of the following.

a. $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$
b. $A = \{0, 2, 4, 6, \ldots\}$, $B = \{1, 3, 5, 7, \ldots\}$
c. $A = \{2, 4, 6, 8, \ldots\}$, $B = \{1, 2, 3, 4, \ldots\}$

Solution

a. $A \cap B = \{3, 4\}$
b. $A \cap B = \varnothing$; therefore A and B are disjoint.
c. $A \cap B = A$ because all the elements of A are also in B.

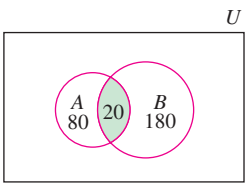
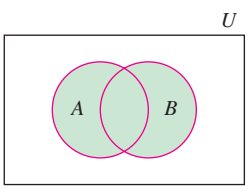


Figure 13

Suppose A represents all students enrolled in art classes and B all students enrolled in biology classes. We may use a Venn diagram to illustrate the previous statement taking into account that some students may be enrolled in both subjects. If we know that 100 students are enrolled in art and 200 in biology and that 20 of these students are enrolled in both art and biology, then $100 - 20$, or 80, students are enrolled in art but not in biology and $200 - 20$, or 180, are enrolled in biology but not art. We record this information as in Figure 13. The total number of students in set A is 100, and the total in set B is 200.

Set Union



$A \cup B$

Figure 14

If A is the set of students taking art courses during the fall quarter and B is the set of students taking biology courses during the fall quarter, then the set of students taking art *or* biology or both during the fall quarter is the *union* of sets A and B. The **union** of sets A and B is pictured in Figure 14.

Definition of Set Union

The **union** of two sets A and B, written $A \cup B$, is the set of all elements in A or in B;
$A \cup B = \{x | x \in A \text{ or } x \in B\}$.

For all sets A and B, $A \subseteq A \cup B$, $B \subseteq A \cup B$, and $A \cap B \subseteq A \cup B$. The key word in the definition of *union* is *or* (see Section 2-1). In mathematics, *or* usually means "one or the other or both." This is known as the *inclusive or*.

Example 18

Find $A \cup B$ for each of the following.

a. $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$
b. $A = \{0, 2, 4, 6, \ldots\}$, $B = \{1, 3, 5, 7, \ldots\}$
c. $A = \{2, 4, 6, 8, \ldots\}$, $B = \{1, 2, 3, 4, \ldots\}$

Solution

a. $A \cup B = \{1, 2, 3, 4, 5, 6\}$ **b.** $A \cup B = \{0, 1, 2, 3, 4, \ldots\}$
c. Because every element of A is already in B, we have $A \cup B = B$.

Example 19

Find each of the following if $A = \{a, b, c\}$.

a. $A \cap \varnothing$ **b.** $A \cup \varnothing$ **c.** $\varnothing \cap \varnothing$ **d.** $\varnothing \cup \varnothing$

Solution

a. $\varnothing$ **b.** A **c.** $\varnothing$ **d.** $\varnothing$

Set Difference

During the fall quarter if A is the set of students taking art classes and B is the set of students taking biology classes, then the set of all students taking biology but not art is the **complement of A relative to B**, or the **set difference** of B and A.

Definition of Relative Complement

The **complement of A relative to B**, written $B - A$, is the set of all elements in B that are not in A; $B - A = \{x | x \in B \text{ and } x \notin A\}$.

Note that $B - A$ is not read as "B minus A." The symbol used here is the same for both subtraction and set difference, but it should not be read as "minus." *Set difference* is an operation on sets. A Venn diagram representing $B - A$ is shown in Figure 15(a). The shaded region represents all the elements that are in B but not in A. A Venn diagram for $B \cap \overline{A}$ is given in Figure 15(b). The shaded region represents all the elements that are in B *and* in $\overline{A}$. Notice that $B - A = B \cap \overline{A}$ because $B \cap \overline{A}$ is, by definitions of intersection and complement, the set of all elements in B and not in A.

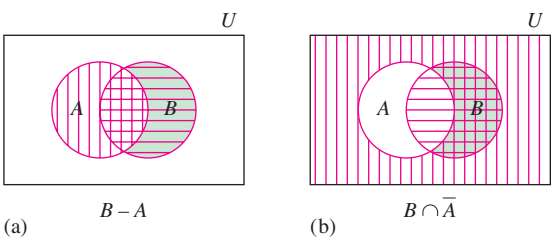


Figure 15

Example 20

Let $A = \{d, e, f\}$, $B = \{a, b, c, d, e, f\}$, and $C = \{a, b, c\}$. Find or answer each of the following.

a. $A - B$
b. $B - A$
c. $B - C$
d. $C - B$
e. To answer parts (a)–(d), does it matter what the universal set is?
f. Let $D = \{c, d, e, f\}$. Find $C - D$.

Solution

a. $A - B = \varnothing$
b. $B - A = \{a, b, c\}$
c. $B - C = \{d, e, f\}$
d. $C - B = \varnothing$
e. Parts (a)–(d) can be answered independently of the universal set. The definition of set difference relates one set to another, independent of the universal set.
f. $C - D = \{a, b\}$

Properties of Set Operations

Because the order of elements in a set is not important, $A \cap B$ is equal to $B \cap A$. It does not matter in which order we write the sets when the intersection of two sets is involved. Similarly, $A \cup B = B \cup A$. These properties are stated formally next.

Theorem 2-7: Commutative Property of Set Intersection and Commutative Property of Set Union

For all sets A and B, $A \cap B = B \cap A$. For all sets A and B, $A \cup B = B \cup A$.

NOW TRY THIS 7

Use Venn diagrams or other means to find whether grouping is important when the same operation is involved. For example, is it always true that $A \cap (B \cap C) = (A \cap B) \cap C$? Investigate similar questions involving union and set difference.

In answering Now Try This 7, the following properties become evident:

Theorem 2-8: Associative Property of Set Intersection and Associative Property of Set Union

For all sets A, B, and C, $A \cap (B \cap C) = (A \cap B) \cap C$. For all sets A and B, $A \cup (B \cup C) = (A \cup B) \cup C$.

Example 21

Is grouping important when two different set operations are involved? For example, is it true that $A \cap (B \cup C) = (A \cap B) \cup C$?

Solution

To investigate this, let $A = \{a, b, c\}$, $B = \{c, d\}$, and $C = \{d, e, f\}$.

$$\begin{aligned} A \cap (B \cup C) &= \{a, b, c\} \cap (\{c, d\} \cup \{d, e, f\}) \\ &= \{a, b, c\} \cap \{c, d, e, f\} \\ &= \{c\} \end{aligned}$$

$$\begin{aligned} (A \cap B) \cup C &= (\{a, b, c\} \cap \{c, d\}) \cup \{d, e, f\} \\ &= \{c\} \cup \{d, e, f\} \\ &= \{c, d, e, f\} \end{aligned}$$

In this case, $A \cap (B \cup C) \neq (A \cap B) \cup C$. We have found a counterexample, that is, an example illustrating that the general statement is not always true.

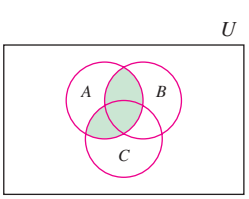


Figure 16

To discover an expression that is equal to $A \cap (B \cup C)$, consider the Venn diagram for $A \cap (B \cup C)$ shown by the shaded region in Figure 16. In the figure, $A \cap B$ and $A \cap C$ are subsets of the shaded region. The union of $A \cap B$ and $A \cap C$ is the entire shaded region. Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Theorem 2-9: Distributive Property of Set Intersection over Set Union

For all sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Theorems 2-7, 2-8, and 2-9 can be proved using the definition of set intersection and set union and the rules of logic.

NOW TRY THIS 8

In the distributive property of set intersection over set union, if the symbol $\cap$ is replaced by $\cup$ and the symbol $\cup$ is replaced by $\cap$, is the new property true? Explain why. What should this property be called?

Example 22

Use set notation to describe the shaded portions of the Venn diagrams in Figure 17.

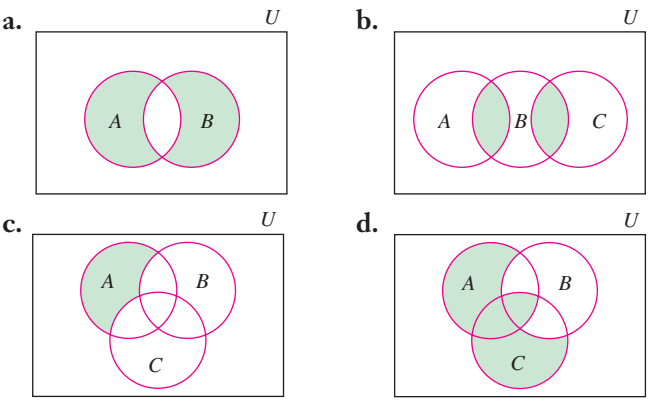


Figure 17

Solution

The solutions can be described in many different, but equivalent, forms. Some possible answers follow.

a. $(A \cup B) - (A \cap B)$ or $(A \cup B) \cap \overline{A \cap B}$ or $(A - B) \cup (B - A)$
b. $(A \cap B) \cup (B \cap C)$ or $B \cap (A \cup C)$
c. $(A - B) - C$ or $A - (B \cup C)$ or $(A - (A \cap B)) - (A \cap C)$
d. $((A \cup C) - B) \cup (A \cap B \cap C)$ or $(A - (B \cup C)) \cup (C - (A \cup B)) \cup (A \cap C)$

Using Venn Diagrams as a Problem-Solving Tool



Venn diagrams can be used as a problem-solving tool for modeling information, as shown in the following examples and on the student page shown on page 72.

Example 23

Suppose M is the set of all students taking mathematics and E is the set of all students taking English. Identify in words and using set notation the students described by each region in Figure 18.

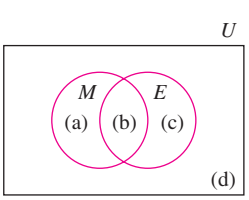
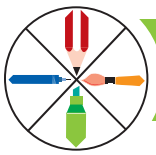


Figure 18

Solution

Answers vary.

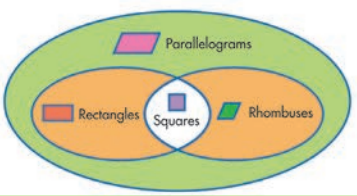
Region (a) contains all students taking mathematics but not English or $M - E$.
Region (b) contains all students taking both mathematics and English or $M \cap E$.
Region (c) contains all students taking English but not mathematics or $E - M$.
Region (d) contains all students taking neither mathematics nor English or $\overline{M \cup E}$ or $\overline{M} \cap \overline{E}$.



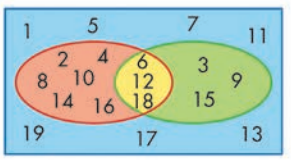
School Book Page — Venn Diagrams

Venn Diagrams

A Venn diagram uses circles, loops, or other shapes to show relationships between sets. Sets are described by the data or objects they contain. Circles overlap, or intersect, when some data belong to more than one set. The Venn diagram below relates parallelograms. Notice that squares are a subset of both rectangles and rhombuses.



Example: Describe the sets shown in the Venn diagram below.



Counting numbers less than 20 are shown in the rectangle. The numbers in the red loop are divisible by 2. The numbers in the green loop are divisible by 3. The numbers divisible by both 2 and 3 are a subset of both loops.

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Example 24

In a survey that investigated the high school backgrounds of 110 college freshmen, the following information was gathered.

25 took physics
45 took biology
48 took mathematics
10 took physics and mathematics
8 took biology and mathematics
6 took physics and biology
5 took all three subjects

a. How many students took biology but neither physics nor mathematics?
b. How many took physics, biology, or mathematics?
c. How many did not take any of the three subjects?

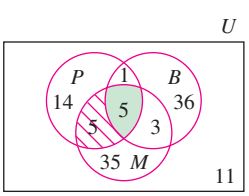


Figure 19

Solution

To solve this problem, we *build a model* using sets. Because there are three distinct subjects, we should use three circles. In Figure 19, P is the set of students taking physics, B is the set of students taking biology, and M is the set of students taking mathematics. The shaded region represents the 5 students who took all three subjects. The lined region represents the students who took physics and mathematics, but who did not take biology.

Part (a) asks for the number of students in the subset of B that has no element in common with either P or M, that is, $B - (P \cup M)$. Part (b) asks for the number of elements in $P \cup B \cup M$. Finally, part (c) asks for the number of students in $\overline{P} \cup \overline{B} \cup \overline{M}$, or $U - (P \cup B \cup M)$. Our strategy is to find the number of students in each of the eight nonoverlapping regions.

One mind set to be aware of in this problem is thinking that the 25 students who took physics took only physics. That is not necessarily the case.

a. Because a total of 10 students took physics and mathematics and 5 of those also took biology, $10 - 5 = 5$ students took physics and mathematics but not biology. Similarly, because 8 students took biology and mathematics and 5 took all three subjects, $8 - 5 = 3$ took biology

and mathematics but not physics. Also $6 - 5 = 1$ student took physics and biology but not mathematics. To find the number of students who took biology but neither physics nor mathematics, we subtract from 45 (the total number who took biology) the number of those that are in the distinct regions that include biology and other subjects; that is, $1 + 5 + 3 = 9$. Because $45 - 9 = 36$, we know that 36 students took biology but neither physics nor mathematics.

b. To find the number of students in all the remaining distinct regions in P, M, or B, we proceed as follows. The number of students who took physics but neither mathematics nor biology is $25 - (1 + 5 + 5) = 14$. The number of students who took mathematics but neither physics nor biology is $48 - (5 + 5 + 3) = 35$. Hence the number of students who took mathematics, physics, or biology is $35 + 14 + 36 + 3 + 5 + 5 + 1 = 99$.

c. Because the total number of students is 110, the number of students who did not take any of the three subjects is $110 - 99 = 11$.

E-Manipulative Activity
Additional work with Venn Diagrams can be found in the *Venn Diagrams* activity on the E-Manipulatives disk (or MML).

Cartesian Products

Another way to produce a set from two given sets is by forming the **Cartesian product**. This formation pairs the elements of one set with the elements of another set in a specific way to create elements in a new set. Suppose a person has three pairs of pants, $P = \{\text{blue, white, green}\}$, and two shirts, $S = \{\text{blue, white}\}$. According to the Fundamental Counting Principle, there are $3 \cdot 2 = 6$ possible different pants-and-shirt pairs, as shown in Figure 20.

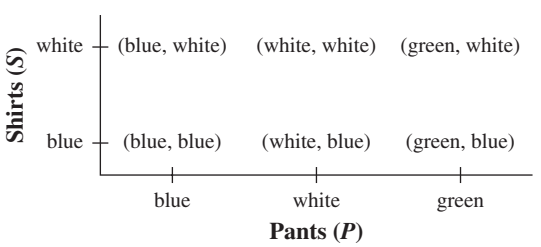


Figure 20

The pants-shirt combinations are elements of the set of all possible pairs in which the first member of the pair is an element of set P and the second member is an element of set S. The set of all possible pairs is given in Figure 20. Because the first component in each pair represents pants and the second component in each pair represents shirts, the order in which the components are written is important. Thus (white, blue) represents white pants and a blue shirt, whereas (blue, white) represents blue pants and a white shirt. Therefore, the two pairs represent different outfits. Because the order in each pair is important, the pairs are **ordered pairs**. The positions that the ordered pairs occupy within the set of outfits is immaterial. Only the order of the **components** within each pair is significant. The pants-and-shirt pairs suggest the following definition of **equality for ordered pairs**: $(x, y) = (m, n)$ *if, and only if, the first components are equal and the second components are equal*. In symbols, $(x, y) = (m, n)$ if, and only if, $x = m$ and $y = n$. The set of six ordered pairs shown in Figure 20, denoted $P \times S$, is an example of a Cartesian product. A formal definition follows.

Definition of Cartesian Product

For any sets A and B, the **Cartesian product** of A and B, written $A \times B$, is the set of all ordered pairs such that the first component of each pair is an element of A and the second component of each pair is an element of B.

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

$A \times B$ is commonly read as "A cross B" and should never be read "A times B."

Example 25

If $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, find each of the following:

a. $A \times B$
b. $B \times A$
c. $A \times A$

Solution

a. $A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$
b. $B \times A = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c)\}$
c. $A \times A = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

It is possible to form a Cartesian product involving the empty set. Suppose $A = \{1, 2\}$. Because there are no elements in $\varnothing$, no ordered pairs (x, y) with $x \in A$ and $y \in \varnothing$ are possible, so $A \times \varnothing = \varnothing$. This is true for all sets A. Similarly, $\varnothing \times A = \varnothing$ for all sets A. There is an analogy between the last equation and the multiplication fact that $0 \cdot a = 0$, where a is a whole number. In Chapter 3 we use the concept of Cartesian product to define multiplication of whole numbers.

Assessment 2-3A

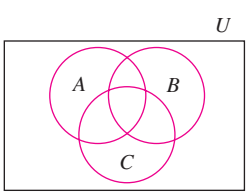
1. If $N = \{1, 2, 3, 4, \ldots\}$, $A = \{x | x = 2n - 1, n \in N\}$, $B = \{x | x = 2n, n \in N\}$, and $C = \{x | x = 2n + 1, n = 0 \text{ or } n \in N\}$, find the simplest possible expression for each of the following.
 a. $A \cup C$
 b. $A \cup B$
 c. $A \cap B$

2. Decide whether the following pairs of sets are always equal.
 a. $A - A$ and $\varnothing$
 b. $B - A$ and $B \cap \overline{A}$
 c. $B - A$ and $B \cap A$ if $A \subseteq B$
 d. $A \cup A$ and $A \cup \varnothing$

3. Tell whether each of the following is true for all sets A and B. If false, give a counterexample.
 a. $A \cup \varnothing = A$
 b. $A - B = B - A$
 c. $\overline{A} \cap \overline{B} = \overline{A} \cap \overline{B}$
 d. $(A \cup B) - A = B$
 e. $(A - B) \cup A = (A - B) \cup (B - A)$

4. If $B \subseteq A$, find a simpler expression for each of the following.
 a. $A \cap B$
 b. $A \cup B$

5. For each of the following, shade the portion of the Venn diagram that illustrates the set.
 a. $(A \cap B) \cup (A \cap C)$
 b. $(A \cup B) \cap \overline{C}$
 c. $(A \cap B) \cup C$

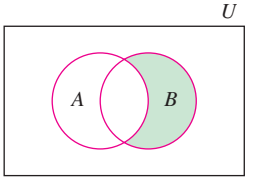


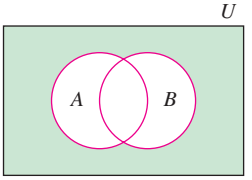
6. If S is a subset of universe U, find each of the following.
 a. $S \cup \overline{S}$
 b. $\overline{U}$
 c. $S \cap \overline{S}$
 d. $\varnothing \cap S$

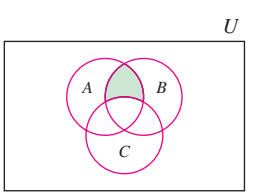
7. For each of the following conditions, find $A - B$.
 a. $A \cap B = \varnothing$
 b. $B = \varnothing$
 c. $B = U$

8. If for sets A and B we know that $A - B = \varnothing$, is it necessarily true that $A \subseteq B$? Justify your answer.

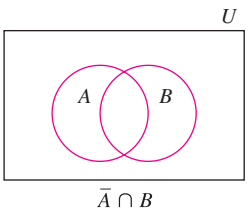
9. Use set notation to identify each of the following shaded regions:

a.


b.


c.


10. **a.** In the following, shade the portion of the Venn diagram that represents the given set.



$\overline{A} \cap B$

b. Find another expression for $\overline{A} \cap B$.

11. Use Venn diagrams to determine if each of the following is true.
a. $A \cup (B \cap C) = (A \cup B) \cap C$
b. $A - (B - C) = (A - B) - C$

12. **a.** If an element is in $A \cap B$, must it be in $A \cup B$? Explain your answer with an example.
b. If an element is in $\overline{A - B}$, can it be in A? Explain your answer with an example.

13. **a.** If A has three elements and B has two elements, what is the greatest number of elements possible in (i) $A \cup B$? (ii) $A \cap B$? (iii) $B - A$? (iv) $A - B$?
b. If A has n elements and B has m elements, what is the greatest number of elements possible in (i) $A \cup B$? (ii) $A \cap B$? (iii) $B - A$? (iv) $A - B$?

14. If $n(A) = 4, n(B) = 5$, and $n(C) = 6$, what is the greatest and least number of elements possible in
a. $A \cup B \cup C$?
b. $A \cap B \cap C$?

15. In 2012, the population of the United States was about 321.8 million; of that population about 2.6 million were Muslim. A total of the number of people who were identified as being of some faith was about 150.6 million. Find the number of each of the following in 2012.
a. The non-Muslims in the United States
b. The people of some faith who were non-Muslims

16. Given that the universe is the set of all humans,
$B = \{x | x \text{ is a college basketball player}\}$, and
$S = \{x | x \text{ is a college student more than 200 cm tall}\}$,
describe each of the following in words:
a. $B \cap S$
b. $\overline{S}$
c. $B \cup S$
d. $\overline{B \cup S}$
e. $\overline{B} \cap S$
f. $B \cap \overline{S}$

17. Of the eighth graders at the Paxson School, 7 played basketball, 9 played volleyball, 10 played soccer, 1 played basketball and volleyball only, 1 played basketball and soccer only, 2 played volleyball and soccer only, and 2 played volleyball, basketball, and soccer. How many played one or more of the three sports?

18. In a sorority with 40 members, 18 take mathematics, 5 take both mathematics and biology, and 8 take neither mathematics nor biology. How many take biology but not mathematics?

19. In Paul's bicycle shop, 50 bicycles were inspected. If 20 needed new tires and 30 needed gear repairs, answer the following.
a. What is the greatest number of bikes that could have needed both?
b. What is the least number of bikes that could have needed both?
c. What is the greatest number of bikes that could have needed neither?

20. The Red Cross looks for three types of antigens in blood tests: A, B, and Rh. When the antigen A or B is present, it is listed, but if both of these antigens are absent, the blood is type O. If the Rh antigen is present, the blood is positive; otherwise, it is negative. If a laboratory technician reports the following results after testing the blood samples of 200 people, how many were classified as O negative?

Number of Samples	Antigen in Blood
50	A
18	B
82	Rh
5	A and B
31	A and Rh
10	B and Rh
4	A, B, and Rh

21. John has applied for the registrar's job at a small college. He submitted the following report to the hiring committee on a survey he did of 100 students: 45 take mathematics; 40 take chemistry; 47 take physics; 20 take mathematics and chemistry; 15 take chemistry and physics; 10 take mathematics and physics; 8 take all three of these subjects; and 10 students take none of these three subjects. Do you think John should be hired on the basis of his report? Explain why.

22. Three announcers each try to predict the winners of next Sunday's professional football games. The only team not picked that is playing Sunday is the Giants. The choices for each person were as follows:

Phyllis: Cowboys, Steelers, Vikings, Bills
Paula: Steelers, Packers, Cowboys, Redskins
Rashid: Redskins, Vikings, Jets, Cowboys
If the only teams playing Sunday are those just mentioned, which teams will play which other teams?

23. The table below contains the 10 most popular candy bars in the United States in 2013 with the primary ingredients listed.

Favorite Candies	Chocolate (*C*)	Nuts/Peanut Butter (*N*)	Caramel (*R*)
Snickers	Yes	Yes	Yes
Toblerone	Yes	No	No
Cadbury Chocolate Bars	Yes	No	No
Hershey	Yes	No	No
Butterfinger	Yes	Yes	No
Baby Ruth	Yes	Yes	Yes
Mars Bar	Yes	No	Yes
Milky Way	Yes	No	Yes
3 Musketeers	Yes	No	No
KitKat	Yes	No	No

a. Draw a Venn diagram sorting the brands according to the three listed ingredients with the universal set (U) being the set of candy bars.
b. How many candy bars are in each of the following sets? What bars are in each set?
i. $\overline{C} \cap R$
ii. $\overline{C} - N$
iii. Of the bars listed, what is the most popular of the listed ingredients? The most unpopular?
iv. If you were a manufacturer of candy bars and using only this list, write an argument for which ingredients your new candy bar would contain.

24. When the *Titanic* sank, the following data about the 324 first-class passengers were reported in newspapers.
- 146 women and children were first-class passengers who survived with only 4 lost.
- 117 men were lost.

How many men who were first-class passengers survived?

25. Two families, each having three children, are at a birthday party. Each of the six children has either blue or brown eyes and brown or blond hair. In a family, children differ by at most one characteristic. The following are descriptions of the six children.
- Tom: blue eyes, brown hair
- Dick: brown eyes, blond hair
- Mary: brown eyes, brown hair
- Harry: blue eyes, blond hair
- Jane: blue eyes, brown hair
- Abby: blue eyes, blond hair

Separate the children into families.

26. Let $A = \{x, y\}$ and $B = \{a, b, c\}$. Find each of the following.
a. $A \times B$
b. $B \times A$
c. Does $A \times B = B \times A$?

27. For each of the following, the Cartesian product $C \times D$ is given by the sets listed. Find C and D.
a. $\{(a, b), (a, c), (a, d), (a, e)\}$
b. $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$
c. $\{(0, 1), (0, 0), (1, 1), (1, 0)\}$

Assessment 2-3B

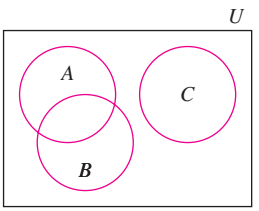
1. If $W = \{0, 1, 2, 3, \ldots\}$, $A = \{x | x = 2n + 1 \text{ and } n \in W\}$, $B = \{x | x = 2n \text{ and } n \in W\}$, and $N = \{1, 2, 3, \ldots\}$, find the simplest possible expression for each of the following.
a. $W - A$
b. $A \cap B$
c. $W \cap N$

2. Decide whether the following pairs of sets are always equal. If not equal, give a counterexample.
a. $A - B$ and $B - A$
b. $A - B$ and $A \cap \overline{B}$
c. $B - A$ and $B \cup A$ if $B \subseteq A$
d. $B \cup \varnothing$ and $B \cap B$

3. Tell whether each of the following is true for all sets A, B, and C. If false, give a counterexample.
a. $A - B = A - \varnothing$
b. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
c. $A \cap (B \cup C) = (A \cap B) \cup C$
d. $(A - B) \cap A = A$
e. $A - (B \cap C) = (A - B) \cap (A - C)$

4. If $X \subseteq Y$, find a simpler expression for each of the following.
a. $X - Y$
b. $X \cap \overline{Y}$

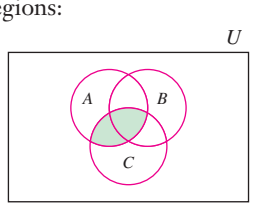
5. For each of the following, shade the portion of the Venn diagram that illustrates the set.
a. $A \cap \overline{C}$
b. $(A \cap B) \cup (B \cap C)$
c. $A \cup (B \cap C)$
d. $A \cup \overline{B \cap C}$

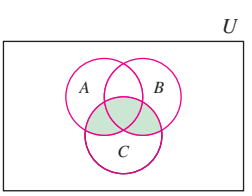


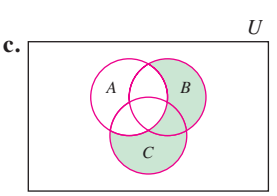
6. If A is a subset of universe U, find each of the following.
a. $A \cup U$
b. $U - A$
c. $A - \varnothing$
d. $\overline{\varnothing} \cap A$

7. For each of the following conditions, find $B - A$.
a. $A = B$
b. $B \subseteq A$

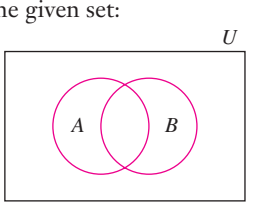
8. Use set notation to identify each of the following shaded regions:

a.


b.


c.


9. **a.** In the following, shade the portion of the Venn diagram that represents the given set:



$\overline{A \cup B}$

b. Find another expression for $\overline{A \cup B}$.

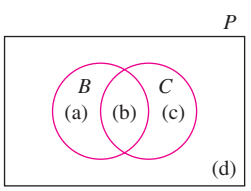
10. Use Venn diagrams to determine whether each of the following is true.
a. $A - (B \cap C) = (A - B) \cap (A - C)$
b. $A - (B \cup C) = (A - B) \cup (A - C)$

11. In a police bulletin, it was reported that a suspect had blond hair, green eyes, and a beard. A second report said that the suspect had blond hair and green eyes. Use your knowledge of set operations to explain which report should have the fewest number of people fitting the descriptions.

12. **a.** If $n(A \cup B) = 23$, $n(A \cap B) = 9$, and $n(B) = 12$, find $n(A)$.
b. If $n(A) = 9$, $n(B) = 13$, and $n(A \cap B) = 5$, find $n(A \cup B)$.

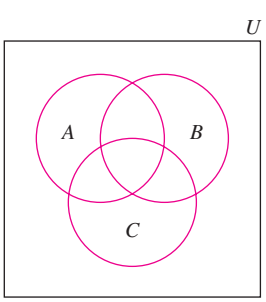
13. The equation $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and a similar equation for $\overline{A \cap B}$ are referred to as *De Morgan's Laws* for sets (see Section 2-1).
a. Discover an equation involving $\overline{A \cap B}$, $\overline{A}$, and $\overline{B}$. Use Venn diagrams to show that the equation holds.
b. Verify the equation in (a) for specific sets.

14. Suppose P is the set of all eighth-grade students at the Paxson School, with B the set of all students in the band and C the set of all students in the choir. Identify in words the students described by each region of the following figure.



15. Fill in the Venn diagram with the appropriate numbers based on the following information.

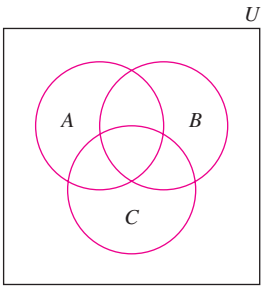
$n(A) = 26$ $\quad n(B \cap C) = 12$
$n(B) = 32$ $\quad n(A \cap C) = 8$
$n(C) = 23$ $\quad n(A \cap B \cap C) = 3$
$n(A \cap B) = 10$ $\quad n(U) = 65$



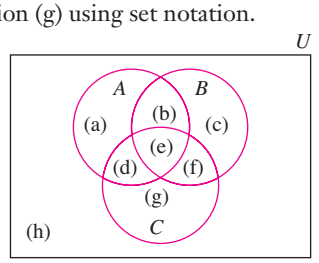
16. Write the letters in the appropriate sections of the following Venn diagram using the following information (ignore the case of the letters).

Set A contains the letters in the word *Iowa*.
Set B contains the letters in the word *Hawaii*.
Set C contains the letters in the word *Ohio*.

The universal set U contains the letters in the word *Washington*.



17. Students at Hellgate High School were asked if they were taking courses in algebra (A), biology (B), or chemistry (C). Use the diagram to answer the following questions.
a. Which region (or regions) represents the students who do not take algebra? (List the letters.)
b. Which region (or regions) represents the students who take biology or chemistry? (List the letters.)
c. A student who took both algebra and biology could be in which region (or regions)? (List the letters.)
d. How would you describe region (a) in sentence form?
e. How would you describe region (f) in sentence form?
f. Describe region (f) using set notation.
g. If we were to combine regions (d) and (g) to form one large region, describe it using set notation.
h. Describe region (g) using set notation.



18. A pollster interviewed 600 university seniors who owned credit cards. She reported that 290 owned Goldcard, 300 had Supercard, and 270 had Thriftcard. Of those seniors, the report said that 80 owned only a Goldcard and a Supercard, 70 owned only a Goldcard and a Thriftcard, 60 owned only a Supercard and a Thriftcard, and 50 owned all three cards. When the report was submitted for publication in the local campus newspaper, the editor refused to publish it, claiming the poll was not accurate. Was the editor right? Why or why not?

19. In a survey of 1000 investors, it was determined that 600 invested in stocks, 575 in bonds, and 300 in both stocks and bonds.
a. How many invested only in stocks?
b. How many invested in stocks or bonds?
c. How many did not invest in either stocks or bonds?

20. In a survey of 150 students, 90 were taking algebra and 30 were taking biology.
a. What is the least number of students who could have been taking both courses?
b. What is the greatest number of students who could be taking both courses?
c. What is the greatest number of students who could have been taking neither course?

21. In 2012, the population of the United States was about 321.8 million (denoted U); of that population about 4.2 million identified themselves as religious Jews. A total of the number of people who were identified as being of some

faith was about 150.6 million. Find the number of each of the following in 2012.
 a. The people of some faith who were not religious Jews
 b. The non-faith people in the United States

22. At the end of a tour, several guides discussed the people on the latest Canadian-American tour. The guides could not remember the total number in the group; however, they compiled the following data about the group.
- 26 Canadian females
- 17 American women
- 17 American males
- 29 girls
- 44 Canadian citizens
- 29 women
- 24 Canadian adults

Find the total number of people in the group.

23. Tell whether each of the following is true or false and explain why:
 a. $(2, 5) = (5, 2)$
 b. $(2, 5) = \{2, 5\}$

24. Answer each of the following.
 a. If A has five elements and B has four elements, how many elements are in $A \times B$?
 b. If A has m elements and B has n elements, how many elements are in $A \times B$?
 c. If A has m elements, B has n elements, and C has p elements, how many elements are in $(A \times B) \times C$?

Mathematical Connections 2-3

Reasoning

1. Answer each of the following and justify your answer.
 a. If $a \in A \cup B$, is it true that $a \in A \cap B$?
 b. If $A \cup B = A \cup C$, then is $B = C$?
 c. If $A \cap B = A \cap C$, then is $B = C$?

2. Given two sets identify conditions under which the complements of the two sets could be equal.

3. Is the operation of forming Cartesian products commutative? Explain why or why not.

4. If A and B are sets, is it always true that $n(A - B) = n(A) - n(B)$? Explain.

5. Explain whether the following could be true: In a survey of 220 students, 110 were taking algebra, 90 were taking biology, and 20 were taking neither of the two subjects.

6. If you were looking for a book in the library on mathematical biology, name the first two items you might type in the search line of a browser. Explain how a search engine might use operations on sets to locate a book.

Open-Ended

7. Describe a real-life situation that can be represented by each of the following:
 a. $A \cap \overline{B}$
 b. $A \cap B \cap C$
 c. $A - (B \cup C)$

Cooperative Learning

8. A set of attribute blocks consists of 32 blocks. Each block is identified by its own shape, size, and color. The four shapes in a set are square, triangle, rhombus, and circle; the four colors are red, yellow, blue, and green; the two sizes are large and small. In addition to the blocks, each set contains a group of 20 cards. Ten cards specify one of the attributes of the blocks (for example, red, large, square). The other 10 cards are negation cards and specify the lack of an attribute (for example, not green, not circle). Many set-type problems can be studied with these blocks. For example, let A be the set of all green blocks and B be the set of all large blocks. As a group using the set of all blocks as the universal set, describe elements in each set listed here to determine which are equal.
 a. $A \cup B$ and $B \cup A$
 b. $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$
 c. $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$
 d. $A - B$ and $A \cap \overline{B}$

9. In small groups, identify three characteristics that at least one person in the group has. Write these on paper. Distribute the papers to the other groups. Have the group that wrote the characteristics stand and see if the rest of the class can identify each person based on the written characteristics. Explain how this activity uses set operations.

Connecting Mathematics to the Classroom

10. A student asks, "If $A = \{a, b, c\}$ and $B = \{b, c, d\}$, why isn't it true that $A \cup B = \{a, b, c, b, c, d\}$?" What is your response?

11. A student is asked to find all one-to-one correspondences between two given sets. He finds the Cartesian product of the sets and claims that his answer is correct because it includes all possible pairings between the elements of the sets. How do you respond?

12. A student argues that adding two sets $A + B$, and taking the union of two sets $A \cup B$ is the same thing. How do you respond?

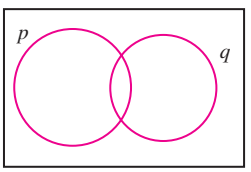
13. The symbol "$\leq$" is read as "less than or equal to." How could this be explained to students using set concepts and terms?

Review Problems

14. Describe the contrapositive of the contrapositive of a statement.

15. Write a conclusion for each of the following.
 a. If Mary is elected class president, she will change the lunch menu. Mary is elected class president.
 b. Samuel is a student in the class play. All the students in the class play stay after school.
 c. If the lake is frozen, we can go ice skating. We cannot go ice skating.

16. In the diagram, suppose statements p and q are represented by the circles with each region of circle p being where p is true, and similarly the regions of circle q showing where q is true. Write a paragraph showing that the diagram could be used to show all possible truth values of combinations of p and q.



17. **a.** Write $\{4, 5, 6, 7, 8, 9\}$ using set-builder notation.
b. Write $\{x \mid x = 5n, n = 3, 6, \text{ or } 9\}$ using the listing method.

18. Find the number of elements in the following sets:
a. $\{x \mid x$ is a letter in the phrase *common core*$\}$
b. The set of letters appearing in the word *committee*

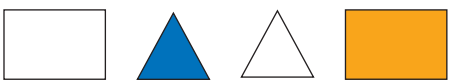
19. If $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$, answer the following questions.
a. How many subsets of A do not contain the element 1?
b. How many subsets of A contain the element 1?
c. How many subsets of A contain either the element 1 or 2?
d. How many subsets of A contain neither the element 1 nor 2?
e. How many subsets of B contain the element 5 and how many do not?

20. Give examples from real life for each of the following:
a. A one-to-one correspondence between two sets
b. A correspondence between two sets that is not one-to-one

21. José has three pairs of slacks, four shirts, and five sweaters. From how many combinations can he choose if he chooses a pair of slacks, a shirt, and a sweater each day?

National Assessments

National Assessment of Educational Progress (NAEP) Question



Melissa chose one of the figures above.

- The figure she chose was shaded.
- The figure she chose was *not* a triangle.

Which figure did she choose?
A. *A*
B. *B*
C. *C*
D. *D*

NAEP, Grade 4, 2007

Hint for Solving the Preliminary Problem

As a first step to answering the questions, organize the data using a Venn diagram with circles representing Alpha Pi, Beta Zeta, and Gamma Iota. Sorting the information into the various sections representing each bit of information will help determine the information Maria needs.

Chapter 2 Summary

KEY CONCEPTS	DEFINITIONS, DESCRIPTIONS, AND THEOREMS
Section 2-1	
Statement (p. 40)	A sentence that is either true or false but not both.
Negation of a statement (p. 40)	A statement that has the opposite truth value of a given statement. The negation of statement p is $\sim p$, which means not p.
Quantifiers (p. 41)	*Universal quantifiers*—words that refer to each and every element in a set, such as *all*, *every*, and *no*. *Existential quantifiers*—words or phrases that refer to one or more, or possibly all, of the elements in a set, such as *some* and *there exists at least one*.
Compound statement (p. 42)	A statement formed by combining two or more statements.
Truth table (p. 42)	A chart that shows all possible true-false patterns for statements.
Conjunction (p. 42)	A compound statement in the form "p *and* q," which uses the connective *and* (denoted by $\wedge$). The *conjunction* $p \wedge q$ is defined to be true if, and only if, both p and q are true.
Disjunction (p. 42)	A compound statement in the form "p *or* q," which uses the connective *or* (denoted by $\vee$). The *disjunction* $p \vee q$ is defined to be true if, and only if, either p or q, or both, are true.
Logically equivalent statements (p. 42)	Statements with the same truth value in every possible situation (denoted by $\equiv$).
De Morgan's laws for logic (p. 43)	• $\sim(p \wedge q) \equiv \sim p \vee \sim q$ • $\sim(p \vee q) \equiv \sim p \wedge \sim q$
Conditional statement (p. 43)	A compound statement in the form "*if* p, *then* q" (denoted by $p \rightarrow q$). The *conditional*, or *implication*, $p \rightarrow q$ is defined to be true unless p is true and q is false. The conditional $p \rightarrow q$ has three related implication statements. • *Converse*: If q, then p (denoted by $q \rightarrow p$). • *Inverse*: If not p, then not q (denoted by $\sim p \rightarrow \sim q$). • *Contrapositive*: If not q, then not p (denoted by $\sim q \rightarrow \sim p$). A conditional and its contrapositive are logically equivalent.
Biconditional (p. 45)	The conjunction of a conditional statement and its converse: $(p \rightarrow q) \wedge (q \rightarrow p)$. This compound statement can be written as $p \leftrightarrow q$, which is read "p, if, and only if, q," or equivalently "q if, and only if, p."
Law of detachment (*modus ponens*) (p. 47)	If the statement "*if* p, *then* q" is true and p is true, then q must be true.
Law of denying the consequent (*modus tollens*) (p. 48)	If the statement "*if* p, *then* q" is true and q is not true, then p is not true.
Tautology (p. 48)	A statement that is always true.
Chain rule (transitivity) (p. 48)	If the statements "*if* p, *then* q" and "*if* q, *then* r" are both true, then the statement "*if* p, *then* r" is true. If $(p \rightarrow q) \wedge (q \rightarrow r)$, then $p \rightarrow r$ is a tautology.

Section 2-2

Set (p. 53)	Any collection of objects.
Element (member) (p. 53)	Any individual object in a set (denoted by $\in$).
Well-defined set (p. 53)	Given a set and some particular object, it must be possible to say whether the object does or does not belong to the set.
Describing a set (p. 54)	*Listing*, or *roster method:* $A = \{1, 2, 3\}$ *Set-builder notation:* $A = \{x \mid x \in N \text{ and } x < 4\}$
Equal sets (p. 55)	Sets that have exactly the same elements, or equivalently $A = B$, if, and only if, $A \subseteq B$ and $B \subseteq A$.
One-to-one correspondence between two sets (p. 55)	Pair the elements of sets A and B so that for each element of A there is exactly one element of B and for each element of B there is exactly one element of A.
Fundamental Counting Principle (p. 57)	If event M can occur in m ways and, after it has occurred, event N can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.
Equivalent sets (p. 57)	Sets that can be placed in a one-to-one correspondence.
Cardinal number of a set (p. 58)	The number of elements in a set S, denoted by $n(S)$.
Special sets (p. 58)	• *Empty* (or *null*) *set*—a set with no elements. • *Universal set* (or the *universe*)—the set that contains all of the elements under discussion.
Complement of a set (p. 59)	The set of elements in the universe that are not in set A, denoted by $\overline{A}$.
Subset of a set (p. 60)	Set B is a subset of set A, written $B \subseteq A$, if, and only if, every element of set B is an element of set A. For all sets A, $\varnothing \subseteq A$.
Proper subset (p. 60)	Set B is a proper subset of A, written $B \subset A$, if, and only if, $B \subseteq A$ and there exists at least one element of A not in B; that is, $B \neq A$. The empty set is a proper subset of every set except itself.
Number of subsets of a finite set (p. 63)	If a set has n elements, then it has 2^n subsets.
Inequalities (defined by sets) (p. 61)	• *Less than*—If A and B are finite sets such that $A \subset B$, then A has fewer elements than B; that is, $n(A) < n(B)$. • *Greater than*—B has more elements than A, that is, $n(B) > n(A)$ if, and only if, $n(A) < n(B)$.

Section 2-3

Intersection of sets *A* and *B* (p. 67)	For any two sets A and B, the set of all elements common to both A and B. The intersection of A and B is denoted by $A \cap B$. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
Union of sets *A* and *B* (p. 68)	For any two sets A and B, the set of all elements in either A or B, or both. The union of A and B is denoted by $A \cup B$. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
Complement of *A* relative to *B* (p. 69)	The set of all elements in B that are not in A. The complement of A relative to B is the *set difference* of B and A, denoted by $B - A$. $B - A = \{x \mid x \in B \text{ and } x \notin A\}$.

Section 2-3	
Properties of set operations (p. 69)	• *Commutative property*—for all sets A and B, *set intersection:* $A \cap B = B \cap A$ *set union:* $A \cup B = B \cup A$ • *Associative property*—for all sets A, B, and C, *set intersection:* $A \cap (B \cap C) = (A \cap B) \cap C$ *set union:* $A \cup (B \cup C) = (A \cup B) \cup C$ • *Distributive property*—for all sets A, B, and C, *set intersection over set union:* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *set union over set intersection:* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Venn diagram (p. 71)	A problem-solving tool using sets to model information.
Cartesian product of two sets (p. 73)	For any sets A and B, the *Cartesian product* of A and B (written as $A \times B$ and read as "A cross B") is the set of all ordered pairs such that the first component of each pair is an element of A and the second component of each pair is an element of B: $A \times B = \{(x, y) \mid x \in A \text{ and } x \in B\}$.

Chapter 2 Review

1. Write an example of a statement and an example of a sentence that is not a statement.

2. Explain the difference between the following two statements: (i) All students made at least C's on the final exam. (ii) Some students made at least C's on the final exam.

3. Which of the following are statements?
- **a.** $3 + 5 = 9$
- **b.** $n + 5 = 15$
- **c.** Some men have Ph.D.'s in mathematics.

4. Negate each of the following:
- **a.** No women smoke.
- **b.** $3 + 5 \neq 8$
- **c.** Bach wrote only classical music.

5. Write the converse, inverse, and contrapositive of the following.
If the whole world is tweeting, someone will read a tweet.

6. Use a truth table to show that $(\sim p \rightarrow q) \equiv (\sim q \rightarrow p)$.

7. Construct truth tables for each of the following.
- **a.** $(p \vee \sim q) \wedge (p \vee q)$
- **b.** $[(p \vee \sim q) \wedge \sim q)] \rightarrow p$

8. Find valid conclusions for the following hypotheses.
- **a.** All Eurasians love Mom and apple pie.
 Alfinia is a Eurasian.
- **b.** Marble and granite eventually crack.
 The Washington Monument is made of marble and granite.
- **c.** Josef passed the math for elementary teachers course or Josef dropped out of school.
 Josef did not drop out of school.

9. Write the following argument symbolically and then determine its validity.
If you pass your classes, your parents will allow you to go to the dance.
If your parents allow you to go to the dance, you will not sit in a corner.
You do sit in a corner.
Conclusion: You do not pass your classes.

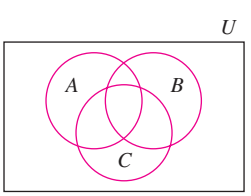
10. State whether the conclusion is valid and explain your answer:
If Bob passes the course, he scored at least 75 on the final exam.
Bob scored less than 75 on the final exam.
Conclusion: Bob did not pass the course.

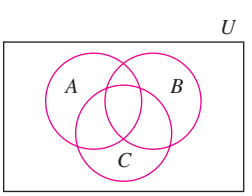
11. How many proper subsets does $A = \{a, b, c, d\}$ have?

12. List all the subsets of $\{m, a, t, h\}$.

13. Let
$U = \{u, n, i, v, e, r, s, a, l\}$
$A = \{r, a, v, e\}$ $C = \{l, i, n, e\}$
$B = \{a, r, e\}$ $D = \{s, a, l, e\}$
Find each of the following.
- **a.** $A \cup B$
- **b.** $C \cap D$
- **c.** $\overline{D}$
- **d.** $A \cap \overline{D}$
- **e.** $\overline{B \cup C}$
- **f.** $(B \cup C) \cap D$
- **g.** $(\overline{A} \cup B) \cap (C \cap \overline{D})$
- **h.** $(C \cap D) \cap A$
- **i.** $n(B - A)$
- **j.** $n(\overline{C})$
- **k.** $n(C \times D)$

14. Indicate the following sets by shading part of the figure.
- **a.** $A \cap (B \cup C)$
- **b.** $\overline{A \cup B \cap C}$





15. Suppose you are playing a word game with five distinct letters. How many different arrangements of the five letters can there be?

16. **a.** If $D = \{t, h, e\}$ and $E = \{e, n, d\}$, show one possible one-to-one correspondence between sets D and E.
b. How many one-to-one correspondences between sets D and E are possible?

17. Use a Venn diagram to determine whether $A \cup (B - C) = (A \cup B) - C$ for all sets A, B, and C.

18. If $A \subseteq B$, which of the following are always true? Give a counterexample if false.
a. $\overline{A} \subseteq \overline{B}$
b. $\overline{B} \subseteq \overline{A}$
c. $A \cup B = B$
d. $A \cap B = A$
e. $\overline{A} \cap \overline{B} = \overline{B}$
f. $\overline{A} \cup \overline{B} = \overline{A}$

19. Tell whether each of the following is true or false. If false, give a counterexample.
a. If $A - B = A - C$, then $B = C$.
b. If $A \times B = A \times C$, then $B = C$.

20. Simplify each of the following:
a. $(A \cap B) \cup (A \cap \overline{B})$
b. $(A \cap B) \cup \overline{B}$

21. Classify each of the following as true or false. If false, tell why.
a. For all sets A and B, either $A \subseteq B$ or $B \subseteq A$.
b. The empty set is a proper subset of every set.
c. For all sets A and B, if $A \sim B$, then $A = B$.
d. The set $\{5, 10, 15, 20, \ldots\}$ is a finite set.
e. No set is equivalent to a proper subset of itself.
f. If A is an infinite set and $B \subseteq A$, then B also is an infinite set.
g. For all finite sets A and B, if $A \cap B \neq \varnothing$, then $n(A \cup B) \neq n(A) + n(B)$.
h. If A and B are sets such that $A \cap B = \varnothing$, then $A = \varnothing$ or $B = \varnothing$.

22. Decide whether each of the following is always true for finite sets A and B.
a. $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
b. $n(A \cup B) = n(A - B) + n(B) = n(B - A) + n(A)$

23. Suppose P and Q are equivalent sets and $n(P) = 17$.
a. What is the minimum number of elements in $P \cup Q$?
b. What is the maximum number of elements in $P \cup Q$?
c. What is the minimum number of elements in $P \cap Q$?
d. What is the maximum number of elements in $P \cap Q$?

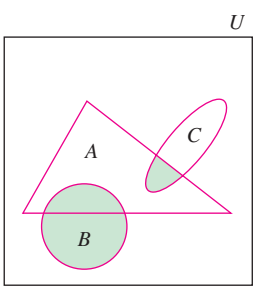
24. If $n(A) = 3$, $n(B) = 4$, and $n(C) = 2$, find $n(A \times B \times C)$.

25. Case Eastern Junior College awarded 26 varsity letters in crew, 15 in swimming, and 16 in soccer. If awards went to 46 students and only 2 lettered in all sports, how many students lettered in just two of the three sports?

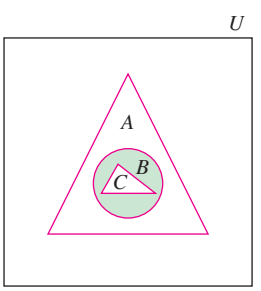
26. According to a student survey, 16 students liked history, 19 liked English, 18 liked mathematics, 8 liked mathematics and English, 5 liked history and English, 7 liked history and mathematics, 3 liked all three subjects, and every student liked at least one of the subjects. Draw a Venn diagram describing this information and answer the following questions.
a. How many students were in the survey?
b. How many students liked only mathematics?
c. How many students liked English and mathematics but not history?

27. Describe, using symbols, the shaded portion in each of the following figures:

a.



b.



28. Heidi has a brown pair and a gray pair of slacks; a brown blouse, a yellow blouse, and a white blouse; and a blue sweater and a white sweater. How many different outfits does she have if each outfit she wears consists of slacks, a blouse, and a sweater?

29. Using the definitions of less than or greater than, show that each of the following inequalities is true.
a. $3 < 13$
b. $12 > 9$

Chapter 3

Numeration Systems and Whole Number Operations

3-1 Numeration Systems

3-2 Addition and Subtraction of Whole Numbers

3-3 Multiplication and Division of Whole Numbers

3-4 Addition and Subtraction Algorithms, Mental Computation, and Estimation

3-5 Multiplication and Division Algorithms, Mental Computation, and Estimation



Preliminary Problem

If the sums of the values of each symbol are shown in the respective rows and columns, find the value of each symbol.

□	△	□	35
○	□	□	32
△	□	○	37
△	△	□	40
52	50	42	

If needed, see Hint on page 168.

In the introduction to the *Common Core Standards,* we find the following comment assessing coherence of the standards.

… to be coherent, a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, including simple math facts and routine computational procedures associated with whole numbers and fractions) to deeper structures inherent in the discipline. …

These Standards endeavor to follow such a design, not only by stressing conceptual understanding of key ideas, but also by continually returning to organizing principles such as place value or the properties of operations to structure those ideas. (p. 4)

In Chapter 3, the conceptual understanding of whole numbers and operations on whole numbers are developed along with estimation and mental mathematics techniques. This foundation lays the groundwork for later work involving the real number system and its properties. We begin the chapter by introducing several counting systems that allow students to compare numeration systems throughout the ages.

3-1 Numeration Systems

3-1 Objectives

Students will be able to understand and explain

- Numbers, their origin, and their representation in numerals and models.
- Different numeration systems including the Hindu-Arabic system.
- Place value and counting in base ten and other bases.
- Issues in learning with different numeration systems.

We introduce various numeration systems to allow comparison to the system used in the United States. Comparing ancient systems helps to develop a clearer appreciation of the U.S. system. The commonly used base-ten system relies on 10 digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Grouping into sets of 10 probably occurred because we have 10 "digits" on our two hands. The word *digit* means "finger" or "toe" in ancient Latin. The written symbols for the digits, such as 2 or 5, are **numerals**. Different cultures developed different numerals for representing numbers. Table 1 shows other representations for the digits 0 through 9.

Table 1

Babylonian												
Egyptian												
Mayan												
Greek		α	β	γ	δ	ϵ	ϕ	ζ	η	υ	ι	
Roman		I	II	III	IV	V	VI	VII	VIII	IX	X	
Hindu												
Arabic												
Hindu-Arabic	0	1	2	3	4	5	6	7	8	9	10	

Table 1 shows rudiments of different sets of numbers. A **numeration system** is a collection of properties and symbols agreed upon to represent numbers systematically. Through the study of various numeration systems, we explore the evolution of the Hindu-Arabic system in U.S. usage.

Hindu-Arabic Numeration System

The **Hindu-Arabic numeration system** was developed by the Hindus and transported to Europe by the Arabs—hence, the name *Hindu-Arabic*. The Hindu-Arabic system relies on the following properties:

1. All numerals are constructed from the 10 digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
2. Place value is based on powers of 10, the *number base* of the system.

IMAP Video

Watch Zenaida discuss the place value of 32, 120 and 316 and watch Maryann's students discuss place value.

Because the Hindu-Arabic system is based on powers of 10, the system is a base ten or *decimal system*. **Place value** assigns a value to a digit depending on its placement in a numeral. To find the value of a digit, placed in a numeral, we multiply the place value of the digit by its **face value**. For example, in the numeral 65,984, the 6 has place value "ten thousand," the 5 has place value "thousand," the 9 has place value "hundred," the 8 has place value "ten," and the 4 has place value "one" as seen in Figure 1. The values of the respective digits are $6 \cdot 10{,}000 = 60{,}000$, $5 \cdot 1000 = 5000$, $9 \cdot 100 = 900$, $8 \cdot 10 = 80$, and $4 \cdot 1 = 4$.

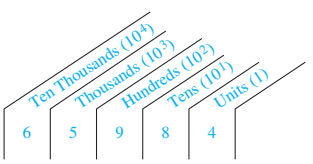


Figure 1

The number 65,984 is read "sixty-five thousand, nine hundred eighty-four." The word *and* does not appear in the name of the number as it is reserved for decimals to separate the decimal portion from the rest of a number. A comma is used to separate groups of three digits. Internationally, commas are omitted and single spaces are used to separate groups of three digits. In some countries commas are used as the United States uses decimal points. We could write 65,984 in **expanded form** as $6 \cdot 10^4 + 5 \cdot 10^3 + 9 \cdot 10^2 + 8 \cdot 10 + 4 \cdot 1$. The expanded form of 65,984 uses exponents. For example, 10,000, or $10 \cdot 10 \cdot 10 \cdot 10$ is written as 10^4. In this case, 10 is a **factor** of the product 10^4. In general, we have the following:

Definition of a^n

If a is any number and n is any natural number, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}.$$

If $a \neq 0$, then $a^0 = 1$.

A set of base-ten blocks (or Dienes blocks), shown in Figure 2, can be used in representing place value. The blocks consist of *units*, *longs*, *flats*, and *blocks*, representing 1's, 10's, 100's, and 1000's, respectively.

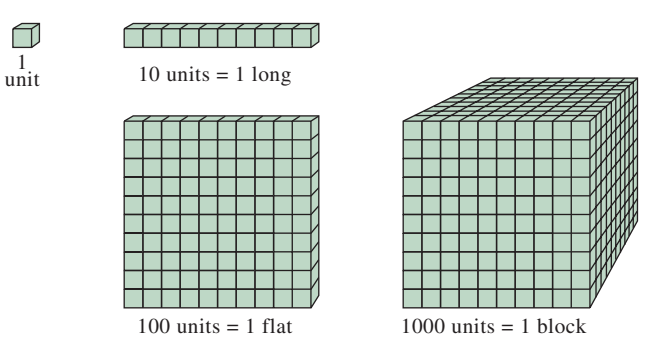


Figure 2

$$1 \text{ unit} \rightarrow 10^0 = 1 \text{ unit}$$
$$1 \text{ long} \rightarrow 10^1 = 10 \text{ units}$$
$$1 \text{ flat} \rightarrow 10^2 = 10 \text{ longs, or } 100 \text{ units}$$
$$1 \text{ block} \rightarrow 10^3 = 10 \text{ flats, or } 100 \text{ longs, or } 1000 \text{ units}$$

Students trade blocks by regrouping. That is, they take a set of base-ten blocks representing a number and trade them until they have the fewest possible pieces representing the same number. For example, suppose we have 58 units and want to trade them for other base-ten blocks. We start trading the units for as many longs as possible. Five sets of 10 units each can be traded for 5 longs. Thus, 58 units can be traded for 5 longs and 8 units. This is analogous to rewriting 58 as $5 \cdot 10 + 8 \cdot 1$.

Example 1

Fair exchange 11 flats, 17 longs, and 16 units to obtain the fewest blocks possible.

Solution

	11 flats	17 longs	~~16 units~~	(16 units = 1 long and 6 units)
		1 long	6 units	(Trade)
	11 flats	18 longs	6 units	(After the first trade)
	11 flats	~~18 longs~~	6 units	(18 longs = 1 flat and 8 longs)
	1 flat	8 longs		(Trade)
	12 flats	8 longs	6 units	(After the second trade)
	~~12 flats~~	8 longs	6 units	(12 flats = 1 block and 2 flats)
1 block	2 flats			(Trade)
1 block	2 flats	8 longs	6 units	(After the third trade)

As a result of the trading we obtain 1 block, 2 flats, 8 longs, and 6 units. This exchange yields the fewest number of pieces, $1 + 2 + 8 + 6 = 17$.

NOW TRY THIS 1

a. Trade base-ten blocks (as shown in Figure 2) to write 3 blocks, 12 flats, 11 longs, and 17 units with the fewest number of blocks. What Hindu-Arabic numeral is represented by these blocks?
b. Write 3282 in expanded form.

Other numeration systems provide a historical perspective and help us better understand the United States system.

Tally Numeration System

The **tally numeration system** uses single strokes, or tally marks, to represent each object counted; for example, the first 10 counting numbers are:

|, ||, |||, ||||, |||||, ||||||, |||||||, ||||||||, |||||||||, ||||||||||

Historical Note



The invention of the Hindu-Arabic numeration system is one of the most important developments in mathematics. The system was introduced in India and then transmitted by the Arabs to North Africa and Spain and then to the rest of Europe. Historians trace the use of zero as a placeholder to the fourth century BCE (Before the Common Era). Arab mathematicians extended the decimal system to include fractions.

A tally system has a one-to-one correspondence between the marks and the items being counted. The system is simple, but it requires many symbols when the numbers are large. As numbers become greater, the tally marks for them are harder to read. The tally system can be improved by *grouping*. Tallies are grouped into fives by placing a diagonal across four tallies to make a group of five. Grouping makes it easier to read the numeral.



Egyptian Numeration System

The Egyptian numeration system, dating to about 3400 BCE, used tally marks for the first nine numerals. Egyptians improved on the system based only on tally marks by developing a *grouping system*. For example, the Egyptians used a heel bone symbol, ∩, to stand for a grouping of 10 tally marks.



Table 2 shows other numerals that the Egyptians used. Some of the symbols from the Karnak temple in Luxor are depicted in Figure 3.

Table 2

Egyptian Numeral	Description	Hindu-Arabic Equivalent
	Vertical staff	1
	Heel bone	10
	Scroll	100
	Lotus flower	1000
	Pointing finger	10,000
	Polliwog or burbot	100,000
	Astonished man	1,000,000



Figure 3

In its simplest form, the Egyptian system involved an **additive property** where the value of a number was the sum of the face values of the numerals. The Egyptians customarily wrote the

numerals in decreasing order from left to right, as in 𓆐𓍢𓎆𓐁𓐁. This number can be converted to base ten as shown below:

𓆐	represents	100,000	
𓍢𓍢𓍢	represents	300	(100 + 100 + 100)
𓎆𓎆	represents	20	(10 + 10)
𓐁𓐁	represents	2	(1 + 1)
𓆐𓍢𓍢𓍢𓎆𓐁𓐁	represents	100,322	

NOW TRY THIS 2

a. Use the Egyptian system to represent 1,312,322.
b. Use the Hindu-Arabic system to represent 𓆐𓆐𓂭𓂭𓂭𓎆𓎆𓐁𓐁𓐁𓐁.
c. What disadvantages do you see in the Egyptian system compared to the Hindu-Arabic system?

Babylonian Numeration System

The Babylonian numeration system was developed about the same time as the Egyptian system. The symbols in Table 3 were made using a stylus either vertically or horizontally on clay tablets.

Table 3

Babylonian Numeral	Hindu-Arabic Equivalent
𒁹	1
𒌋	10

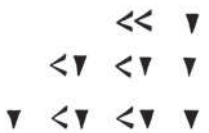
The clay tablets were heated and dried to preserve a permanent record. Babylonian symbols on a clay tablet are pictured in Figure 4.



Figure 4

Babylonian numerals 1 through 59 were similar to Egyptian numerals, but the vertical staff and the heel bone were replaced by symbols shown in Table 3. For example,  represented 23.

The Babylonian numeration system used a *place-value system*. Numbers greater than 59 were represented by repeated groupings of 60, much as we use groupings of 10 today. For example, represents $2 \cdot 60 + 20 \cdot 1 = 140$. The space indicates that represents $2 \cdot 60$ rather than 2. Numerals immediately to the left of a second space have a value $60 \cdot 60 = 60^2$ times their face value, and so on.



represents $20 \cdot 60 + 1 \cdot 1 = 1201$

represents $11 \cdot 60 \cdot 60 + 11 \cdot 60 + 1 \cdot 1 = 11 \cdot 60^2 + 11 \cdot 60 + 1 \cdot 1 = 40{,}261$

represents $1 \cdot 60 \cdot 60 \cdot 60 + 11 \cdot 60 \cdot 60 + 11 \cdot 60 + 1 \cdot 1 = 1 \cdot 60^3 + 11 \cdot 60^2 + 11 \cdot 60 + 1 \cdot 1 = 256{,}261$

The initial Babylonian system was inadequate by today's standards. For example, the symbol could have represented 2 or $2 \cdot 60$. Later, the Babylonians introduced the symbol  as a placeholder for missing position values. Using this symbol, represented $10 \cdot 60 + 21 \cdot 1$ and represented $10 \cdot 60^2 + 0 \cdot 60 + 21 \cdot 1$. In this sense,  represented 0.

NOW TRY THIS 3

a. Use the Babylonian system to represent 12,321.
b. Use the Hindu-Arabic system to represent .
c. What advantages does the Hindu-Arabic system have over the Babylonian system?

Mayan Numeration System

In the early development of numeration systems, people frequently used parts of their bodies to count. Fingers could be matched to objects to stand for one, two, three, four, or five objects. Two hands could then stand for a set of ten objects. In warmer climates where people went barefoot, people may have used their toes as well as their fingers for counting. The Mayans introduced an attribute that was not present in the Egyptian or early Babylonian systems, namely, a symbol for zero. The Mayan system used only three symbols shown in Table 4, and based their system primarily on 20 with vertical groupings.

Table 4

Mayan Numeral	Hindu-Arabic Equivalent
	0
•	1
—	5

The symbols for 11 numbers (beginning with zero) in the Mayan system are shown in Table 1. Notice the groupings of five, where each horizontal bar represents a group of five. Thus, the symbol for 19 was , or three 5s and four 1s. The symbol for 20 was , which represents one group of 20 plus zero 1s. Remember that vertical groupings are used. In Figure 5(a), we have $2 \cdot 5 + 3 \cdot 1 = 13$ groups of 20 plus $2 \cdot 5 + 1 \cdot 1 = 11$, for a total of $260 + 11 = 271$. In Figure 5(b), we have $3 \cdot 5 + 1 \cdot 1 = 16$ groups of 20 and zero 1s, for a total of 320.

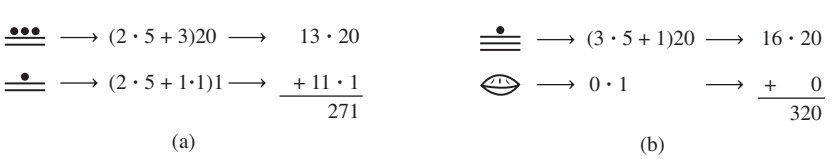


Figure 5

For a number written using three rows in a true base-twenty system, the place value of the symbols in the top row should represent one group of 20^2, or 400. However, the Mayans used $20 \cdot 18 = 360$ instead of 400. (The number 360 is an approximation of the length of a calendar year, which consisted of 18 months of 20 days each, plus 5 "unlucky" days.) Thus, instead of place values of 1, 20, 20^2, 20^3, 20^4, and so on, the Mayans used 1, 20, $20 \cdot 18$, $20^2 \cdot 18$, $20^3 \cdot 18$, and so on. For example, in Figure 6(a), we have $5 + 1 = 6$ groups of 360, plus $2 \cdot 5 + 2 = 12$ groups of 20, and $5 + 4 = 9$ groups of 1, for a total of $2400 + 9 = 2409$. In Figure 6(b), we have $2 \cdot 5$ (or 10) groups of 360, plus 0 groups of 20, plus two 1s, for a total of 3602. Spacing is important in the Mayan system. For example, if two horizontal bars are placed close together, as in $=$, the symbols represent $5 + 5 = 10$. If the bars are spaced apart, as in $\overline{\underline{\ }}$, then the value is $5 \cdot 20 + 5 \cdot 1 = 105$.

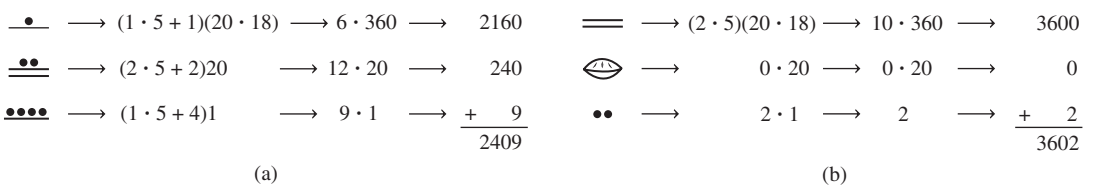


Figure 6

Roman Numeration System

The Roman numeration system was used in Europe in its early form from the third century BCE. It is still seen today on cornerstones, on the opening pages of books, and on the faces of some clocks. The Roman system uses only the symbols shown in Table 5.

Table 5

Roman Numeral	Hindu-Arabic Equivalent
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

Roman numerals can be written using an additive property. For example, MDCLXVI represents $1000 + 500 + 100 + 50 + 10 + 5 + 1 = 1666$, CCCXXVIII represents 328, and VI represents 6.

To avoid repeating a symbol more than three times, as in IIII, a **subtractive property** was introduced in the Middle Ages. For example, I is less than V, so if it is to the immediate left of V, it is subtracted. Thus, IV has a value of $5 - 1 = 4$, and XC represents $100 - 10 = 90$. Some extensions of the subtractive property could lead to ambiguous results. For example, IXC could be 91 or 89. By custom, 91 is written XCI and 89 is written LXXXIX. In general, only one lesser number symbol can be to the left of a greater number symbol, and the pair must be one of those listed in Table 6.

Table 6

Roman Numeral	Hindu-Arabic Equivalent
IV	$5 - 1 = 4$
IX	$10 - 1 = 9$
XL	$50 - 10 = 40$
XC	$100 - 10 = 90$
CD	$500 - 100 = 400$
CM	$1000 - 100 = 900$

In the Middle Ages, a bar was placed over a Roman numeral to multiply it by 1000. The use of bars is based on a **multiplicative property**. For example, $\overline{V}$ represents $5 \cdot 1000 = 5000$, and $\overline{CDX}$ represents $410 \cdot 1000 = 410{,}000$. To indicate even greater numbers, more bars appear. For example, $\overline{\overline{V}}$ represents $(5 \cdot 1000)1000 = 5{,}000{,}000$; $\overline{\overline{CXI}}$ represents $111 \cdot 1000^3 = 111{,}000{,}000{,}000$; and $\overline{CXI}$ represents $110 \cdot 1000 + 1 = 110{,}001$.

Several properties might be combined to represent some numbers, for example:

$$\overline{D}CLIX = \underbrace{\underbrace{(500 \cdot 1000)}_{\text{Multiplicative}} + \underbrace{(100 + 50)}_{\text{Additive}} + \underbrace{(10 - 1)}_{\text{Subtractive}}}_{\text{Additive}} = 500{,}159$$

NOW TRY THIS 4

a. Write CCXLIX as a Hindu-Arabic numeral.
b. Use Roman numerals to represent each of the following
(i) 1634 **(ii)** 5280 **(iii)** 88
c. Use Mayan numerals to represent each of the following
(i) 684 **(ii)** 164

Other Number-base Systems

To investigate some problems students encounter learning the Hindu-Arabic system, we investigate systems that have different number bases.

Base Five

The Luo peoples of Kenya used a *quinary*, or base-five, system. A system of this type can be modeled by counting with only one hand. The digits available for counting are 0, 1, 2, 3, and 4. In the "one-hand system," or base-five system, we count 0, 1, 2, 3, 4, 10, where 10 *represents one hand and no additional fingers*. Counting in base five proceeds as shown in Figure 7. We write the subscript "five" as a reminder that the numeral is written in base five. **If no base is written, a numeral is assumed to be in base ten**. Also note that 0, 1, 2, 3, 4 are the same, and have the same meaning, in both base five and base ten.

One-Hand System	Base-Five Symbol	Base-Five Blocks
0 fingers	0_{five}	
1 finger	1_{five}	
2 fingers	2_{five}	
3 fingers	3_{five}	
4 fingers	4_{five}	
1 hand and 0 fingers	10_{five}	
1 hand and 1 finger	11_{five}	
1 hand and 2 fingers	12_{five}	
1 hand and 3 fingers	13_{five}	
1 hand and 4 fingers	14_{five}	
2 hands and 0 fingers	20_{five}	
2 hands and 1 finger	21_{five}	

Figure 7

Counting in base five is similar to counting in base ten. Because we have only five digits, 4_{five} plays a role similar to 9 in base ten. Figure 8 shows how to find the number that comes after 34_{five} using base-five blocks. If we add one more unit block to 34_{five} and perform a trade as shown in Figure 8, we obtain 40_{five}. This numeral is read "four zero base five" and not "forty base five" because forty is a base-ten word.

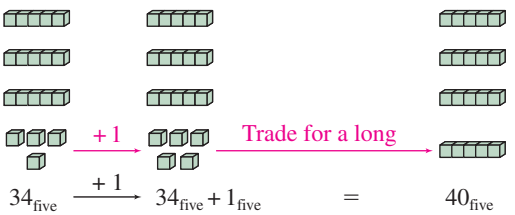


Figure 8

What number follows 44_{five}? There are no more two-digit numerals in the system after 44_{five}. In base ten, the same situation occurs at 99, where we use 100 to represent ten 10s, or one hundred. In the base-five system, we need a numeral to represent five 5s. To continue the analogy with base ten, we use 100_{five} to represent one group of five 5s or 5^2, zero groups of five, and zero units. The name for 100_{five} is read "one zero zero base five." The numeral 100 means $1 \cdot 10^2 + 0 \cdot 10^1 + 0 \cdot 1$, whereas the numeral 100_{five} means $(1 \cdot 10^2 + 0 \cdot 10^1 + 0)_{\text{five}}$, or $1 \cdot 5^2 + 0 \cdot 5^1 + 0 = 25$.

Examples of base-five numerals along with their base-five block representations and conversions to base ten are given in Figure 9.

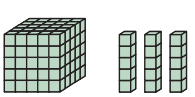
Base-Five Numeral	Base-Five Blocks	Base-Ten Numeral
14_{five}		$1 \cdot 5 + 4 = 9$
124_{five}		$1 \cdot 5^2 + 2 \cdot 5 + 4 = 39$
1030_{five}		$1 \cdot 5^3 + 0 \cdot 5^2 + 3 \cdot 5 + 0 \cdot 1 = 140$

Figure 9

Example 2

Convert 11244_{five} to base ten.

Solution

$$\begin{aligned} 11244_{\text{five}} &= 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 5^1 + 4 \cdot 1 \\ &= 1 \cdot 625 + 1 \cdot 125 + 2 \cdot 25 + 4 \cdot 5 + 4 \cdot 1 \\ &= 625 + 125 + 50 + 20 + 4 \\ &= 824 \end{aligned}$$

Example 2 suggests a method for changing a base-five numeral to a base-ten numeral using powers of 5. To convert 824 to base five, we consider how to write 824 using powers of 5. We first determine the greatest power of 5 less than or equal to 824. Because $5^4 = 625$ and $5^5 = 3125$, the greatest power is 5^4. How many 5^4s are contained in 824? There is only one, so we have $824 = 1 \cdot 5^4 + 199$. Next we find the greatest power of 5 in 199. How many 5^3s are in 199? There is only one giving us $824 = 1 \cdot 5^4 + 1 \cdot 5^3 + 74$. Similarly, there are two 5^2s in 74 with 24 remaining, giving $824 = 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 24$. Because there are four 5s in 24 with

4 remaining, we have $824 = 1 \cdot 5^4 + 1 \cdot 5^3 + 2 \cdot 5^2 + 4 \cdot 5^1 + 4 \cdot 1$. Therefore, $824 = 11244_{\text{five}}$. A shorthand method for recording this conversion follows.

$5^4 = 625 \rightarrow 625 \overline{)\,824\,} \; 1$ -625 How many groups of 625 in 824?

$5^3 = 125 \rightarrow 125 \overline{)\,199\,} \; 1$ -125 How many groups of 125 in 199?

$5^2 = 25 \rightarrow 25 \overline{)\,74\,} \; 2$ -50 How many groups of 25 in 74?

$5^1 = 5 \rightarrow 5 \overline{)\,24\,} \; 4$ -20 How many groups of 5 in 24?

$5^0 = 1 \rightarrow 1 \overline{)\,4\,} \; 4$ -4 0 How many 1s in 4?

Thus, $824 = 11244_{\text{five}}$.

NOW TRY THIS 5

A different method of converting 824 to base five is shown using successive divisions by 5. The quotient in each case is placed below the dividend and the remainder is placed on the right, on the same line with the quotient. The answer is read from bottom to top, that is, as 11244_{five}. Use this method to convert 728 to base five.

$5 \overline{)\,824}$
$5 \overline{)\,164}\;\; 4$
$5 \overline{)\,32}\;\; 4$
$5 \overline{)\,6}\;\; 2$
$1 \;\; 1$

Base Two

Historians tell of early tribes that used base two. Some aboriginal tribes still count "one, two, two and one, two twos, two twos and one, ..." Because base two has only two digits, it is called the **binary system**. In a technological setting, such as a computer or when sending data over the internet, one of the two digits can be represented by the presence of an electrical signal and the other by the absence of an electrical signal. Although base two works well for some purposes, it is inefficient for everyday use because multidigit numbers are reached very rapidly in counting in this system.

Conversions from base two to base ten, and vice versa, can be accomplished in a manner similar to that used for base five conversions.

Example 3

a. Convert 10111_{two} to base ten.
b. Convert 27 to base two.

Solution

a. $10111_{\text{two}} = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1$
$= 16 + 0 + 4 + 2 + 1$
$= 23$

b.
$$\begin{array}{r|r|l} 16 & 27 & 1 \\ & -16 & \\ \hline 8 & 11 & 1 \\ & -8 & \\ \hline 4 & 3 & 0 \\ & -0 & \\ \hline 2 & 3 & 1 \\ & -2 & \\ \hline 1 & 1 & 1 \\ & -1 & \\ \hline & 0 & \end{array}$$

How many groups of $2^4 = 16$ in 27?

How many groups of $2^3 = 8$ in 11?

How many groups of $2^2 = 4$ in 3?

How many groups of 2 in 3?

How many 1s in 1?

Thus, 27 is equivalent to 11011_{two}.

Alternative Solution:

$$\begin{array}{rrr} 2 & 27 & \\ 2 & 13 & 1 \\ 2 & 6 & 1 \\ 2 & 3 & 0 \\ & 1 & 1 \end{array}$$

Base Twelve

Another commonly used number-base system is the base twelve, or the duodecimal ("dozens"), system. Eggs are bought by the dozen, and pencils are bought by the *gross* (a dozen dozen). In base twelve, there are 12 digits, just as there are 10 digits in base ten, 5 digits in base five, and 2 digits in base two. In base twelve, new symbols are needed to represent the following groups of *x*'s:

$$\underbrace{xxxxxxxxxx}_{10\ x\text{'s}} \quad \text{and} \quad \underbrace{xxxxxxxxxxx}_{11\ x\text{'s}}$$

The new symbols chosen are T and E, respectively, so that the base twelve digits are $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E)_{\text{twelve}}$. Thus, in base twelve we count $(1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, 10, 11, 12, \ldots, 17, 18, 19, 1T, 1E, 20, 21, 22, \ldots, 28, 29, 2T, 2E, 30, \ldots)_{\text{twelve}}$.

Example 4

a. Convert $E2T_{\text{twelve}}$ to base ten.
b. Convert 1277 to base twelve.

Solution

a.
$$\begin{aligned} E2T_{\text{twelve}} &= 11 \cdot 12^2 + 2 \cdot 12^1 + 10 \cdot 1 \\ &= 11 \cdot 144 + 24 + 10 \\ &= 1584 + 24 + 10 \\ &= 1618 \end{aligned}$$

b.
$$\begin{array}{r|r|l} 144 & 1277 & 8 \\ & -1152 & \\ \hline 12 & 125 & T \\ & -120 & \\ \hline 1 & 5 & 5 \\ & -5 & \\ \hline & 0 & \end{array}$$

How many groups of $12^2 = 144$ in 1277?

How many groups of 12 in 125?

How many 1s in 5?

Thus, $1277 = 8T5_{\text{twelve}}$.

Example 5

Rob used base twelve to write the following:

$$g36_{\text{twelve}} = 1050_{\text{ten}}$$

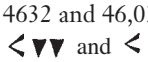
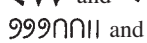
What is the value of g? Check your answer.

Solution We could write the following equations:

$$\begin{aligned} g \cdot 12^2 + 3 \cdot 12 + 6 \cdot 1 &= 1050 \\ 144g + 36 + 6 &= 1050 \\ 144g + 42 &= 1050 \\ 144g &= 1008 \\ g &= 7 \end{aligned}$$

Check $736_{\text{twelve}} = 7 \cdot 12^2 + 3 \cdot 12 + 6 \cdot 1 = 1050$

Assessment 3-1A

1. Write the following numerals in words.
 a. 56,283,914
 b. 5,365,295,234
2. Indicate which digit in 234,518 has place value
 a. hundreds
 b. ten thousands.
3. For each of the following base-ten numerals, give the place value of the underlined digit.
 a. 827,367
 b. 8,421,000
4. Rewrite each of the following as a base-ten numeral.
 a. $3 \cdot 10^6 + 4 \cdot 10^3 + 5$
 b. $2 \cdot 10^4 + 1$
5. If □, △, and ○ each represent different non-zero base-ten digits, order the numerals from least to greatest.
 a. □; △□○; □△
 b. △□○; □□; □□△△
6. How many digits are there in the standard form of the numeral representing $625 \cdot 10^{12}$?
7. A certain three-digit natural number has the following properties: The hundreds digit is greater than 7; the tens digit is an odd number; and the sum of the three digits is 10. What could the number be?
8. For each of the following pairs of numerals tell which numeral represents the greater number.
 a. $\overline{\text{M}}$CDXXIV and $\overline{\overline{\text{M}}}$CDXXIV
 b. 4632 and 46,032
 c.  and 
 d.  and 
 e.  and 
9. For each of the following numerals write both the succeeding and the preceding numerals (one more and one less).
 a. MCMXLIX
 b. 
 c. 
 d.
10. A cornerstone representing when a building was built reads MCMXXII. When was this building built?
11. Write each of the following numerals in Roman symbols.
 a. 121
 b. 42
 c. 91
 d. 2014
12. Complete the following table, which compares symbols for numbers in different numeration systems:

	Hindu-Arabic	Babylonian	Egyptian	Roman	Mayan
a.	72				
b.					
c.					

13. In the counting frame, the value of each dot is represented by the number in the box below the dot. For example, the following figure represents the number 154.

• •	• • •	• •
64	8	1

What numbers are represented in the following frames?

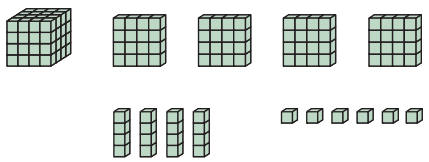
a.

• • •	• •	•
25	5	1

b.

•		•	•
8	4	2	1

14. Write the base-four numeral for the representation shown.



15. Write the first 15 counting numerals for each of the following bases.
a. Two **b.** Four

16. How many different digits are needed for the base-twenty system?

17. Write 2032_{four} in expanded notation and then convert it to base ten.

18. Determine the greatest three-digit number in each of the following bases.
a. Base two **b.** Base twelve

19. Find the numeral preceding and succeeding each of the following numerals.
a. $EE0_{twelve}$
b. 100000_{two}
c. 555_{six}

20. What, if anything, is wrong with the following numerals?
a. 204_{four}
b. 607_{five}

21. What is the least number of base-four blocks needed to represent 214?

22. Draw base-five blocks to represent 231_{five}.

23. Use only quarters, nickels, and pennies to answer the following questions.
a. What is the fewest number of quarters, nickels, and pennies you can receive in a fair exchange for two quarters, nine nickels, and eight pennies?
b. How could you use the approach in (a) to write 73 in base five?

24. Without converting to base ten, tell which is the lesser in each of the following pairs.
a. 3030_{four} or 3100_{four}
b. $E0TE_{twelve}$ or $E0ET_{twelve}$

25. Recall that with base-ten blocks: 1 long = 10 units, 1 flat = 10 longs, and 1 block = 10 flats. Make all possible exchanges to obtain the fewest number of pieces, and write the corresponding number in the given base in the following.
a. Ten flats in base ten
b. Twenty flats in base twelve

26. Convert each of the following base-ten numerals to a numeral in the indicated bases.
a. 456 in base five
b. 1782 in base twelve
c. 32 in base two

27. Write each of the following numerals in base ten.
a. 432_{five}
b. 101101_{two}
c. $92E_{twelve}$

28. Represent $abcd_{six}$, a base-six numeral, in expanded form.

29. In what base does $12_b + 26_b = 41_b$?

30. The hexadecimal numeration system in computer programming is a base-sixteen system using the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *A*, *B*, *C*, *D*, *E*, *F*. Convert the following to base ten.
a. $256_{sixteen}$ **b.** $3ACD_{sixteen}$

31. Write 584 in base sixteen. (See exercises 30.)

32. You are asked to distribute $900 in prize money. The dollar amounts for the prizes are $625, $125, $25, $5, and $1. How should this $900 be distributed in order to give the fewest number of prizes?

33. Convert each of the following quantities.
a. 58 days to weeks and days
b. 29 hours to days and hours

34. For each of the following, find *b* if possible. If not possible, tell why.
a. $b2_{seven} = 44$ **b.** $5b2_{twelve} = 734$

35. Write the following numerals in the indicated base without multiplying out the powers.
a. $3{\cdot}5^4 + 3{\cdot}5^2$ in base five
b. $2 \cdot 12^5 + 8 \cdot 12^3 + 12$ in base twelve

36. In a game called WIPEOUT, we are to "wipe out" digits from a calculator's display without changing any of the other digits. "Wipe out" in this case means to replace the chosen digit(s) with a 0. For example, if the initial number is 54,321 and we are to wipe out the 4, we could subtract 4000 to obtain 50,321. Complete the following two parts and then try other numbers or challenge another person to wipe out a number from the number you have placed on the screen.
a. Wipe out the 2s from 32,420.
b. Wipe out the 5 from 67,357.

Assessment 3-1B

1. Write the following in words.
a. 253,243,810
b. 62,355,288,432

2. Indicate which digit in 2,345,925 has place value
a. thousands
b. hundred thousands

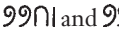
3. For each of the following base-ten numerals, give the place value of the underlined digit.
a. 9<u>7</u>,998
b. <u>8</u>10,485

4. Rewrite each of the following as a base-ten numeral.
a. $3 \cdot 10^3 + 5 \cdot 10^2 + 6 \cdot 10$
b. $9 \cdot 10^6 + 9 \cdot 10 + 9$

5. If □, △, and ○ represent different non-zero base-ten digits, order the numerals from least to greatest.
a. ○○; □△○; ○
b. □□□□; △△; □△○

6. How many digits are there in the standard form of $5250 \cdot 10^{15}$?

7. A two-digit number has the property that the units digit is 4 less than the tens digit, and the tens digit is twice the units digit. What is the number?

8. For each of the following pairs, tell which represents the greater number and why.
- **a.** $\overline{M}$DCXXIV and $\overline{M}$CDXXIV
- **b.** 3456 and 30,456
- **c.**  and
- **d.**  and
- **e.**  and

9. For each of the following name both the preceding and the succeeding numbers (one more and one less).
- **a.** $\overline{M}$I
- **b.** CMXCIX
- **c.** 
- **d.** 
- **e.** 

10. On the United States one-dollar bill, the number MDCCLXXVI is written on the base of the pyramid. What year does this represent?

11. Write each of the following in Roman numerals.
- **a.** 89
- **b.** 5202
- **c.** 32
- **d.** 102

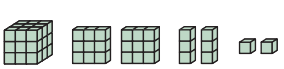
12. Complete the following table, which compares symbols for numbers in different numeration systems.

	Hindu-Arabic	Babylonian	Egyptian	Roman	Mayan
a.	78				
b.					
c.					

13. A number is represented on the following counting frame. What might the number be? Explain your reasoning.

•	• •	• •
27	9	1

14. Write the base-three numeral for the base-three representation shown.



15. Write the first 10 counting numerals for each of the following bases.
- **a.** Base three
- **b.** Base eight

16. How many different digits are used in base eighteen?

17. Write 2022_{three} in expanded form and then convert it to base ten.

18. Determine the greatest three-digit number in each of the following bases.
- **a.** Base three
- **b.** Base seven

19. Find the numeral preceding and succeeding each of the following numerals.
- **a.** 100_{seven}
- **b.** 1000_{two}
- **c.** 101_{two}

20. What, if anything, is wrong with the following.
- **a.** 306_{four}
- **b.** 1023_{two}

21. What is the fewest number of base-three blocks needed to represent 79?

22. Draw base-two blocks to represent 1001_{two}.

23. Using a number system based on dozen and gross, how would you describe the representation for 277?

24. Without converting to base ten, tell which is the greater for each of the following pairs and explain why.
- **a.** $EET9E_{twelve}$ or $EOT9E_{twelve}$
- **b.** 1011011_{two} or 101011_{two}
- **c.** 50555_{six} or 51000_{six}

25. What is the fewest number of multibase blocks that can be used to represent the corresponding numeral in the given base?
- **a.** 10 longs in base four
- **b.** 10 longs in base three

26. Convert each of the following base-ten numerals to a numeral in the indicated base.
- **a.** 234 in base four
- **b.** 1876 in base twelve
- **c.** 303 in base three
- **d.** 22 in base two

27. Write each of the following numerals in base ten.
- **a.** 432_{six}
- **b.** 11011_{two}
- **c.** $E29_{twelve}$

28. Represent $abcd_{eight}$ as a base-ten numeral in expanded form.

29. In what base does $54_b + 45_b = 132_b$?

30. The hexadecimal numeration system is used in computer programming. It is a base-sixteen system and uses the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *A*, *B*, *C*, *D*, *E*, *F*. Convert the following to base ten.
- **a.** $864_{sixteen}$
- **b.** $2CDE_{sixteen}$

31. Write 256 in base sixteen. (See exercises 30.)

32. *Who Wants the Money*, a game show, distributes prizes that are powers of 2. What is the minimum number of prizes that could be distributed from $900?

33. A coffee shop sold 1 cup, 1 pint, and 1 quart of coffee. Express the number of cups sold in base two.

34. For each of the following numerals find *b*, if possible. If not possible, tell why.
- **a.** $b3_{four} = 31_{ten}$
- **b.** $1b2_{twelve} = 1534_{six}$

35. Using only the number keys on a calculator, fill the display to show the greatest four-digit number if each key can be used only once.

36. The English word MIX has value 1000 + 9 = 1009 in Roman numerals. Find other English words that can be converted to base ten if the letters are Roman numerals.

Mathematical Connections 3-1

Reasoning

1. Explain why a number and a numeral are considered different.
2. If a numeration system has place value, what does this mean?
3. Explain the difference between zero and nothing.
4. What are the major drawbacks to each of the following systems?
 - **a.** Egyptian
 - **b.** Babylonian
 - **c.** Roman
5. **a.** Why are large numbers in the United States written with commas separating groups of three digits?
 b. Find examples from other countries that do not use commas to separate groups of three digits.
6. In the Roman numeral system explain **(a)** when you add values, **(b)** when you subtract values, and **(c)** when you multiply values. Give an example to illustrate each case.

Open-Ended

7. Research an abacus and discuss how it uses place value.
8. An inspector of weights and measures uses a special set of weights to check the accuracy of scales. Various weights are placed on a scale to check accuracy of any amount from 1 oz through 15 oz. What is the fewest number of weights the inspector needs? What weights are needed to check the accuracy of scales from 1 oz through 15 oz? From 1 oz through 31 oz? Discuss the pattern forming here and find possible extensions.

Cooperative Learning

9. **a.** Create a numeration system with unique symbols, and write a paragraph explaining the properties of the system.
 b. Complete the following table using the system in part (a).

Hindu-Arabic Numeral	Your System Numeral
1	
5	
10	
50	
100	
5000	
10,000	
115,280	

Connecting Mathematics to the Classroom

10. Joanne asks if the year is 2015, why isn't it in the 20th century? How do you respond?
11. Allie added the Babylonian numbers below as shown.

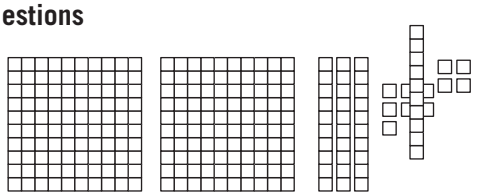


How would you help her?

12. Betty asks how the two 3s in the numeral 32,356 are different. How do you respond?
13. A student claims that the Roman system is a base-ten system since it has symbols for 10, 100, and 1000. How do you respond?
14. When using Roman numerals, a student asks whether it is correct to write $\bar{\text{I}}\text{I}$, as well as MI for 1001. How do you reply?

National Assessments

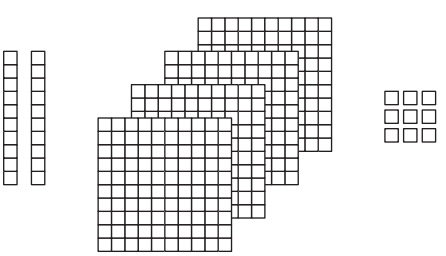
National Assessment of Educational Progress (NAEP) Questions



The figure above represents 237. Which number is more than 237?

A. 244
B. 249
C. 251
D. 377

NAEP, Grade 4, 2007



Each small square (□) above is equal to 1. There are 10 small squares in each strip. There are 100 small squares in each large square. What number is shown?

A. 4,029
B. 492
C. 429
D. 249

NAEP, Grade 4, 2003

3-2 Addition and Subtraction of Whole Numbers

3-2 Objectives

Students will be able to understand and explain

- Number relationships including comparing and ordering.
- The meaning of addition and subtraction by studying various models and in turn learn addition and subtraction facts.
- Properties of addition and subtraction and how to use them to develop computational strategies.
- The inverse relationship between addition and subtraction.

CCSS **In the grade 1 *Common Core Standards*, we find the following:**

Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 − 4 = 13 − 3 − 1 = 10 − 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 − 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13). (p. 15)

When 0 is included with the set of natural numbers $N = \{1, 2, 3, 4, 5, \ldots\}$, we have the set of whole numbers, denoted $W = \{0, 1, 2, 3, 4, 5, \ldots\}$. In the first part of this section, we develop the strategies mentioned above for adding and subtracting whole numbers. Later in the section, we show how to use properties of addition and subtraction to create and use additional strategies for these operations. Addition and subtraction are **binary operations** because *two* numbers are combined to form another number.

Addition of Whole Numbers

Counting On

Children encounter addition in their preschool years by combining sets of objects and wanting to know how many objects are in the combined set. They may count the objects to find the cardinal number of the combined set, or they may "count on." *Counting on* is an addition strategy where addition is performed by counting on from one of the numbers, for example, 5 + 3 can be computed by starting at 5 and counting 6, 7, 8. The addition 7 + 4 can be performed by starting at 7 and counting 8, 9, 10, 11.

Set Model

A set model is one way to represent addition of whole numbers. Suppose Jane has 4 blocks in one pile and 3 in another. If she combines the two groups, how many objects are there in the combined group? Figure 10 shows the solution. The combined set of blocks is the union of the disjoint sets of 4 blocks and 3 blocks. *Note the importance of the sets being disjoint or having no elements in common.* If the sets have common elements, then an incorrect conclusion can be drawn when using the set model.

Historical Note

Historians think that the word *zero* originated from the Hindu word *sūnya*, which means "void." Then *sūnya* was translated into the Arabic *sifr*, which when translated to Latin became *zephirum*, from which the word *zero* was derived.

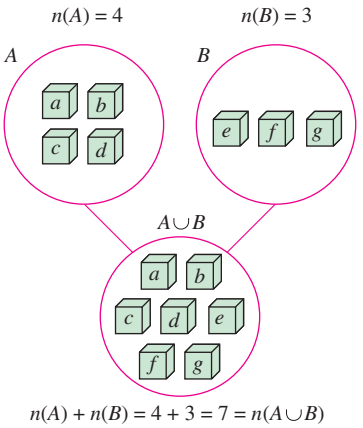


Figure 10

We define addition formally using set terminology.

Definition of Addition of Whole Numbers

Let A and B be two disjoint finite sets. If $n(A) = a$ and $n(B) = b$, then $a + b = n(A \cup B)$.

The numbers a and b are the **addends** and $a + b$ is the **sum**.

NOW TRY THIS 6

If the sets in the preceding definition of addition of whole numbers are not disjoint, explain why the definition is incorrect.

Number Line (Measurement) Model

The set model for addition is not always the best model for addition. For example, consider the following addition problems.

1. Josh has 4 feet of red ribbon and 3 feet of white ribbon. How many feet of ribbon does he have altogether?
2. One day, Gail drank 4 ounces of orange juice in the morning and 3 ounces at lunchtime. If she drank no other orange juice that day, how many ounces of orange juice did she drink for the entire day?

A *number line* can be used to model the preceding and other whole-number additions. Any line marked with two fundamental points, one representing 0 and the other representing 1, can be represented by a number line. The points representing 0 and 1 mark the ends of a *unit segment*. Other points can be marked and labeled, as shown in Figure 11. Any two consecutive whole numbers on the number line in Figure 11 mark the ends of a segment that has the same length as the unit segment.

Historical Note

The symbol "+" first appeared in a 1417 manuscript and was a short way of writing the Latin word *et*, which means "and." The word *minus* means "less" in Latin. First written as an *m*, it was later shortened to a horizontal bar. In a printed book in 1489, the symbols + and − referred only to surpluses and deficits in business problems.

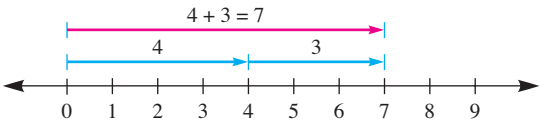


Figure 11

Addition problems may be modeled using directed arrows (*vectors*) on the number line. For example, the sum of 4 + 3 is shown in Figure 11. Arrows representing the addends, 4 and 3, are combined into one arrow representing the sum 4 + 3.

Students need to understand that the sum represented by any two directed arrows can be found by placing the endpoint of the first directed arrow at 0 and then joining to it the directed arrow for the second number with no gaps or overlaps. The sum of the numbers can then be read. We have depicted the addends as arrows (or vectors) above the number line, but students typically concatenate (connect) the arrows directly on the line. Figure 11 poses an inherent problem for students. If an arrow starting at 0 and ending at 3 represents 3, why should an arrow starting at 4 and ending at 7 also represent 3?

NOW TRY THIS 7

A common error is that students represent 3 as an arrow on the number line sometimes starting at 1, as shown in Figure 12. Explain why this is not appropriate.



Figure 12

Ordering Whole Numbers

CCSS **In the kindergarten *Common Core Standards*, we find that students should be able to:**

Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

Compare two numbers between 1 and 10 presented as written numerals. (p. 11)

In Chapter 2, we used the concept of a set and one-to-one correspondence to define *less than* and *greater than* relations. A horizontal number line can also be used to describe **greater than** and **less than** relations on the set of whole numbers. For example, in Figure 11, notice that 4 is to the left of 7 on the number line. We say, "four is less than seven," and we write $4 < 7$. Equivalently we also say "seven is greater than four" and write $7 > 4$. Because 4 is less than 7, there is a natural number that can be added to 4 to get 7, namely 3. Thus, $4 < 7$ because $4 + 3 = 7$. This discussion can be generalized to form the following definition of *less than*.

Definition of Less Than

For any whole numbers a and b, a is **less than** b, written $a < b$, if, and only if, there exists a natural number k such that $a + k = b$.

Sometimes equality is combined with the inequalities, greater than and less than, to give the relations **greater than or equal to** and **less than or equal to**, denoted $\geq$ and $\leq$, respectively. Thus, $a \leq b$ means $a < b$ or $a = b$. The emphasis with respect to these symbols is on the meaning of *or* in logic. Thus $3 \leq 5$, $5 \geq 3$, and $3 \geq 3$ are all true statements.

Whole-Number Addition Properties

CCSS **In the grade 1 *Common Core Standards*, we find that students should:**

use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction. (p. 13)

We next develop properties of whole number addition.

Whenever two whole numbers are added, a unique whole number is obtained. We say that "the set of whole numbers is *closed under addition*."

Theorem 3-1: Closure Property of Addition of Whole Numbers

If a and b are whole numbers, then $a + b$ is a whole number.

The closure property implies that the sum of two whole numbers *exists*, and that the sum is a *unique whole number*; for example, $5 + 2$ is a unique whole number, and that number is 7.

NOW TRY THIS 8

Determine whether each of the following sets is closed under addition: If not, tell why not.

a. $E = \{2n | n \in W\}$
b. $F = \{2n + 1 | n \in W\}$
c. $G = \{0, 2\}$

Figure 13(a) shows two additions. Pictured above the number line is $3 + 5$ and below the number line is $5 + 3$. The sums are the same. Figure 13(b) shows the same sums obtained with colored rods. Both illustrations in Figure 13 demonstrate the idea that when two whole numbers are added in either order, we obtain the same answer. This property is true in general and is the *commutative property of addition of whole numbers*.

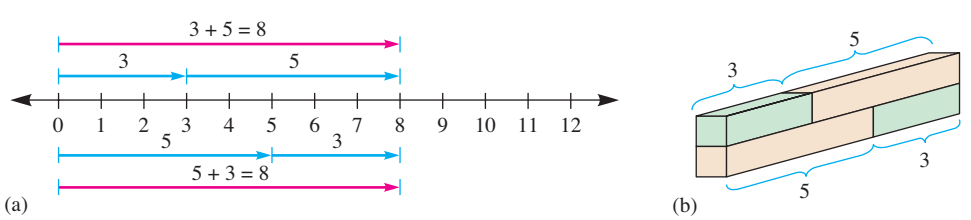


Figure 13

Theorem 3-2: Commutative Property of Addition of Whole Numbers

If a and b are any whole numbers, then $a + b = b + a$.

We say that "addition of whole numbers is commutative." The word *commutative* is derived from *commute*, which means "to interchange."

The commutative property of addition of whole numbers is not obvious to many young children. They may be able to find the sum $9 + 2$ and not be able to find the sum $2 + 9$. Using *counting on*, $9 + 2$ can be computed by starting at 9 and then counting on two more as "ten" and "eleven." To compute $2 + 9$ without the commutative property, the *counting on* is more involved. Students need to understand that $2 + 9$ is another name for $9 + 2$.

NOW TRY THIS 9

a. Use a set model to show the commutative property for $3 + 5 = 5 + 3$.
b. Use a number line model to show that $4 + 2 = 2 + 4$.
c. Which model do you prefer? Why?

IMAP Video
Watch Connor use the associative and commutative properties of addition to find the sum of whole numbers.

Another property of addition is demonstrated when selecting the order in which to add three or more numbers. For example, we could compute $24 + 8 + 2$ by grouping the 24 and the 8 together: $(24 + 8) + 2 = 32 + 2 = 34$. (The parentheses indicate that the first two numbers are grouped together.) We might also recognize that it is easy to add any number to 10 and compute it as $24 + (8 + 2) = 24 + 10 = 34$. This example illustrates the *associative property of addition of whole numbers*. The word *associative* is derived from the word *associate*, which means "to unite."

Theorem 3-3: Associative Property of Addition of Whole Numbers

If a, b, and c are whole numbers, then $(a + b) + c = a + (b + c)$.

When several numbers are being added, the parentheses are usually omitted because the grouping does not alter the result.

Another property of addition of whole numbers is seen when one addend is 0. In Figure 14, set A has 5 blocks and set B has 0 blocks. The union of sets A and B has only 5 blocks.

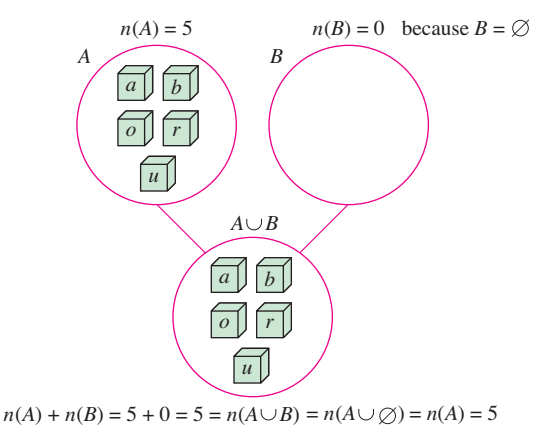


Figure 14

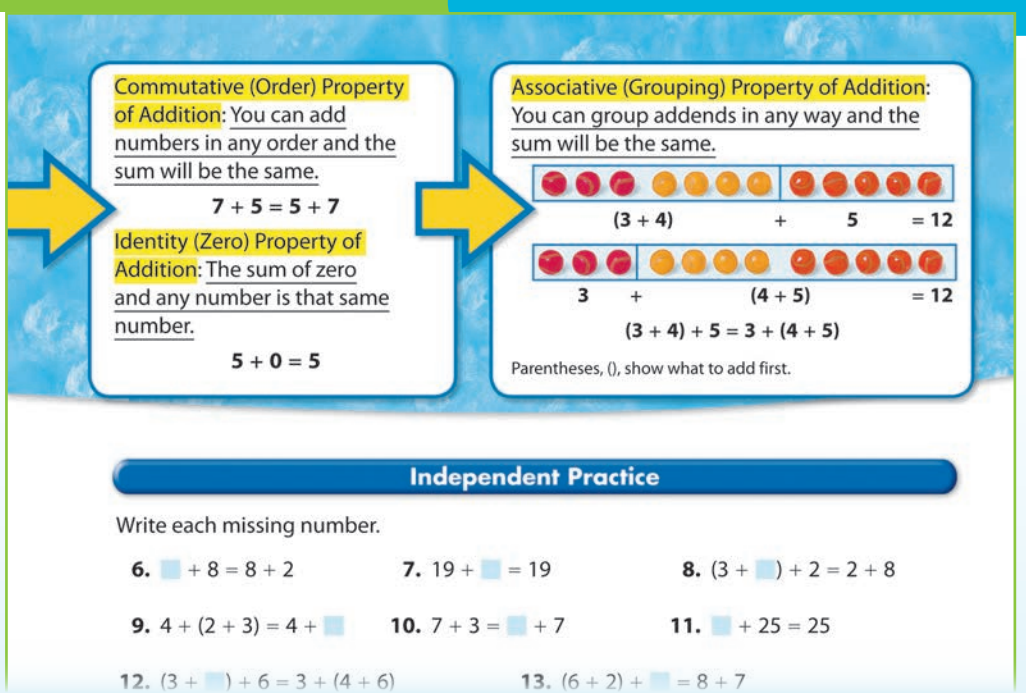
The set model illustrates the following property of whole numbers:

Theorem 3-4: Identity Property of Addition of Whole Numbers

There is a unique whole number 0, called the **additive identity**, such that for any whole number a, $a + 0 = a = 0 + a$.

The commutative, associative, and identity properties are introduced on the following partial grade 3 student page.

School Book Page



Commutative (Order) Property of Addition: You can add numbers in any order and the sum will be the same.

$7 + 5 = 5 + 7$

Identity (Zero) Property of Addition: The sum of zero and any number is that same number.

$5 + 0 = 5$

Associative (Grouping) Property of Addition: You can group addends in any way and the sum will be the same.

$(3 + 4) + 5 = 12$

$3 + (4 + 5) = 12$

$(3 + 4) + 5 = 3 + (4 + 5)$

Parentheses, (), show what to add first.

Independent Practice

Write each missing number.

6. $\square + 8 = 8 + 2$ **7.** $19 + \square = 19$ **8.** $(3 + \square) + 2 = 2 + 8$

9. $4 + (2 + 3) = 4 + \square$ **10.** $7 + 3 = \square + 7$ **11.** $\square + 25 = 25$

12. $(3 + \square) + 6 = 3 + (4 + 6)$ **13.** $(6 + 2) + \square = 8 + 7$

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Example 6

Which properties of whole numbers are illustrated in each of the following statements?

a. $5 + 7 = 7 + 5$
b. $1001 + 733$ is a unique whole number.
c. $(3 + 5) + 7 = (5 + 3) + 7$
d. $(8 + 5) + 2 = 2 + (8 + 5) = (2 + 8) + 5$

Solution

a. Commutative property of addition
b. Closure property of addition
c. Commutative property of addition
d. Commutative and then associative properties of addition

Mastering Basic Addition Facts

Certain mathematical facts are *basic addition facts*. Basic addition facts are those involving a single digit plus a single digit. One method of learning the basic facts is to organize them according to different derived fact strategies. Some stategies are listed below.

1. **Counting on.** The strategy of counting on from one of the addends can be used any time whole numbers are added, but it may be inefficient. It is usually used when one addend is 1, 2, or 3. For example, $9 + 2$ can be computed by starting at 9 and then counting on 10, 11.
2. **Doubles.** Use of *doubles*, such as $3 + 3$, receive special attention. After students master doubles, *doubles* + 1 and *doubles* + 2 can be learned easily. For example, if a student knows $6 + 6 = 12$, then using the associative property $6 + 7 = 6 + (6 + 1) = (6 + 6) + 1$, or 1 more than the double of 6, or 13.
3. **Making 10.** Another strategy is that of *making* 10 and then adding any remaining. For example, think of $8 + 5$ as shown in Figure 15 using the associative property of addition.

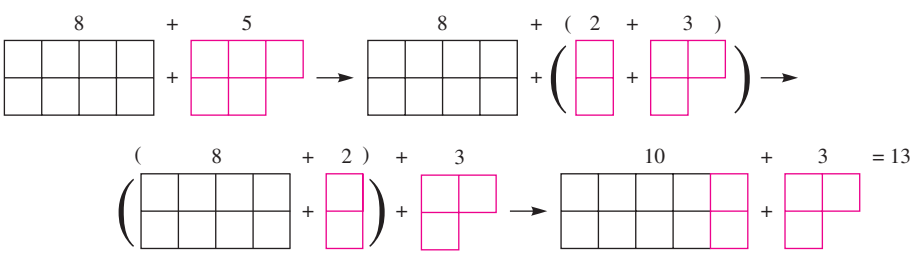


Figure 15

4. **Counting back.** The strategy of *counting back* is usually used when one number is 1 or 2 less than 10. For example, because 9 is 1 less than 10, then $9 + 7$ is 1 less than $10 + 7$. In symbols, this is $9 + 7 = (10 + 7) - 1 = 17 - 1 = 16$.

Many basic facts might be classified under more than one strategy. For $9 + 8$, we use *making* 10 as $9 + (1 + 7) = (9 + 1) + 7 = 10 + 7 = 17$ or use a *double* + 1 as $(8 + 8) + 1$.

Subtraction of Whole Numbers

In elementary school, operations that "undo" each other are **inverse operations**. Subtraction is the inverse operation of addition. It is sometimes hard for children to understand this inverse relationship. Subtraction of whole numbers models include the *take-away* model, the *missing addend* model, the *comparison* model, and the *number line (measurement)* model.

Take-Away Model

To model addition, we imagine a second set of objects as being joined to a first set, but in subtraction, we imagine a subset being *taken away* from a first set. For example, suppose from a set of 8 blocks 3 are taken away. This is illustrated in Figure 16 and the process is recorded as $8 - 3 = 5$.

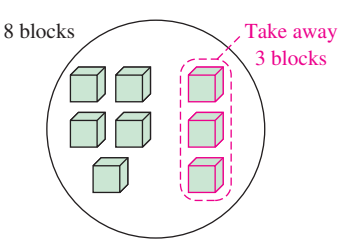


Figure 16

NOW TRY THIS 10

Recall how addition of whole numbers was defined using the concept of union of two disjoint sets. Similarly, write a definition of subtraction of whole numbers using the concepts of subsets and set difference.

Missing Addend Model

The *missing addend* model for subtraction relates subtraction and addition. In Figure 17(a) suppose set A has 8 blocks but only 3 are visible. How many blocks must be in the set to have a total of 8? The answer is $8 - 3$, but this can also be thought of as the number of blocks that could be added to 3 blocks in order to get 8 blocks; that is,

$$3 + \boxed{8 - 3} = 8.$$

The number $\boxed{8 - 3} = 5$ is the **missing addend** in the equation

$$3 + \square = 8.$$

We demonstrate the *missing addend* approach using a number line. The subtraction $8 - 3$ is illustrated in Figure 17(a) using sets and in Figure 17(b) using the number line.

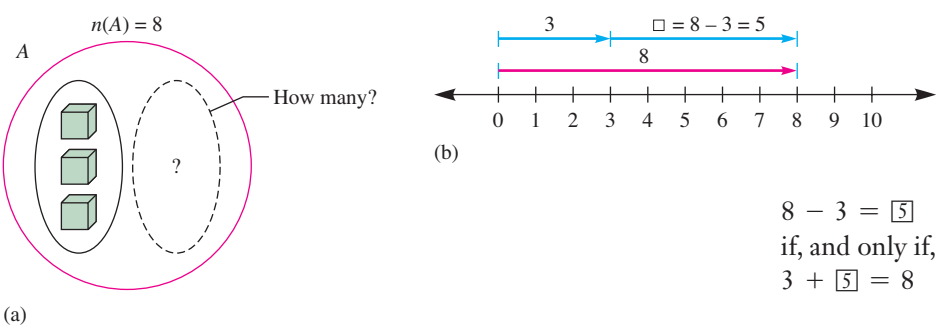


Figure 17

IMAP Video
Watch Arriel solve a problem using the missing-addend model.

The missing addend model gives elementary school students an opportunity to use algebraic thinking because the concept of the unknown plays a key role in this model.

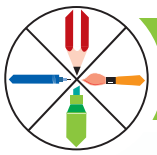
Cashiers often use the missing-addend model. For example, if the bill for a movie is \$8 and you give the cashier \$10, the cashier might calculate the change by saying "8 and 2 is 10." This idea can be generalized: *For any whole numbers a and b such that $a \geq b$, $a - b$ is the unique whole number such that $b + (a - b) = a$.* That is, $a - b$ is the unique solution of the equation $b + \square = a$. The definition can be written as follows:

Definition of Subtraction of Whole Numbers

For any whole numbers a and b such that $a \geq b$, $a - b$ is the unique whole number c such that $b + c = a$.

An alternative but equivalent definition for subtraction can be given without stipulating that $a \geq b$: "For any whole numbers a and b, $a - b$ is the unique whole number c, if it exists, such that $b + c = a$." Notice that "$a \geq b$" has been deleted and "if it exists" added.

Refer to the partial student page shown on page 108 to see how grade 3 students are shown how addition and subtraction are related using a **fact family**.



School Book Page

Another Example **Subtract to find a missing addend.**

Rick plans on making 13 flags. How many more flags does he need?

The parts and the whole show how addition and subtraction are related.



$8 + \square = 13$

You can write a fact family when you know the parts and the whole.

$5 + 8 = 13$ $\quad$ $8 + 5 = 13$

$13 - 8 = 5$ $\quad$ $13 - 5 = 8$

A fact family is a group of related facts using the same numbers.

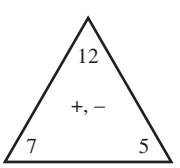
The missing part is 5. This means Rick needs to make 5 more flags.

The difference is the answer when subtracting two numbers.

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NOW TRY THIS 11

The following is an *Addition/Subtraction Fact Triangle*. It emphasizes fact families. It has two addends and a sum.



Fact Family	
$7 + 5 = 12$	$5 + 7 = 12$
$12 - 5 = 7$	$12 - 7 = 5$

Develop an *Addition/Subtraction Fact Triangle* for each of the following and write the fact family for each case

a. 7, 8, 15 **b.** 4, 7, 11

Comparison Model

Another way to consider subtraction is by using a *comparison* model. Suppose Juan has 8 blocks, Susan has 3 blocks, and we want to know how many more blocks Juan has than Susan. We can pair Susan's blocks with some of Juan's blocks, as shown in Figure 18, and determine that Juan has 5 more blocks than Susan. We also write this as $8 - 3 = 5$.

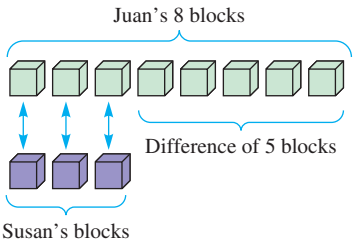


Figure 18

Number Line (Measurement) Model

Subtraction of whole numbers can also be modeled on a number line using directed arrows, as suggested in Figure 19, which shows that $5 - 3 = 2$.

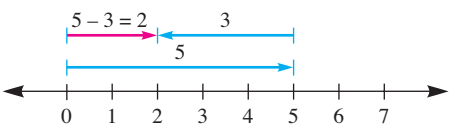


Figure 19

Properties of Subtraction

In an attempt to find $3 - 5$, we use the definition of subtraction: $3 - 5 = c$ if, and only if, $c + 5 = 3$. Since there is no whole number c that satisfies the equation, $3 - 5$ is not meaningful in the set of whole numbers. In general, if $a < b$, then $a - b$ is not meaningful in the set of whole numbers. Therefore, the set of whole numbers is not closed under subtraction.

NOW TRY THIS 12

Which of the following properties hold for subtraction of whole numbers? Explain.

a. Closure property
b. Associative property
c. Commutative property
d. Identity property

Using Whole-Number Addition and Subtraction in Equations

Sentences such as $9 + 5 = \square$ and $12 - \Delta = 4$ can be true or false depending on the values of $\square$ and Δ. For example, if $\square = 10$, then $9 + 5 = \square$ is false. If $\Delta = 8$, then $12 - \Delta = 4$ is true. A value that makes an equation true is a **solution** to the equation. Thus, $8 - 5 = 3$ is the solution to $\square + 5 = 8$.

NOW TRY THIS 13

Find the solution for each of the following where x is a whole number.

a. $x + 8 = 13$
b. $15 - x = 8$
c. $x > 9$ and $x < 11$
d. $x + b = a$ if $a \geq b$, a and b are whole numbers

Assessment 3-2A

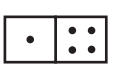
1. A student uses the *counting on* strategy for adding two numbers. She says "thirteen, fourteen, fifteen, sixteen, seventeen." What two numbers is she adding?
2. For which of the following sets is it true that $n(A) + n(B) = n(A \cup B)$?
 a. $A = \{a, b, c\}, B = \{d, e\}$
 b. $A = \{a, b, c\}, B = \{b, c\}$
 c. $A = \{a, b, c\}, B = \varnothing$
3. If $n(A) = 3$, $n(B) = 5$, and $n(A \cup B) = 6$, what do you know about $n(A \cap B)$?
4. Give an example to show why, in the definition of addition, sets A and B must be disjoint.
5. If $n(A) = 3$ and $n(A \cup B) = 6$, answer the following questions.
 a. What are the possible values for $n(B)$?
 b. If in addition $A \cap B = \varnothing$, what are the possible values of $n(B)$?
6. Which of the following sets are closed under addition? If a set is not closed under addition explain why not.
 a. $B = \{0\}$
 b. $T = \{3n | n \in W\}$
 c. $N = \{1, 2, 3, 4, 5, \ldots\}$
 d. $V = \{3, 5, 7\}$
 e. $A = \{x | x \in W \text{ and } x > 10\}$
 f. $C = \{0, 1\}$
7. Set A is closed under addition and contains the numbers 2 and 3. List six additional numbers, three even and three odd, that must be in A.
8. Set A contains the elements 2 and 3 and is closed under addition. What whole numbers might be missing from A?
9. Each of the following equations illustrates an example of one of the properties of whole-number addition. Fill in the blank to make a true statement and identify the property or properties.
 a. $9 + 1 = __ + 9$
 b. $7 + (3 + 5) = (3 + 5) + __$
 c. $a + __ = a = __ + a$
 d. $7 + (3 + 5) = (3 + __) + 5$
10. Each of the following illustrates a property of addition of whole numbers. Identify the property illustrated.
 a. $6 + 3 = 3 + 6$
 b. $(6 + 3) + 5 = 6 + (3 + 5)$
 c. $(6 + 3) + 5 = (3 + 6) + 5$
 d. $5 + 0 = 5 = 0 + 5$
 e. $5 + 0 = 0 + 5$
 f. $(a + c) + d = a + (c + d)$
 g. $5 + 0$ is a unique whole number
11. In the definition of *less than*, can the natural number k be replaced by the whole number k? Why or why not?
12. Give a definition of *less than* and *greater than* using the concept of subtraction.
13. Find the next three terms in each of the following arithmetic sequences.
 a. 8, 13, 18, 23, 28, __, __, __
 b. 98, 91, 84, 77, 70, 63, __, __, __
14. Complete the following magic square using the whole numbers 1–9. (Each row, column, and diagonal must have the same sum.)

8		
	5	
		2

15. If A, B, and C each stand for a different single digit from 1 to 9, answer the following if

$$A + B = C.$$

 a. What is the greatest digit that C could be?
 b. What is the greatest digit that A could be?
 c. What is the least digit that C could be?
 d. If A, B, and C are even, what number(s) could C be?
 e. If C is 5 more than A, what number(s) could B be?
 f. If A is 3 times B, what number(s) could C be?
 g. If A is odd and A is 5 more than B, what number(s) could C be?
16. A special domino set contains all number pairs from double-0 to double-8, with each number pair occurring only once. For example, the following domino counts as 1-4 and 4-1. How many dominos are in the set?



17. a. At a volleyball game, the players stood in a row ordered by height. If Kent is shorter than Mischa, Sally is taller than Mischa, and Vera is taller than Sally, who is the tallest and who is the shortest?
 b. Write possible heights for the players in part (a).
18. Rewrite each of the following subtraction problems as an equivalent addition problem:
 a. $9 - 7 = x$
 b. $x - 6 = 3$
 c. $9 - x = 2$
19. Illustrate $8 - 5 = 3$ using each of the following models:
 a. Take-away
 b. Missing addend
 c. Comparison
 d. Number line
20. Write a word problem that corresponds well to each of the following models for subtraction.
 a. Take-away
 b. Number line
 c. Comparison
 d. Missing addend
21. a. Solve for x in each of the following equations.
 i. $15 - 9 = 15 - 10 + x$
 ii. $27 - 9 = 27 - 10 + x$
 iii. $32 - 9 = 32 - 10 + x$
 b. What strategy does this suggest for subtracting 9 from a whole number?
22. a. One fact in a fact family is $6 + 5 = 11$. What are the other three members of this fact family?
 b. Draw four number line models to illustrate each of the number facts in part (a).

23. What conditions, if any, must be placed on a, b, and c in each of the following to make sure that the operations can be performed within the set of whole numbers?
- **a.** $a - b$
- **b.** $a - (b - c)$

24. Show that each of the following is true. Give a property of addition to justify each step in your argument.
- **a.** $x + (y + z) = z + (x + y)$
- **b.** $x + (y + z) = y + (x + z)$

25. Find the solution in the set W for each of the following equations
- **a.** $3 + (4 + 7) = (3 + x) + 7$
- **b.** $8 + 0 = x$
- **c.** $5 + 8 = 8 + x$
- **d.** $x + 8 = 12 + 5$
- **e.** $x + 8 = 5 + (x + 3)$
- **f.** $x - 2 = 9$
- **g.** $x - 3 = x + 1$
- **h.** $0 + x = x + 0$

26. Kelsey has a marbles, Gena has b marbles, and Noah has c marbles. If Kelsey has more marbles than Gena and Noah combined, write an expression (using a, b, and c) that shows how many more marbles Kelsey has than Gena and Noah combined.

27. Is set $B = \{5, 10, 15, 20, 25, \ldots\}$ or $B = \{5n | n \in W\}$ closed with respect to addition? Why?

Assessment 3-2B

1. A student uses the *counting on* strategy for adding two numbers. He says "fifteen, sixteen, seventeen, eighteen." What two numbers is he adding?

2. For which of the following sets is it true that $n(A) + n(B) = n(A \cup B)$?
- **a.** $A = \{a, b\}, B = \{d, e\}$
- **b.** $A = \{a, b, c\}, B = \{b, c, d\}$
- **c.** $A = \{a\}, B = \varnothing$

3. If $n(A) = 3$, $n(B) = 5$, and $n(A \cap B) = 1$, what do you know about $n(A \cup B)$?

4. If $n(A) - n(B)$ is defined as $n(A - B)$, what must $A \cap B$ equal?

5. If $n(B) = 4$ and $n(A \cup B) = 6$, answer the following.
- **a.** What are the possible values of $n(A)$?
- **b.** If in addition $n(A \cap B) = 0$, what are the possible values of $n(A)$?

6. Explain whether the following sets are closed under addition.
- **a.** $B = \{0, 1\}$
- **b.** $T = \{0, 4, 8, 12, 16, \ldots\}$
- **c.** $F = \{5, 6, 7, 8, 9, 10, \ldots\}$
- **d.** $H = \{x | x \in W \text{ and } x > 100\}$

7.
- **a.** Set A is closed under addition and contains the numbers 2, 5, and 8. List six other elements that must be in A.
- **b.** Set A contains the element 1. What other whole numbers must be in set A for it to be closed under addition?

8. The set A contains the elements 2 and 4 and is closed under addition. What whole numbers must necessarily be elements of A?

9. Each of the following equations is an example of one of the properties of whole-number addition. Fill in the blank to make a true statement, and identify the property.
- **a.** $3 + 4 = __ + 3$
- **b.** $5 + (4 + 3) = (4 + 3) + __$
- **c.** $8 + __ = 8$
- **d.** $3 + (4 + 5) = (3 + __) + 5$

10. Each of the following illustrates a property of addition of whole numbers. Identify the property illustrated.
- **a.** $6 + 8 = 8 + 6$
- **b.** $(6 + 3) + 0 = 6 + 3$
- **c.** $(6 + 8) + 2 = (8 + 6) + 2$
- **d.** $(5 + 3) + 2 = 5 + (3 + 2)$

11. Complete the following statement: $a \leq b$ if, and only if, $b = a + k$ for some _____ number k.

12. Use the concept of subtraction to define $a \geq b$.

13. Find the next three terms in each of the following arithmetic sequences.
- **a.** 5, 12, 19, 26, 33, __ __ __
- **b.** 63, 59, 55, 51, 47, __ __ __

14. Complete the following magic square using the whole numbers 1–9. (Each row, column, and diagonal must have the same sum.)

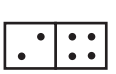
6		
1		
8		

15. If A, B, C, and D each represent a different single digit from 1 to 9, answer each of the following if

$$\begin{array}{r} A \\ +\ B \\ \hline CD \end{array}$$

- **a.** What is the value of C?
- **b.** Can D be 1? Why?
- **c.** If D is 7, what values can A be?
- **d.** If A is 6 greater than B, what is the value of D?

16.
- **a.** A domino set contains all number pairs from double-0 to double-6, with each number pair occurring only once; for example, the following domino counts as 2-4 and 4-2. How many dominos are in the set?



- **b.** When considering the sum of all dots on a single domino in an ordinary set of dominos, explain how the commutative property might be important.

17. If $a < b$, $c > b$, $d > c$, and $c < e < d$, order the letters from the least to the greatest.

18. Rewrite each of the following subtraction problems as an equivalent addition problem.
- **a.** $9 - 3 = x$
- **b.** $x - 5 = 8$
- **c.** $11 - x = 2$

19. Illustrate $7 - 3 = 4$ using each of the following models.
- **a.** Take-away
- **b.** Missing addend
- **c.** Comparison
- **d.** Number line

20. Write a word problem that corresponds to each of the following models for subtraction.
- **a.** Take-away
- **b.** Number line
- **c.** Comparison
- **d.** Missing addend

21. Use the *doubles plus* strategy for adding the following.
- **a.** $11 + 7$
- **b.** $24 + 28$
- **c.** $35 + 37$

22. **a.** One fact in a fact family is $11 + 5 = 16$. What are the other three members of this fact family?
b. Draw four number line models to illustrate each of the number facts in part (a).

23. What conditions, if any, must be placed on a, b, and c in each of the following expressions to make sure the operations can be performed within the set of whole numbers?
- **a.** $b - a$
- **b.** $b - (a - 3)$

24. Show that each of the following is true. Give a property of addition to justify each step.
- **a.** $a + (b + c) = c + (a + b)$
- **b.** $a + (b + c) = (c + b) + a$

25. Find the solution in the set W for each of the following equations.
- **a.** $12 - x = x + 6$
- **b.** $(9 - x) - 6 = 1$
- **c.** $3 + x = x + 3$
- **d.** $11 - x = 0$
- **e.** $14 - x = 7 - x$
- **f.** $x - 3 = 17$
- **g.** $x + 3 = x - 1$
- **h.** $0 + x = x - 0$

26. Rob has 11 pencils. Kelly has 5 pencils. Which number sentence shows how many more pencils Rob has than Kelly?
(i) $11 + 5 = 16$
(ii) $16 - 5 = 11$
(iii) $11 - 5 = 6$
(iv) $11 - 6 = 5$

27. Is the set $B = \{7, 14, 21, 28, 35, \ldots\}$ or $B = \{7n | n \in W\}$ closed with respect to addition? Why?

Mathematical Connections 3-2

Reasoning

1. In a survey of 52 students, 22 said they were taking algebra and 30 said they were taking biology. Are there necessarily 52 students taking algebra or biology? Why?

2. To find $9 + 7$, a student says she thinks of $9 + 7$ as $9 + (1 + 6) = (9 + 1) + 6 = 10 + 6 = 16$. What property or properties is she using?

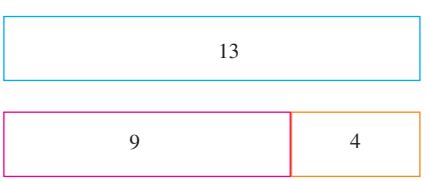
3. Explain in two different ways why if $a > b$ and c is a whole number, then $a + c > b + c$.

4. When subtraction and addition appear in an expression without parentheses, it is agreed that the operations are performed in order of their appearance from left to right. Taking this into account, and assuming that all expressions are meaningful, use an appropriate real-world model for subtraction to explain why each of the following statements is true.
- **a.** $a - b - c = a - c - b$
- **b.** $a - b - c = a - (b + c)$
- **c.** $(a + b) - c = a + (b - c)$
- **d.** $a + b - c = a - c + b$

5. Explain whether it is important for elementary students to learn more than one model for the operations of addition and subtraction.

6. Do elementary students still have to learn their basic facts when the calculator is part of the curriculum? Explain.

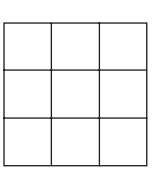
7. Explain how the model shown can be used to illustrate each of the following family of facts.
- **a.** $9 + 4 = 13$
- **b.** $4 + 9 = 13$
- **c.** $4 = 13 - 9$
- **d.** $9 = 13 - 4$



8. How are addition and subtraction related? Explain.

9. Is there a non-empty subset of the whole numbers that is closed under subtraction? If no, tell why not. If yes, give an example.

10. Use the figure below to design an *unmagic square*. That is, use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 exactly once so that every column, row, and diagonal adds to a different sum. Explain how you constructed your square.



Open-Ended

11. Describe any model not in this text that you might use to teach addition to students.

12. Suppose $A \subseteq B$. If $n(A) = a$ and $n(B) = b$, then $b - a$ is defined as $n(B - A)$. Choose two sets A and B and illustrate this definition.

13. **a.** Create a word problem in which the set model would be appropriate to show $25 + 8 = 33$.

b. Create a word problem in which the number-line (measurement) model would be appropriate to show $25 + 8 = 33$.
c. Create a word problem or a diagram that explains why $a - (b - c) = a - b + c$ (where all operations result in whole numbers).

Cooperative Learning

14. Use the basic addition fact table for whole numbers shown and discuss with your group each of the following questions.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

a. How does the table illustrate the closure property?
b. How does the table illustrate the commutative property?
c. How does the table illustrate the identity property?
d. How do the addition properties help students learn their basic facts?

15. Suppose that a mathematical system uses only four symbols, a, b, c, and d, and the operation Δ as shown in the table. Discuss with your group each of the following questions.

Δ	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

a. Is the system closed? Why?
b. Is the operation commutative? Why?
c. Does the operation have an identity? If so, what is it?
d. Use several examples to investigate whether the operation is associative.

16. Have each person in your group choose a different grade textbook that does whole number operations and report on when and how subtraction of whole numbers is introduced. Compare with the different ways subtraction is introduced in this section.

Connecting Mathematics to the Classroom

17. A student says that 0 is the identity for subtraction because if $a \in W$, then $a - 0 = a$. How do you respond?
18. A student asks why we use subtraction to determine how many more pencils Sam has than Karly if nothing is being taken away. How do you respond?
19. A student claims that the set of whole numbers is closed with respect to subtraction. To show this is true, she shows that $8 - 5 = 3$, $5 - 2 = 3$, $6 - 1 = 5$, and $12 - 7 = 5$ and says she can show examples like this all day that yield whole numbers when the subtraction is performed. How do you respond?
20. John claims that he can get the same answer to the problem below by adding up (begin with $4 + 7$) or by adding down (begin with $8 + 7$). He wants to know why and if this works all the time. How do you respond?

$$\begin{array}{r} 8 \\ 7 \\ +\,4 \\ \hline \end{array}$$

21. A first grader claims that $2 - 4 = 0$ because you can't have less than nothing. How do you respond?
22. A first grader claims that $2 - 4 = 2$ because you just turn the numbers around. How do you respond?
23. Al claims that if he has 8 cookies and eats 3 of them and later eats 2 more, this is the same as having 8 cookies and eating 2 of them and later eating 3 of them. He says that this is just the commutative property of subtraction. How do you respond?
24. Jane and Mark started with the same number of pencils. Jane gave Mark two pencils and claims that Mark now has two more pencils than she has. How do you respond?

Review Problems

25. Write 5280 in expanded form.
26. What is the value of MCDX in Hindu-Arabic numerals?
27. Convert each of the following to base ten:
a. $E0T_{\text{twelve}}$
b. 1011_{two}
c. 43_{five}
28. Write $12^4 + 12^2 + 13$ in base twelve.

National Assessments

National Assessment of Educational Progress (NAEP) Questions

$\square - 8 = 21$

What number should be put in the box to make the number sentence above true?

Answer: _____

NAEP, Grade 4, 2009

Fill in the four missing numbers on the number line above.

NAEP, Grade 4, 2005

3-3 Multiplication and Division of Whole Numbers

3-3 Objectives

Students will be able to understand and explain

- Meanings of multiplication and division by examining various models.
- The inverse relationship between multiplication and division.
- Properties of multiplication and division and how to use them to develop computational strategies.
- The special cases of multiplication and division by 0 and 1.

CCSS **In the grade 3 *Common Core Standards*, we find:**

Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division. (p. 21)

This quote sets the tone and agenda for this section. If multiplication and division facts are only memorized, they may be forgotten. If students have a conceptual understanding of the basic facts, then the basic facts can be determined even if not automatically recalled. We discuss models used to help students understand the meanings of multiplication and division. We then develop the distributive property of multiplication over addition and the relationship of multiplication and division as inverse operations.

Repeated-Addition and Number-Line (Measurement) Models

CCSS **In the grade 3 *Common Core Standards*, we find that students should be able to:**

Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as 5 × 7.* (p. 23)

One way to interpret products of whole numbers uses a *repeated-addition* model.

The student page shown on page 115 shows that if there are 3 groups of 8 goldfish, addition can put the groups together. When equal-sized groups are put together, we can use multiplication. Think of this as combining 3 sets of 8 objects into a single set. The 3 sets of 8 suggest the following addition:

$$\underbrace{8 + 8 + 8}_{\text{three 8}s} = 24$$

Historical Note



William Oughtred (1574–1660), an English mathematician, was the first to introduce the "St. Andrew's cross" (×) as the symbol for multiplication. This symbol was not readily adopted because, as Gottfried Wilhelm von Leibniz (1646–1716) objected, it was too easily confused with the letter *x*. Leibniz used the dot (·) for multiplication, which has become common.

Lesson 4-1

Common Core

3.OA.1 Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each.... Also 3.OA.3, 3.OA.5

Multiplication as Repeated Addition

How can you find the total number of objects in equal groups?

Jessie used 3 bags to bring home the goldfish she won at a Fun Fair. She put the same number of goldfish in each bag. How many goldfish did she win?



The counters show 3 groups of 8 goldfish.



You can use addition to join equal groups.

8 + 8 + 8 = 24

Multiplication is an operation that gives the total number when you join equal groups.

What You Say 3 times 8 equals 24

What You Write 3 × 8 = 24

factor factor product

Factors are the numbers that are being multiplied. The product is the answer to a multiplication problem.

Addition sentence: 8 + 8 + 8 = 24

Multiplication sentence: 3 × 8 = 24

So, 8 + 8 + 8 = 3 × 8.

Jessie won 24 goldfish.

Guided Practice*

MATHEMATICAL PRACTICES

Do you know HOW?

Copy and complete. Use counters.

1. 

2 groups of ☐
4 + 4 = ☐
2 × ☐ = ☐

2. 

☐ groups of 5
5 + ☐ + ☐ = ☐
3 × ☐ = ☐

Do you UNDERSTAND?

3. Can you write 3 + 3 + 3 + 3 as a multiplication sentence? Explain.

4. **Reason** Can 3 + 5 + 6 = 14 be written as a multiplication sentence? Explain.

5. **Model** Write an addition sentence and a multiplication sentence to solve this problem:

Jessie bought 4 packages of colorful stones to put in the fish bowl. There were 6 stones in each package. How many stones did Jessie buy?

Independent Practice

Use Tools Copy and complete. Use counters or draw a picture to help.

6. 

2 groups of ☐
6 + ☐ = ☐
2 × ☐ = ☐

7. 

3 groups of ☐
7 + ☐ + ☐ = ☐
3 × ☐ = ☐

Animated Glossary, eTools
www.pearsonsuccessnet.com

*For another example, see Set A on page 110.

Copy and complete each number sentence. Use counters or draw a picture to help.

8. 2 + 2 + 2 + 2 = 4 × ☐

9. ☐ + ☐ + ☐ = 3 × 7

10. 9 + ☐ + ☐ = ☐ × 9

11. 6 + 6 + 6 + 6 + 6 = ☐ × ☐

Persevere Write +, −, or × for each ☐.

12. 4 ☐ 3 = 12

13. 3 ☐ 6 = 9

14. 4 ☐ 4 = 0

15. 6 ☐ 4 = 10

16. 5 ☐ 3 = 2

17. 2 ☐ 4 = 8

Problem Solving

MATHEMATICAL PRACTICES

18. What number sentence shows how to find the total number of erasers?



A 5 + 5 = ☐
B 15 − 5 = ☐
C 15 + 5 = ☐
D 3 × 5 = ☐

19. **Model** Write an addition sentence and a multiplication sentence to solve the problem below.

Maria has 6 new flashlights. Each flashlight takes 3 batteries. How many batteries will Maria need for the flashlights?

20. **Communicate** Luke says that you can add or multiply to join groups. Is he correct? Explain.

21. Which picture shows 3 groups of 2?

A 
B 
C
D 

Lesson 4-1

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We write $8 + 8 + 8$ as $3 \cdot 8$ and say "three times eight" or "eight multiplied by three." The advantage of the multiplication notation over repeated addition is evident when the number of addends is great; for example, with 25 groups of 8 goldfish, we could find the total number of goldfish by adding twenty-five 8s or writing this as $25 \cdot 8$. As seen on the student page the numbers 3 and 8 are **factors** and 24 is the **product**. Work through the exercises on the student page.

The *repeated-addition* model can be illustrated in several ways. For example, using colored rods of length 4, the combined length of five rods can be found by joining the rods end-to-end, as in Figure 20(a). Figure 20(b) shows the process using arrows on a number line.

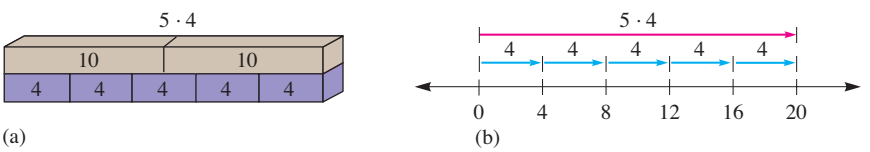


Figure 20



The constant feature on a calculator can help relate multiplication to addition. Students can find products on the calculator without using the $\boxed{\times}$ key. For example, if a calculator has the *constant feature*, then 5×3 can be found by starting with 0 and pressing $\boxed{+}\ \boxed{3}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$. Each press of the equal sign adds 3 to the display. (Some calculators work differently.)

The repeated-addition model motivates the following definition.

Definition of Multiplication of Whole Numbers: Repeated-Addition Model

For any whole numbers a and natural number n,

$$n \cdot a = \underbrace{a + a + a + \ldots + a}_{n \text{ terms}}.$$

Rectangular-Array and Area Models

Another representation useful in exploring multiplication of whole numbers is an *array*. An array is suggested when objects are arranged with the same numbers of objects in each row and column as in Figure 21.



Figure 21

In Figure 22(a), sticks are crossed to create intersection points, thus forming an array of points. The number of intersection points on a single vertical stick is 4 and there are 5 sticks, forming a total of $5 \cdot 4$ points in the array. In Figure 22(b), the *area* model is shown as a 4-by-5 grid. The number of unit squares required to fill in the grid is 20.

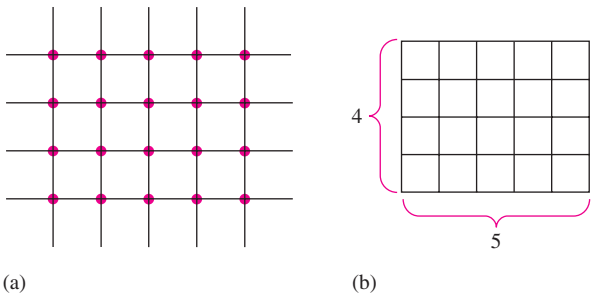


Figure 22

Cartesian-Product Model

The *Cartesian-product* model offers another way to discuss multiplication. Suppose you can order a soyburger on light or dark bread with one condiment: mustard, catsup, or horseradish. To show the number of different soyburger orders that a waiter could write for the cook, we use a *tree diagram*. The ways of writing the order are listed in Figure 23, where the bread is chosen from the set $B = \{\text{light, dark}\}$ and the condiment is chosen from the set $C = \{\text{mustard, catsup, horseradish}\}$.

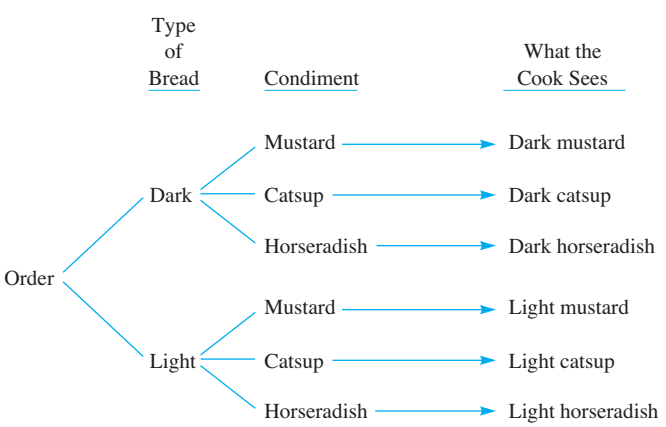


Figure 23

Table 7

Bread	Condiment: m	c	h
l	lm	lc	lh
d	dm	dc	dh

Each order can be written as an ordered pair, for example, (dark, mustard). The set of ordered pairs forms the Cartesian product $B \times C$. The Fundamental Counting Principle tells us that the number of ordered pairs in $B \times C$ is $2 \cdot 3$ or 6.

A table is another way to model this example. Let $B = \{l, d\}$ and $C = \{m, c, h\}$ where the elements of B represent the types of bread (light, dark) and the elements of C represent the set of condiments (mustard, catsup, horseradish). Table 7 shows the six possible combinations.

The preceding discussion demonstrates how multiplication of whole numbers can be defined in terms of Cartesian products. Thus, an equivalent definition of multiplication of whole numbers follows:

Definition of Multiplication of Whole Numbers: Cartesian Product Model

For finite sets A and B, if $n(A) = a$ and $n(B) = b$, then $a \cdot b = n(A \times B)$.

In this definition, sets A and B do not have to be disjoint. The expression $a \cdot b$, or simply ab, is the **product** of a and b, and a and b are **factors**. Note that $A \times B$ indicates the Cartesian product, not multiplication. Numbers are multiplied, sets are not.

Properties of Whole-Number Multiplication

The grade 3 *Common Core Standards* recognize the importance of applying the properties of whole numbers to multiply and divide.

Apply properties of operations as strategies to multiply and divide. Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.) (p. 23)

Next we discuss properties of whole-number multiplication.

The set of whole numbers is *closed* under multiplication. That is, if we multiply any two whole numbers, the result is a unique whole number. This property is referred to as the *closure property of multiplication of whole numbers*. Multiplication on the set of whole numbers, like addition, has the closure, commutative, associative, and identity properties.

Theorem 3-5: Properties of Multiplication of Whole Numbers

Closure property of multiplication of whole numbers For whole numbers a and b, $a \cdot b$ is a unique whole number.

Commutative property of multiplication of whole numbers For whole numbers a and b, $a \cdot b = b \cdot a$.

Associative property of multiplication of whole numbers For whole numbers a, b, and c, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Identity property of multiplication of whole numbers There is a unique whole number 1, the *multiplicative identity*, such that for every whole number a, $a \cdot 1 = a = 1 \cdot a$.

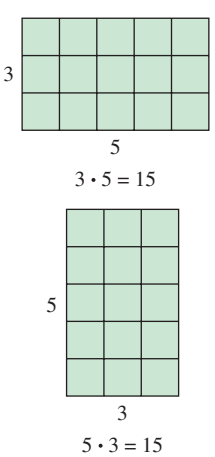


Figure 24

E-Manipulative Activity

Practice using the area model to perform multiplication can be found in the *Multiplication* activity.

The *commutative property of multiplication of whole numbers* is illustrated easily by building a 3-by-5 grid and then turning it sideways, as shown in Figure 24. We see that the number of 1×1 squares present in either case is 15; that is, $3 \cdot 5 = 15 = 5 \cdot 3$. The commutative property can be verified by showing that for sets with $n(A) = a$ and $n(B) = b$, $n(A \times B) = n(B \times A)$.

The *associative property of multiplication of whole numbers* can be illustrated as follows. Suppose $a = 3$, $b = 5$, and $c = 4$. In Figure 25(a), we see a picture of $3(5 \cdot 4)$ blocks. In Figure 25(b), we see the same blocks, this time arranged as $4(3 \cdot 5)$. By the commutative property, this can be written as $(3 \cdot 5)4$. Because both sets of blocks in Figure 25(a) and Figure 25(b) compress to the set shown in Figure 25(c), we see that $3(5 \cdot 4) = (3 \cdot 5)4$.

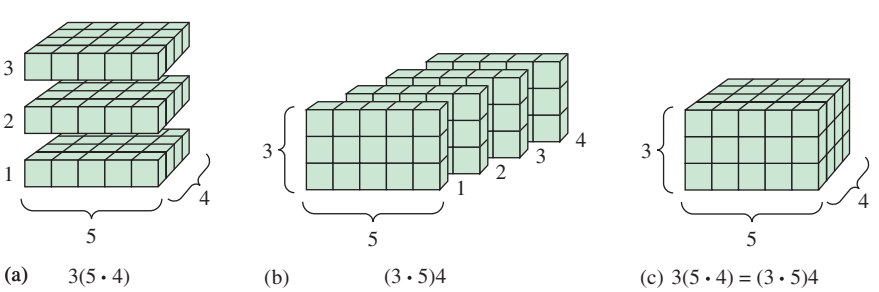


Figure 25

The associative property is useful in computations such as the following:

$$4 \cdot 60 = 4(6 \cdot 10) = (4 \cdot 6)10 = 24 \cdot 10 = 240$$

The *multiplicative identity for whole numbers* is 1. For example, $3 \cdot 1 = 1 + 1 + 1 = 3$. In general, for any whole number a,

$$a \cdot 1 = \underbrace{1 + 1 + 1 + \ldots + 1}_{a \text{ terms}} = a.$$

Thus, $a \cdot 1 = a$, and using the commutative property for multiplication, we have $a \cdot 1 = a = 1 \cdot a$. Cartesian products can also be used to show that $a \cdot 1 = a = 1 \cdot a$.

Next, consider multiplication involving 0. For example, $0 \cdot 6$ by definition means we have zero 6s or 0. Also $6 \cdot 0 = 0 + 0 + 0 + 0 + 0 + 0 = 0$. Thus, we see that multiplying 0 by 6 or 6 by 0 yields a product of 0. This is an example of the *multiplication property of* 0. This property can also be verified by using the definition of multiplication in terms of Cartesian products. In algebra, $3x$ means 3 x's or $x + x + x$. Similarly, $0 \cdot x$ means 0 x's or 0. Thus the following is true.

Theorem 3-6: Multiplication Property of 0 for Whole Numbers

Multiplication property of 0 for whole numbers For every whole number a, $a \cdot 0 = 0 = 0 \cdot a$.

IMAP Video

Watch Ally use the distributive property of multiplication over addition to find the product of two whole numbers.

The Distributive Property of Multiplication over Addition and Subtraction

Operations of multiplication and addition combine to form the basis for understanding multiplication algorithms for whole numbers. The area of the large rectangle in Figure 26 equals the sum of the areas of the two smaller rectangles and therefore $5(3 + 4) = 5 \cdot 3 + 5 \cdot 4$.

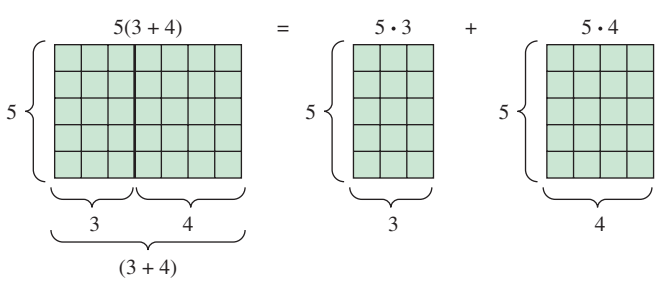


Figure 26

Properties of addition and the definition of multiplication of whole numbers also can be used to justify the above result:

$$5(3 + 4) = \underbrace{(3 + 4) + (3 + 4) + (3 + 4) + (3 + 4) + (3 + 4)}_{\text{Five terms}}$$

$$= (3 + 3 + 3 + 3 + 3) + (4 + 4 + 4 + 4 + 4)$$

Commutative and associative properties of addition

$$= 5 \cdot 3 + 5 \cdot 4$$ Definition of multiplication

This example illustrates the *distributive property of multiplication over addition* for whole numbers. A similar property of multiplication over subtraction is also true. The distributive property of multiplication over addition and the distributive property of multiplication over subtraction are stated as follows.

Theorem 3-7: Distributive Property of Multiplication over Addition for Whole Numbers

For all whole numbers a, b, and c,

$$a(b + c) = ab + ac.$$

Theorem 3-8: Distributive Property of Multiplication over Subtraction for Whole Numbers

For all whole numbers a, b, and c with $b \geq c$,

$$a(b - c) = ab - ac.$$

Theorem 3-7 can be proved using the definition of multiplication, and Theorem 3-8 can be proved using Theorem 3-7. Note that in Theorem 3-8 the condition $b \geq c$ is included to ensure that the subtraction is defined in whole numbers. Because the commutative property of multiplication of whole numbers holds, the distributive property of multiplication over addition can be rewritten as $(b + c)a = ba + ca$. The distributive property can also be generalized to any finite number of terms. For example, $a(b + c + d) = ab + ac + ad$.

The distributive property can be written as

$$ab + ac = a(b + c).$$

This is commonly referred to as *factoring*. Thus, the factors of $ab + ac$ are a and $(b + c)$.

Students find the distributive property of multiplication over addition useful when doing mental computation. For example, $13 \cdot 7 = (10 + 3)7 = 10 \cdot 7 + 3 \cdot 7 = 70 + 21 = 91$. The distributive property of multiplication over addition is important in the study of algebra and in developing algorithms for arithmetic operations. For example, it is used to combine like terms when we work with variables, as in $3x + 5x = (3 + 5)x = 8x$.

Example 7

a. Use an area model to show that $(x + y)(z + w) = xz + xw + yz + yw$.

b. Use the distributive property of multiplication over addition to justify the result in part (a).

Solution

a. Consider the rectangle in Figure 27, whose height is $x + y$ and whose length is $z + w$. The area of the entire rectangle is $(x + y)(z + w)$. If we divide the rectangle into smaller rectangles as shown, we notice that the sum of the areas of the four smaller rectangles is $xz + xw + yz + yw$. Because the area of the original rectangle equals the sum of the areas of the smaller rectangles, the statement is true.

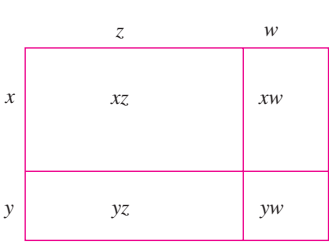


Figure 27

b. To apply the distributive property of multiplication over addition, think of $x + y$ as one number (closure property) and proceed as follows:

$$\begin{aligned}(x + y)(z + w) &= (x + y)z + (x + y)w && \text{The distributive property of multiplication over addition}\\ &= (xz + yz) + (xw + yw) && \text{The distributive property of multiplication over addition}\\ &= xz + xw + yz + yw && \text{The commutative and associative properties of addition}\end{aligned}$$

Example 8

Prove that $A = \{6n | n \in W\}$ is closed under addition.

Proof Suppose $a, b \in A$. Then $a = 6j$ and $b = 6k$, for some $j, k \in W$. By using the distributive property of multiplication over addition, $a + b = 6j + 6k = 6(j + k)$. Because $a + b \in A$, A is closed under addition.

The properties of whole-number multiplication reduce the 100 basic multiplication facts involving numbers 0–9. For example, 19 facts involve multiplication by 0, and 17 more have a factor of 1. Therefore, knowing the multiplication property of 0 and the identity property of multiplication allows students to know 36 facts. The commutative property cuts this number in half, because if students know $7 \cdot 9$, then they know $9 \cdot 7$. Knowing some multiplication facts, students can use the associative and distributive properties to figure out other products. For example, $6 \cdot 5$ can be thought of as $(5 + 1)5 = 5 \cdot 5 + 1 \cdot 5 = 25 + 5 = 30$.

Division of Whole Numbers

We use three models for division: the *set (partition)* model, the *missing-factor* model, and the *repeated-subtraction* model.

In the grade 3 *Common Core Standards*, we find that students should be able to:

Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.* (p. 23)

IMAP Video
Watch Cheyenne use the set (partition) model to answer a story problem.

E-Manipulative Activity
Practice using the set (partition) model to perform divisions can be found in the *Divisions* activity.

We introduce division using the set (partition) model.

Set (Partition) Model

Suppose we have 18 cookies and want to give an equal number of cookies to each of 3 friends: Bob, Dean, and Charlie. How many should each person receive? If we draw a picture, we can see that we can divide (or partition) the 18 cookies into 3 sets, with an equal number of cookies in each set. Figure 28 shows that each friend receives 6 cookies.

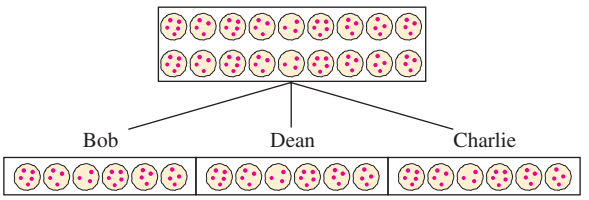


Figure 28

The answer may be symbolized as $18 \div 3 = 6$. Thus, $18 \div 3$ is the number of cookies in each of three disjoint sets whose union has 18 cookies. In this approach to division, we partition a set into a number of equivalent subsets.

Missing-Factor Model

Another strategy for dividing 18 cookies among 3 friends is to use the *missing-factor* model. If each friend receives c cookies, then the 3 friends receive $3c$, or 18, cookies. Hence, $3c = 18$. Because $3 \cdot 6 = 18$, $c = 6$. We have modeled a division problem by using multiplication. This leads to the following definition of division of whole numbers:

Definition of Division of Whole Numbers: Missing-Factor Model

For any whole numbers a and b, with $b \neq 0$, $a \div b = c$ if, and only if, c is the unique whole number such that $b \cdot c = a$.

The number a is the **dividend**, b is the **divisor**, and c is the **quotient**. Note that $a \div b$ can also be written as $\frac{a}{b}$, $b\overline{)a}$ or a/b.

Repeated-Subtraction Model

Suppose we have 18 cookies and want to package them in cookie boxes that hold 6 cookies each. How many boxes are needed? We could reason that if one box is filled, then we have $18 - 6$ (or 12) cookies left. If one more box is filled, then there are $12 - 6$ (or 6) cookies left. Finally, we place the last 6 cookies in a third box. This discussion is summarized by writing $18 - 6 - 6 - 6 = 0$. We found by repeated subtraction that $18 \div 6 = 3$. Treating division as repeated subtraction works well if there are no cookies remaining. If there are cookies remaining, a non-zero remainder arises.



Calculators illustrate the repeated subtraction operation. For example, consider $135 \div 15$. If the calculator has a constant key, [K], press [1] [5] [−] [K] [1] [3] [5] [=] . . . and then count how many times the [=] key must be pressed to make the display read 0. Calculators with a different constant feature may require a different sequence of entries. For example, on some calculators, we press [1] [3] [5] [−] [1] [5] [=] and count the number of times the [=] key is pressed to make the display read [0].

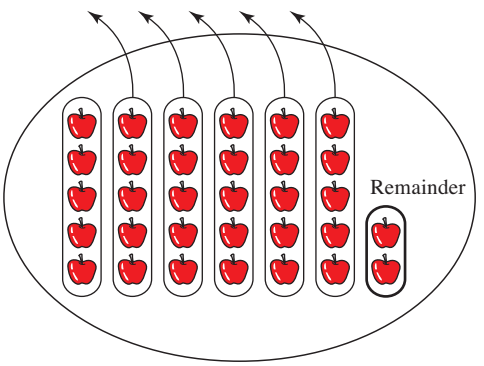
The Division Algorithm

The set of whole numbers is not closed under division. For example, to find $27 \div 5$, look for a whole number c such that $5c = 27$. Table 8 shows several products of whole numbers times 5. Since 27 is between 25 and 30, there is no whole number c such that $5c = 27$. Because no whole number c satisfies this equation, $27 \div 5$ has no meaning in the set of whole numbers, and the set of whole numbers is not closed under division.

Table 8

$5 \cdot 1$	$5 \cdot 2$	$5 \cdot 3$	$5 \cdot 4$	$5 \cdot 5$	$5 \cdot 6$
5	10	15	20	25	30

Even though the set of whole numbers is not closed under division, practical applications with whole number divisions are common. For example, if 32 apples were to be divided among 6 students, each student would receive 5 apples and 2 apples would remain. The number 2 is the **remainder**. Thus, 32 contains six 5s with a remainder of 2. Observe that the remainder is a whole number less than 6. This operation is illustrated in Figure 29. The concept illustrated is the **division algorithm**.



$32 = 6 \cdot 5 + 2$ with $0 \le 2 < 6$

Figure 29

The above is frequently written as $32 \div 6 = 5$ R2, which means $32 = 5 \cdot 6 + 2$. In general as shown below when we divide a by $b, b \neq 0$, we get a quotient q and a remainder r.

Division Algorithm

Given any whole numbers a and b with $b \neq 0$, there exist unique whole numbers q (quotient) and r (remainder) such that

$$a = bq + r \qquad \text{with } 0 \le r < b.$$

When a is divided by b and the remainder is 0, we say a is *divisible* by b, or b is a *divisor* of a, or b *divides* a. By the division algorithm, a is divisible by b if $a = bq$ for a unique whole number q. Thus, 63 is divisible by 9 because $63 = 9 \cdot 7$ and the remainder is 0. Notice that 63 is also divisible by 7 and the remainder is 0.

Example 9

If 123 is divided by a number and the remainder is 13, what are the possible divisors?

Solution If 123 is divided by b and the quotient is q, then by the division algorithm

$$123 = bq + 13 \quad \text{and} \quad b > 13.$$

Therefore, $bq = 123 - 13$, and hence $110 = bq$. Now we look for two numbers whose product is 110, where one number is greater than 13. Table 9 shows the pairs of whole numbers whose product is 110.

Table 9

1	110
2	55
5	22
10	11

The only possible values for b are 110, 55, and 22 because each is greater than 13.

Relating Multiplication and Division as Inverse Operations



In Section 3-2, subtraction and addition were related as inverse operations. In a similar way, division with remainder 0 and multiplication are related. Division is the inverse of multiplication. This relationship can be seen by looking at fact families as shown on the grade 4 student page shown on page 125. Answer the questions on the student page.

Next consider how the four operations of addition, subtraction, multiplication, and division are related on the set of whole numbers. This is shown in Figure 30. Note that addition and subtraction are inverses of each other, as are multiplication and division with remainder 0. Also multiplication can be viewed as repeated addition, and division can be accomplished using repeated subtraction when the remainder is 0.

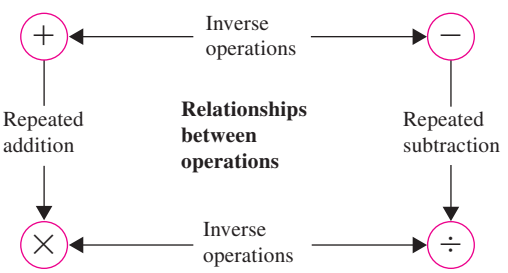


Figure 30

In Section 3-2, it was shown that the set of whole numbers is closed under addition that addition is commutative and associative, and there is an identity. On the other hand, subtraction did not have these properties. In this section, we have seen that multiplication has some of the same properties that hold for addition. Does it follow that division behaves like subtraction? Investigate this question in Now Try This 14.

NOW TRY THIS 14

a. Provide counterexamples to show that the set of whole numbers is not closed under division and that division is neither commutative nor associative.
b. Why is 1 not the identity for division?

Division by 0 and 1

Division by 0 and by 1 are frequently misunderstood by students. Try to find the values of the following three expressions.

1. $3 \div 0$ **2.** $0 \div 3$ **3.** $0 \div 0$

Consider the following explanations:

1. By definition, $3 \div 0 = c$ if, and only if, c is the unique whole number such that $0 \cdot c = 3$. Since the multiplication property of 0 states that $0 \cdot c = 0$ for any whole number c, there is no whole number c such that $0 \cdot c = 3$. Because there is no answer to the equivalent multiplication problem, $3 \div 0$ is undefined.
2. By definition, $0 \div 3 = c$ if, and only if, c is the unique whole number such that $3 \cdot c = 0$. Because $3 \cdot 0 = 0$, $c = 0$ and $0 \div 3 = 0$. Note that $c = 0$ is the only number that satisfies $3 \cdot c = 0$.
3. By definition, $0 \div 0 = c$ if, and only if, c is the unique whole number such that $0 \cdot c = 0$. Notice that for *any* c, $0 \cdot c = 0$. According to the definition of division, c must be unique. Since there is no *unique* number c such that $0 \cdot c = 0$, it follows that $0 \div 0$ is undefined.

Lesson 1-7

Common Core

4.OA.1 Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

Relating Multiplication and Division

Operations that undo each other are inverse operations. Multiplying by 3 and dividing by 3 are inverse operations.

Each trading card sheet has 3 rows with 2 pockets in each row. How many pockets are on each sheet?

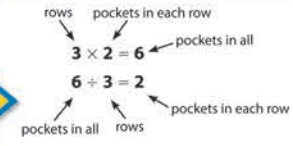


A fact family shows all the related multiplication and division facts for a set of numbers. You can use fact families to help you remember division facts.

This is the fact family for 2, 3, and 6:

$2 \times 3 = 6$ $\quad 6 \div 2 = 3$

$3 \times 2 = 6$ $\quad 6 \div 3 = 2$



Each has 6 pockets.

Guided Practice*

MATHEMATICAL PRACTICES

Do you know HOW?

In **1** and **2**, copy and complete each fact family.

1. $8 \times \square = 32$ $\quad \square \times 8 = 32$ $\quad 32 \div 8 = \square$ $\quad 32 \div \square = 8$

2. $6 \times 9 = \square$ $\quad 9 \times 6 = \square$ $\quad 54 \div 9 = \square$ $\quad 54 \div \square = 9$

In **3** and **4**, write the fact family for each set of numbers.

3. 3, 6, 18

4. 5, 7, 35

Do you UNDERSTAND?

5. Why are there four number sentences in the example above?

6. Is $2 \times 6 = 12$ part of the fact family from the example above?

7. Why is $3 + 3 = 6$ **NOT** in the fact family of 2, 3, and 6?

8. Reason If you know $7 \times 9 = 63$, what division facts do you know?

Independent Practice

Leveled Practice In **9** through **12**, copy and complete each fact family.

9. $5 \times \square = 35$ $\quad \square \times 5 = 35$ $\quad 35 \div 7 = \square$ $\quad 35 \div \square = 5$

10. $9 \times \square = 72$ $\quad \square \times 9 = 72$ $\quad 72 \div 8 = \square$ $\quad 72 \div \square = 9$

11. $3 \times \square = 18$ $\quad \square \times 3 = 18$ $\quad 18 \div 6 = \square$ $\quad 18 \div \square = 6$

12. $2 \times \square = 24$ $\quad \square \times 2 = 24$ $\quad 24 \div 12 = \square$ $\quad 24 \div \square = 12$

Animated Glossary
www.pearsonsuccessnet.com

In **13** through **20**, write a fact family for each set of numbers.

13. 7, 8, 56

14. 2, 8, 16

15. 6, 7, 42

16. 6, 6, 36

17. 3, 8, 24

18. 7, 10, 70

19. 6, 5, 30

20. 5, 8, 40

Problem Solving

MATHEMATICAL PRACTICES

21. Persevere How many years did it take to release all 50 quarters? Write a division fact you can use to find this quotient.

State Quarters	
First quarters released	1999
Number of new quarters each year	5



22. In the fact family for the numbers 5, 6, 30, which term does **NOT** describe 5 or 6?

A factor **C** product

B divisor **D** quotient

23. Josh practiced his drums two hours before dinner and three hours after dinner. How many hours did he practice in all?

A 3 hours **C** 5 hours

B 4 hours **D** 6 hours

24. Model Write the fact family that has 9 as a factor and 45 as a product.

25. Reason Why does the fact family for 64 and 8 have only two number sentences?

*For another example, see Set E on page 33.

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Division involving 0 may be summarized as follows. Let n be any non-zero whole number. Then,

1. $n \div 0$ is undefined. **2.** $0 \div n = 0$. **3.** $0 \div 0$ is undefined.

In general, division by 0 is undefined.

Recall that $n \cdot 1 = n$ for any whole number n. Thus, by the definition of division, $n \div 1 = n$. For example, $3 \div 1 = 3, 1 \div 1 = 1$, and $0 \div 1 = 0$.

Order of Operations

Difficulties involving the order of arithmetic operations sometimes arise. For example, many students treat $2 + 3 \cdot 6$ as $(2 + 3)6$, whereas others treat it as $2 + (3 \cdot 6)$. In the first case, the value is 30; in the second case, the value is 20. To avoid confusion, mathematicians agree that when no parentheses are present, multiplications and divisions are performed *before* additions and subtractions. The multiplications and divisions are performed in the order they occur from left to right, and then the additions and subtractions are performed in the order they occur from left to right. Thus, $2 + 3 \cdot 6 = 2 + 18 = 20$. If parentheses are present, the computations in parentheses are done first. This order of operations is not built into calculators that display an incorrect answer of 30.

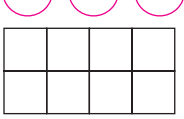
The computation $8 - 9 \div 3 \cdot 2 + 3$ is performed as follows.

$$\begin{aligned} 8 - 9 \div 3 \cdot 2 + 3 &= 8 - 3 \cdot 2 + 3 \\ &= 8 - 6 + 3 \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

Assessment 3-3A

1. Show that $3 \cdot 4 = 12$ using each of the following models.
- **a.** Repeated-addition number line
- **b.** Rectangular array
- **c.** Area
- **d.** Cartesian product

2. If $A = \{a, b\}$ and $B = \{x, y, z\}$, find each of the following.
- **a.** $A \times B$
- **b.** $n(A \times B)$
- **c.** Write a multiplication equation using numerals related to the answers in parts (a) and (b).

3. What multiplication is suggested by the following models?
- **a.** XX XX XX XX
- **b.** 

4. For each of the following find, if possible, a whole number that makes the equation true.
- **a.** $3 \cdot \square = 15$ **b.** $18 = 6 + 3 \cdot \square$
- **c.** $\square \cdot (5 + 6) = \square \cdot 5 + \square \cdot 6$

5. Determine whether the following sets are closed under multiplication.
- **a.** $\{0, 1\}$ **b.** $\{2, 4, 6, 8, 10, \ldots\}$
- **c.** $\{1, 4, 7, 10, 13, \ldots\}$

6. **a.** If 5 is removed from the set of whole numbers, is the set closed with respect to addition? Explain.
- **b.** If 5 is removed from the set of whole numbers, is the set closed with respect to multiplication? Explain.

7. Rename each of the following using the distributive property of multiplication over addition so there are no parentheses in the final answer:
- **a.** $(a + b)(c + d)$
- **b.** $\square(\Delta + \bigcirc)$
- **c.** $a(b + c) - ac$

8. Place parentheses, if needed, to make each of the following equations true.
- **a.** $5 + 6 \cdot 3 = 33$
- **b.** $8 + 7 - 3 = 12$
- **c.** $6 + 8 - 2 \div 2 = 13$
- **d.** $9 + 6 \div 3 = 5$

9. Using the distributive property of multiplication over addition, we can factor as in $x^2 + xy = x(x + y)$. Use the distributive property and other multiplication properties to factor each of the following expressions.
- **a.** $xy + y^2$ **b.** $xy + x$
- **c.** $a^2b + ab^2$

10. For each of the following equations, find whole numbers to make the statement true, if possible.
- **a.** $18 \div 3 = \square$ **b.** $\square \div 76 = 0$
- **c.** $28 \div \square = 7$

11. A sporting goods store has designs for 6 shirts, 4 pairs of pants, and 3 vests. How many different shirt-pants-vest outfits are possible?

12. Which property is illustrated in each of the following equations.
- **a.** $6(5 \cdot 4) = (6 \cdot 5)4$
- **b.** $6(5 \cdot 4) = 6(4 \cdot 5)$
- **c.** $6(5 \cdot 4) = (5 \cdot 4)6$
- **d.** $1 \cdot (5 \cdot 4) = 5 \cdot 4$
- **e.** $(3 + 4) \cdot 0 = 0$
- **f.** $(3 + 4)(5 + 6) = (3 + 4)5 + (3 + 4)6$

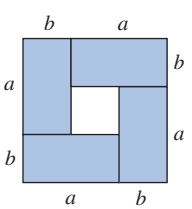
13. Students are overheard making the following statements. What properties justify their statements?
- **a.** I know that $9 \cdot 7$ is either 63 or 69, and I know they can't both be right.
- **b.** I know that $9 \cdot 0$ is 0 because I know that any number times 0 is 0.
- **c.** Any number times 1 is the same as the number we started with, so $9 \cdot 1$ is 9.

14. The product $6 \cdot 14$ can be found by thinking of the problem as $6(10 + 4) = 6 \cdot 10 + 6 \cdot 4 = 60 + 24 = 84$.
- **a.** What properties are being used?
- **b.** Use this technique to mentally compute $32 \cdot 12$.

15. Use the distributive property of multiplication over subtraction to compute each of the following expressions.
- **a.** $9(10 - 2)$
- **b.** $20(8 - 3)$

16. Show that $(a + b)^2 = a^2 + 2ab + b^2$ using
- **a.** the distributive property of multiplication over addition and other properties.
- **b.** an area model.

17. If a and b are whole numbers with $a > b$, use the rectangles in the figure to explain why $(a + b)^2 - (a - b)^2 = 4ab$.



18. Use the property $(a + b)^2 = a^2 + 2ab + b^2$ to compute the following expressions.
- **a.** 51^2
- **b.** 102^2

19. In each of the following equations, show that the left side of the equation is equal to the right side and give a reason for every step.
- **a.** $(ab)c = (ca)b$
- **b.** $(a + b)c = c(b + a)$

20. Factor each of the following expressions.
- **a.** $xy - y^2$
- **b.** $47 \cdot 101 - 47$
- **c.** $ab^2 - ba^2$

21. Rewrite each of the following division problems as a multiplication problem.
- **a.** $40 \div 8 = 5$
- **b.** $326 \div 2 = x$

22. If $108/a = b$, then find $108/b$.

23. Write the complete fact family for $72/8 = 9$.

24. Think of a number. Multiply it by 5. Add 5. Divide by 5 and then subtract 1. How does the result compare with your original number? Will this work all the time? Justify your answer.

25. Show that, in general, each of the following is false if a, b, and c are whole numbers.
- **a.** $(a \div b) \div c = a \div (b \div c)$
- **b.** $a \div (b + c) = (a \div b) + (a \div c)$

26. Suppose all of the operations result in whole numbers. Explain why $(a + b) \div c = (a \div c) + (b \div c)$.

27. Find the solution for each of the following equations.
- **a.** $5x + 2 = 22$
- **b.** $3x + 7 = x + 13$
- **c.** $3(x + 4) = 18$
- **d.** $(x - 5) \div 10 = 9$

28. A new model of car is available in 4 exterior colors and 3 interior colors. Use a tree diagram and specific colors to show how many color schemes are possible for the car.

29. Is it possible to find a whole number less than 100 that when divided by 10 has remainder 4 and when divided by 47 has remainder 17?

30. Students were divided into 10 teams with 12 on each team. Later, the same students were divided into teams with 8 on each team. How many teams were there then?

31. In each of the following, tell what computation must be done last:
- **a.** $5(16 - 7) - 18$
- **b.** $54/(10 - 5 + 4)$
- **c.** $(14 - 3) + (24 \cdot 2)$
- **d.** $21{,}045/345 + 8$

32. Find infinitely many whole numbers that leave remainder 1 upon division by 4.

33. The operation $\odot$ is defined on the set $S = \{a, b, c\}$, as shown in the following table. For example, $a \odot b = b$ and $b \odot a = b$.

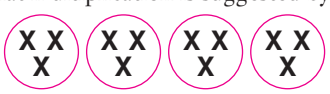
$\odot$	***a***	***b***	***c***
a	a	b	c
b	b	c	a
c	c	a	b

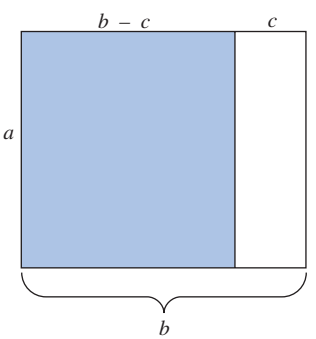
- **a.** Is S closed with respect to $\odot$?
- **b.** Is $\odot$ commutative on S?
- **c.** Is there an identity for $\odot$ on S? If yes, what is it?
- **d.** Try several examples to investigate the associative property for $\odot$ on S.

34. If $a, b \in W$, is the operation # commutative if $a \# b = a + b + 4$? Explain why.

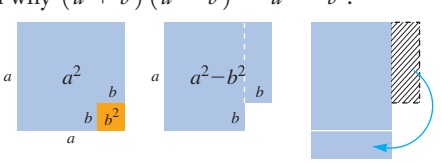
35. At a certain concert, the audience was allowed to enter in a certain way. The first time a bell rang only 1 person was allowed to enter and choose his or her seat. The second time the bell rang 3 people were allowed to enter. On each successive ring the group that enters has two more people than the previous group.
- **a.** How many people have entered after the 25th ring of the bell?
- **b.** How many have entered after the nth ring?
- **c.** After how many rings will there be at least 1000 people?

Assessment 3-3B

1. Show that $5 \cdot 3 = 15$ using each of the following models.
 a. Repeated-addition number line
 b. Rectangular array
 c. Area
 d. Cartesian product model
2. If $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$, find each of the following.
 a. $A \times B$ b. $n(A \times B)$
 c. Write a multiplication equation involving numerals related to the answers in parts (a) and (b).
3. What multiplication is suggested by the following models?
 a. 
 b. 
4. For each of the following, find, if possible, a whole number that makes the equation true.
 a. $8 \cdot \square = 24$ b. $28 = 4 + 6 \cdot \square$
 c. $\square \cdot (8 + 6) = \square \cdot 8 + \square \cdot 6$
5. Determine if the following sets are closed under multiplication.
 a. $\{1, 2\}$
 b. $\{2k + 1 \mid k \in W\}$
 c. $\{2^{k+1} \mid k \in W\}$
6. a. If 2 is removed from the set of whole numbers, is the set closed with respect to addition? Explain.
 b. If 1 is removed from the set of whole numbers, is the set closed with respect to multiplication? Explain.
7. Rename each of the following using the distributive property of multiplication over addition so there are no parentheses in the final answer. Use other properties to simplify when possible.
 a. $3(x + y + 5)$
 b. $(x + y)(x + y + z)$
 c. $x(y + 1) - x$
8. Place parentheses, if needed, to make each of the following equations true:
 a. $4 + 3 \cdot 2 = 14$
 b. $9 \div 3 + 1 = 4$
 c. $5 + 4 + 9 \div 3 = 6$
 d. $3 + 6 - 2 \div 1 = 7$
9. Using the distributive property of multiplication over addition, we can factor as in $x^2 + xy = x(x + y)$. Use the distributive property and other multiplication properties to factor each of the following.
 a. $47 \cdot 99 + 47$
 b. $(x + 1)y + (x + 1)$
 c. $x^2y + zx^3$
10. For each of the following equations, find whole numbers to make the statement true, if possible.
 a. $27 \div 9 = \square$ b. $\square \div 52 = 1$
 c. $13 \div \square = 13$
11. A new car comes in 5 exterior colors and 3 interior colors. How many different looking cars are possible?
12. Which property of whole numbers is illustrated in each of the following equations?
 a. $(5 \cdot 4)0 = 0$
 b. $7(3 \cdot 4) = 7(4 \cdot 3)$
 c. $7(3 \cdot 4) = (3 \cdot 4)7$
 d. $(3 + 4)1 = 3 + 4$
 e. $(3 + 4)5 = 3 \cdot 5 + 4 \cdot 5$
 f. $(1 + 2)(3 + 4) = (1 + 2)3 + (1 + 2)4$
13. Students are overheard making the following statements. What properties justify their statements?
 a. I know if I remember what $7 \cdot 9$ is, then I also know what $9 \cdot 7$ is.
 b. To find $9 \cdot 6$, I just remember that $9 \cdot 5$ is 45, so $9 \cdot 6$ is just 9 more than 45, or 54.
14. The product $5 \cdot 24$ can be found by thinking of the computation as $5(20 + 4) = 5 \cdot 20 + 5 \cdot 4 = 100 + 20 = 120$.
 a. What property is being used?
 b. Use this technique to mentally compute $8 \cdot 34$.
15. Use the distributive property of multiplication over subtraction to compute each of the following.
 a. $15(10 - 2)$ b. $30(9 - 2)$
16. Show that if $b > c$, then $a(b - c) = ab - ac$ using an area model suggested by the given figure (express the shaded area in two different ways).



17. Use an area model suggested by the following figure to explain why $(a + b)(a - b) = a^2 - b^2$.



18. Use the formula $(a + b)(a - b) = a^2 - b^2$ to compute the following.
 a. $19 \cdot 21$ b. $25 \cdot 15$
 c. $99 \cdot 101$ d. $101^2 - 99^2$
19. Show that the left-hand side of the equation is equal to the right-hand side and give a reason for every step.
 a. $(ab)c = b(ac)$ b. $a(b + c) = ac + ab$
20. Factor each of the following:
 a. $xy - y$
 b. $(x + 1)y - (x + 1)$
 c. $a^2b^3 - ab^2$
21. Rewrite each of the following division problems as a multiplication problem.
 a. $48 \div x = 16$
 b. $x \div 5 = 17$
22. If $\frac{64}{a} = b$, then find $\frac{64}{b}$.
23. Write the complete fact family for $30/6 = 5$.

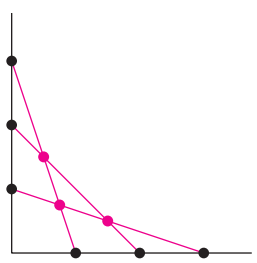
24. Think of a number. Multiply it by 2. Add 2. Divide by 2. Subtract 1. **(a)** How does the result compare with your original number? **(b)** Will this work all the time? Explain your answer.

25. Show that, in general, each of the following is false if a and b are whole numbers.
a. $a \div b = b \div a$ **b.** $a - b = b - a$

26. Suppose all operations result in whole numbers. Explain why $(a - b) \div c = (a \div c) - (b \div c)$.

27. Find the solution for each of the following equations.
a. $5x + 8 = 28$ **b.** $5x + 6 = x + 14$
c. $5(x + 3) = 35$ **d.** $(x - 6) \div 3 = 1$

28. String art is formed by connecting evenly spaced nails on the vertical and horizontal axes by segments of string. Connect the nail farthest from the origin on the vertical axis with the nail closest to the origin on the horizontal axis. Continue until all nails are connected, as shown in the figure that follows. How many intersection points (red dots) are created with 10 nails on each axis?



3 nails per axis
3 intersections

29. Jonah has a large collection of marbles. He notices that if he borrows 5 marbles from a friend, he can arrange the marbles in rows of 13 each. What is the remainder when he divides his original number of marbles by 13?

30. Students were divided into eight teams with nine on each team. Later, the same students were divided into teams with six on each team. How many teams were there then?

31. In each of the following, tell what computation must be done last.
a. $5 \cdot 6 - 3 \cdot 4 + 2$
b. $19 - 3 \cdot 4 + 9 \div 3$
c. $15 - 6 \div 2 \cdot 4$
d. $5 + (8 - 2)3$

32. Find infinitely many whole numbers that leave remainder 3 upon division by 5.

33. The operation $\odot$ is defined on the set $S = \{a, b, c\}$, as shown in the following table. For example, $a \odot b = b$ and $b \odot a = b$.

$\odot$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

a. Is S closed with respect to $\odot$?
b. Is $\odot$ commutative on S?
c. Is there an identity for $\odot$ on S? If yes, what is it?
d. Try several examples to investigate the associative property for $\odot$ on S.

34. If $a, b \in W$, is the operation # commutative if $a \# b = 0$, if either a or b or both are odd and $a \# b = 1$, if a and b are both even?

35. When a marching band was placed in rows of 5, one member remained. When the members were placed in rows of 6, there was still one member remaining. However, when they were placed in rows of 7, nobody remained. What is the least number of members that could have been in the band?

Mathematical Connections 3-3

Reasoning

1. Why is an odd number times an odd number always odd?

2. Can 0 be the identity for multiplication? Explain why or why not.

3. Suppose you forgot the product of $9 \cdot 7$. Give several ways that you could find the product using different multiplication facts and properties.

4. Is $x \div x$ always equal to 1? Explain your answer.

5. Is $x \cdot x$ ever equal to x? Explain your answer.

6. Describe all pairs of whole numbers whose sum and product are the same.

7. In the *Common Core Standards* quote at the beginning of this section it states that students should learn the relationship between multiplication and division. Explain this relationship.

8. Explain how the distributive property of multiplication over addition would be helpful to mentally perform the following computation.

$$37 \cdot 52 + 37 \cdot 48$$

Open-Ended

9. Create word problems using each model for multiplication and division.

10. Describe a real-life situation that could be represented by the expression $3 + 2 \cdot 6$.

Cooperative Learning

11. Multiplication facts that most children have memorized can be stated in the following table that is partially filled.

×	1	2	3	4	5	6	7	8	9
1									
2									
3									
4				16					
5							35		
6									
7									
8									72
9									81

a. Complete the table of multiplication facts. Find as many patterns as you can. List all the patterns that your group discovered and explain why some of those patterns occur in the table.
b. How can the multiplication table be used to solve division problems?
c. Consider the odd number 35 shown in the multiplication table. Consider all the numbers that surround it. Note that they are all even. Does this happen for all odd numbers in the table? Explain why or why not.

12. Enter a natural number less than 20 on a calculator. If the number is even, divide it by 2; if it is odd, multiply it by 3 and add 1. Next, follow the same directions for the number on the display. Keep repeating the process.
a. Will the display eventually reach 1?
b. Which number less than 20 takes the most steps before reaching 1?
c. Do even or odd numbers reach 1 more quickly?
d. Investigate what happens with numbers greater than 20.

Connecting Mathematics to the Classroom

13. Suppose a student argued that $0 \div 0 = 1$ because every number divided by itself is 1. How would you help that student?
14. Sue claims the following is true by the distributive law, where a and b are whole numbers.

$$3(ab) = (3a)(3b)$$

How might you help her?
15. A student claims that for all whole numbers $(ab) \div b = a$. How do you respond?
16. A student says that 1 is the identity for division because $a \div 1 = a$ for all whole numbers a. How do you respond?
17. Amy claims that no finite set of whole numbers is closed under multiplication. How do you respond?
18. Linda performed the following calculations.

$2 + 3 \cdot 4 = 2 + 12 = 14$ and $2(3 + 4) = 2 \cdot 7 = 14$.

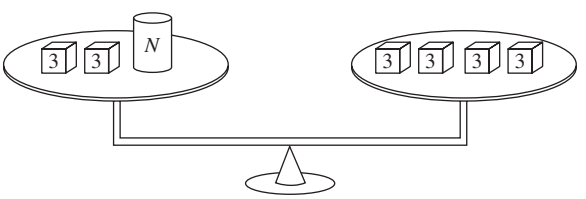
From this she concluded that if $a, b, c \in W$, then $a + bc = a(b + c)$. How do you respond?

Review Problems

19. Give an infinite set of even numbers that is not closed under addition.
20. Is the operation of subtraction for whole numbers commutative? If not, give a counterexample.
21. In base two, does the number "two" exist? Explain.
22. Write 81 in base three.

National Assessments

National Assessment of Educational Progress (NAEP) Question



The weights on the scale above are balanced. Each cube weighs 3 pounds. The cylinder weighs N pounds. Which number sentence best describes this situation?
A. $6 + N = 12$
B. $6 + N = 4$
C. $2 + N = 12$
D. $2 + N = 4$
NAEP, Grade 4, 2007

3-4 Addition and Subtraction Algorithms, Mental Computation, and Estimation

3-4 Objectives

Students will be able to understand and explain

- Models to develop algorithms for addition and subtraction.
- Addition and subtraction algorithms including the standard algorithms and how to use them to solve problems.

(*continued on next page*)

CCSS **The grade 2 *Common Core Standards* state that students at this level should be able to:**

Use place value understanding and properties of operations to add and subtract.

Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

Add up to four two-digit numbers using strategies based on place value and properties of operations.

Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900. (p. 19)

- Number bases other than ten to provide insight into base-ten algorithms.
- Mental addition and subtraction computational skills and estimation techniques to check reasonableness of answers.

Section 3-2 introduced the operations of addition and subtraction of whole numbers, and now it is time to focus on *computational fluency*—having and using efficient and accurate methods for computing. Standard algorithms are one means to achieve this fluency. An **algorithm** is a step-by-step procedure for solving a problem or performing a computation. The word is derived from the name of the Persian mathematician, astronomer, and geographer al-Khwarizmi (780–850 CE).

This section focuses on developing and understanding algorithms involving addition and subtraction along with mental mathematics and estimation associated with those operations.

Addition Algorithms

To help children understand algorithms, they should explore addition using manipulatives. If children can touch and move around items such as chips, bean sticks, and an abacus or use base-ten blocks, they often proceed naturally on their own to the creation of algorithms for addition. In the following, we use base-ten blocks to illustrate the development of the standard algorithm for whole-number addition. The grade 4 *Common Core Standards* state that students should "fluently add and subtract multi-digit whole numbers using the standard algorithm." (p. 29)

To add 14 + 23, we start with a concrete model as in Figure 31(a), move to the expanded algorithm in Figure 31(b), and then to the standard algorithm in Figure 31(c).

IMAP Video

Watch Andrew discuss and algorithm for adding whole numbers he invented himself.

E-Manipulative Activity

Practice using base-ten blocks to perform additions can be found in the *Adding Blocks* activity.

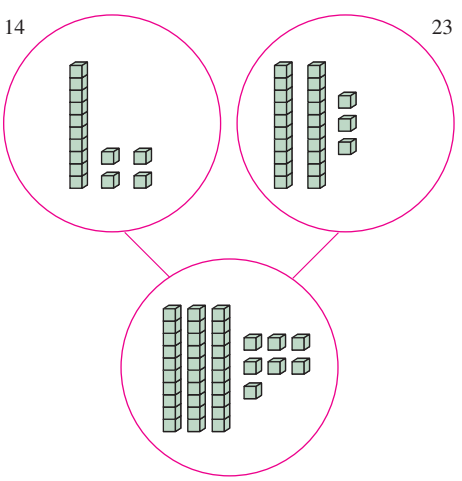


Figure 31

A more formal justification for this addition, not usually presented at the elementary school level, is the following:

$$
\begin{aligned}
14 + 23 &= (1 \cdot 10 + 4) + (2 \cdot 10 + 3) && \text{Expanded form} \\
&= (1 \cdot 10 + 2 \cdot 10) + (4 + 3) && \text{Commutative and associative properties of addition} \\
&= (1 + 2)10 + (4 + 3) && \text{Distributive property of multiplication over addition} \\
&= 3 \cdot 10 + 7 && \text{Single-digit addition facts} \\
&= 37 && \text{Place value}
\end{aligned}
$$

The student page shown on page 132 shows an example of adding two-digit numbers with regrouping using base-ten blocks. Kim's way leads to the expanded algorithm, and Henry's way leads to the standard algorithm. On the student page, students are asked to estimate their answers before performing an algorithm. This is good practice and leads to the development of number sense and also makes students consider whether their answers are reasonable.

School Book Page

Adding Two-Digit Numbers

Lesson 3-1

Key Idea
You can break apart numbers, using place value, to add.

Vocabulary
- regroup

TEST TALK

Think It Through
- I should **estimate** so I will know if my answer is reasonable.
- I can **use place-value blocks** to show addition.



Adding Two-Digit Numbers

WARM UP
Use mental math.
1. 48 + 20 2. 63 + 11
3. 71 + 8 4. 53 + 5

LEARN

How do you add two-digit numbers?

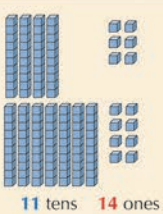
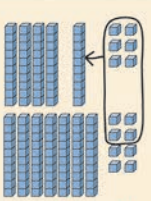
Example

Cal counted 46 ladybugs on a log and 78 more on some bushes. How many ladybugs did he count all together?

Find 46 + 78.

Estimate: 46 rounds to 50. 78 rounds to 80.

50 + 80 = 130, so the answer should be about 130.

What You **Think**		What You **Write**
Kim's Way		
• Add the ones. 6 + 8 = 14 ones • Add the tens. 4 tens + 7 tens = 11 tens = 110 • Find the sum.	 11 tens 14 ones	46 + 78 14 110 124
Henry's Way		
• Add the ones. 6 + 8 = 14 ones • Regroup 14 ones into 1 ten 4 ones. • Add the tens. 1 ten + 4 tens + 7 tens = 12 tens • Find the sum.	 14 ones = 1 ten 4 ones	1 46 + 78 124

Cal counted 124 ladybugs all together.

Talk About It

1. Why did Henry write a small 1 above the 4 in the tens place?
2. Why should you estimate when adding two-digit numbers?

Source: p. 126; From SCOTT FORESMAN-ADDISON WESLEY MATHEMATICS. Copyright © 2008 Pearson Education, Inc., or its affiliates. Used by permission. All Rights Reserved.

Once children have mastered concrete models, they should be ready to use the expanded and standard algorithms. Figure 32 shows the computation 37 + 28 using both algorithms. In Figure 32(b), notice that when there were more than 10 ones, we regrouped 10 ones as a ten and then added the tens. The words *regroup* or *trade* are now commonly used in the elementary classroom to describe what was called *carrying* in the past. The reason for this is that *regroup* and *trade* match more closely the physical actions done with manipulatives.

$$\begin{array}{r} 37 \\ +\ 28 \\ \hline 15 \\ +\ 50 \\ \hline 65 \end{array} \quad \begin{array}{l} \text{(Add ones)} \\ \text{(Add tens)} \end{array}$$

(a) Expanded algorithm

$$\begin{array}{r} {\scriptstyle 1} \\ 37 \\ +\ 28 \\ \hline 65 \end{array} \quad \text{(Add the ones, regroup, and add the tens.)}$$

(b) Standard algorithm

Figure 32

Next we add two three-digit numbers involving two regroupings. Figure 33 shows how to add 186 + 127 using base-ten blocks and how this concrete model relates to the standard algorithm.

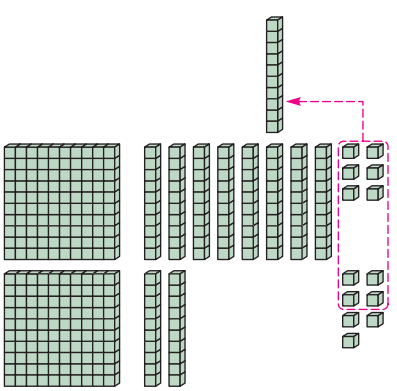
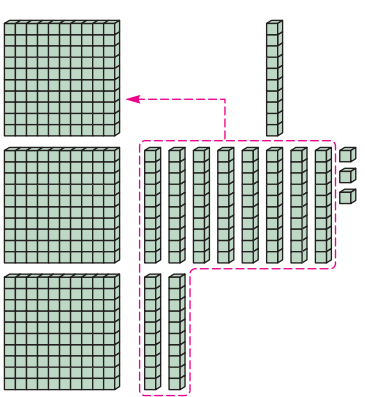
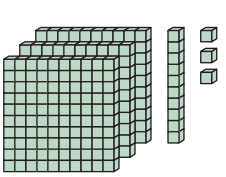
	Concrete Mode	**Standard Algorithm**
1. Add the ones and regroup. 6 ones + 7 ones = 13 ones 13 ones = 1 ten + 3 ones		$\begin{array}{r} {\scriptstyle 1} \\ 186 \\ +\ 127 \\ \hline 3 \end{array}$
2. Add the tens and regroup. 1 ten + 8 tens + 2 tens = 11 tens 11 tens = 1 hundred + 1 ten		$\begin{array}{r} {\scriptstyle 1\,1} \\ 186 \\ +\ 127 \\ \hline 13 \end{array}$
3. Add the hundreds. 1 hundred + 1 hundred + 1 hundred = 3 hundreds 186 + 127 = 313		$\begin{array}{r} {\scriptstyle 1\,1} \\ 186 \\ +\ 127 \\ \hline 313 \end{array}$

Figure 33

Addition of whole numbers using blocks has a natural carryover to the expanded form and trading used earlier. Consider the following addition:

$$\begin{array}{r} 376 \\ 459 \\ +\ 8716 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 3\cdot10^2 + 7\cdot10 + 6 \\ 4\cdot10^2 + 5\cdot10 + 9 \\ +\ 8\cdot10^3 + 7\cdot10^2 + 1\cdot10 + 6 \\ \hline 8\cdot10^3 + 14\cdot10^2 + 13\cdot10 + 21 \end{array}$$

To complete the addition, trading is used.

Consider an analogous algebra problem of adding polynomials:

$$(3x^2 + 7x + 6) + (4x^2 + 5x + 9) + (8x^3 + 7x^2 + x + 6) \quad \text{or} \quad \begin{array}{r} 3x^2 + 7x + 6 \\ 4x^2 + 5x + 9 \\ +\ 8x^3 + 7x^2 + x + 6 \\ \hline 8x^3 + 14x^2 + 13x + 21 \end{array}$$

Note that if $x = 10$, the addition is the same as given earlier. Knowledge of place value in addition problems aids in algebraic thinking. Next we explore several algorithms that have been used throughout history.

IMAP Video

Watch Gilberto use the left-to-right Algorithm to answer a story problem.

Left-to-Right Algorithm for Addition (Partial Sums Algorithm)

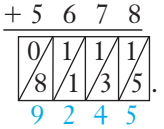
Because children learn to read from left to right, it might seem natural that they add from left to right. When working with base-ten blocks, many children in fact do combine the larger pieces first and then move to combining the smaller pieces. This method has the advantage of emphasizing place value. An example of the use of left-to-right algorithm follows:

$$\begin{array}{rcl} & & \begin{array}{r} 568 \\ +\ 757 \\ \hline \end{array} \\ (500 + 700) & \rightarrow & 1200 \\ (60 + 50) & \rightarrow & 110 \\ (8 + 7) & \rightarrow & 15 \\ & & \hline 1325 \end{array} \longrightarrow \begin{array}{r} 568 \\ +\ 757 \\ \hline 1215 \\ {\scriptstyle 3\ 2} \end{array} \rightarrow 1325$$

Lattice Algorithm for Addition

We demonstrate the lattice algorithm using an addition involving two four-digit numbers. For example,

$$\begin{array}{r} 3\ 5\ 6\ 7 \\ +\ 5\ 6\ 7\ 8 \\ \hline \end{array}$$



To use the algorithm, add the single-digit numbers by place value on top to the single-digit numbers on the bottom and record the results in a lattice. Then add the sums along the diagonals, as shown. This is very similar to the expanded algorithm introduced earlier. This method is useful with some students because it does not matter if you start left, right, or in between; the regrouping is handled by placement and diagonals. For example in $\boxed{\begin{matrix}1\\3\end{matrix}}\boxed{\begin{matrix}1\\5\end{matrix}}$, the $\boxed{\begin{matrix}1\\5\end{matrix}}$ separates 15 units into 1 ten and 5 units, and $\boxed{\begin{matrix}1\\3\end{matrix}}$ separates 130 units into 1 hundred and 3 tens, allowing addition along diagonals.

Scratch Algorithm for Addition

The *scratch algorithm* for addition is often referred to as a *low-stress algorithm* because it allows students to perform complicated additions by doing a series of additions that involve only two single digits. An example follows:

1.
$$\begin{array}{r} 87 \\ 6\not{5}_{2} \\ +\ 49 \\ \hline \end{array}$$
Add the numbers in the units place starting at the top. When the sum is 10 or more, record this sum by scratching a line through the last digit added and writing the number of units next to the scratched digit. For example, since $7 + 5 = 12$, the "scratch" represents 10 and the 2 represents the units.

2.
$$\begin{array}{r} 87 \\ 6\not{5}_{2} \\ +\ 4\not{9}_{1} \\ \hline \end{array}$$
Continue adding the units, including any new digits written. When the addition again results in a sum of 10 or more, as with $2 + 9 = 11$, repeat the process described in step (1).

3.
$$\begin{array}{r} {}^{2}87 \\ 6\not{5}_{2} \\ +\ 4\not{9}_{1} \\ \hline 1 \end{array}$$
When the first column of additions is completed, write the number of units, 1, below the addition line in the proper place value position. Count the number of scratches, 2, and add this number to the second column.

4.
$$\begin{array}{r} {}^{2}\not{8}_{0}7 \\ 6\ \not{5}_{2} \\ +\ \not{4}_{0}\not{9}_{1} \\ \hline 2\ 0\ 1 \end{array}$$
Repeat the procedure for each successive column until the last column with non-zero values. At this stage, sum the scratches in that column and place the number to the left of the current value.

Subtraction Algorithms

As with addition, we use base-ten blocks to provide a concrete model for subtraction. Consider how the base-ten blocks are used to perform the subtraction $243 - 61$. First we represent 243 with 2 flats, 4 longs, and 3 units, as shown in Figure 34.

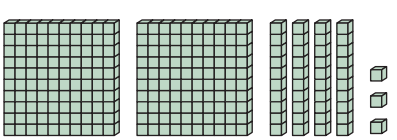


Figure 34

To subtract 61 from 243, we try to remove 6 longs and 1 unit from the blocks in Figure 34. We can remove 1 unit, as in Figure 35.

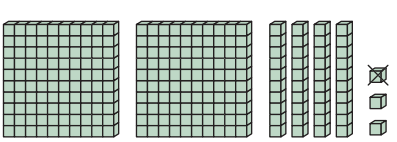


Figure 35

E-Manipulative Activity

Practice using base-ten blocks to perform subtractions can be found in the *Subtracting Blocks* activity.

To remove 6 longs from Figure 35, we have to trade 1 flat for 10 longs, as shown in Figure 36.

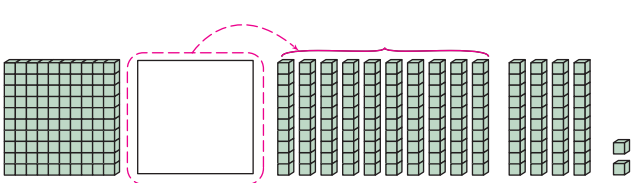


Figure 36

Now we can remove, or "take away," 6 longs, leaving 1 flat, 8 longs, and 2 units, or 182, as shown in Figure 37.

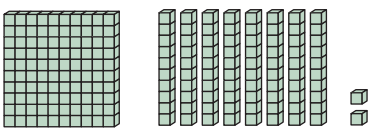


Figure 37



Student work with base-ten blocks along with discussions and recorded work lead to the development of the standard algorithm as seen on the student page shown on page 137.

Subtraction of whole numbers using blocks carries over to the expanded form and trading. For example, consider the following subtraction problem done earlier with blocks:

$$\begin{array}{r} 243 \\ -61 \end{array} \longrightarrow \begin{array}{r} 2\cdot 10^2 + 4\cdot 10 + 3 \\ -(6\cdot 10 + 1) \end{array} \longrightarrow \begin{array}{r} 1\cdot 10^2 + \quad 14\cdot 10 + 3 \\ -\quad (6\cdot 10 + 1) \end{array}$$

$$1\cdot 10^2 + (14 - 6)10 + (3 - 1)$$

$$= 1\cdot 10^2 + 8\cdot 10 + 2$$

$$= 182$$

Note that to complete the subtraction, trading was used.

Equal-Additions Algorithm

The equal-additions algorithm for subtraction is based on the fact that the difference between two numbers does not change if we add the same amount to both numbers. For example, $93 - 27 = (93 + 3) - (27 + 3)$. Thus, the difference can be computed as $96 - 30 = 66$. Using this approach, subtraction could be performed as follows:

$$\begin{array}{r} 255 \\ -163 \end{array} \rightarrow \begin{array}{r} 255 + 7 \\ -(163 + 7) \end{array} \rightarrow \begin{array}{r} 262 \\ -170 \end{array} \rightarrow \begin{array}{r} 262 + 30 \\ -(170 + 30) \end{array} \rightarrow \begin{array}{r} 292 \\ -200 \\ \hline 92 \end{array}$$

NOW TRY THIS 15

Jessica claims that a method similar to *equal additions* for subtraction also works for addition. She says that in an addition problem, "you may add the same amount to one number as you subtract from the other. For example, $68 + 29 = (68 - 1) + (29 + 1)$. Thus, the sum can be computed as $67 + 30 = 97$ or as $(68 + 2) + (29 - 2) = 70 + 27 = 97$." **(i)** Explain why this method is valid and **(ii)** use it to compute $97 + 69$.

Understanding Addition and Subtraction in Bases Other Than Ten

A look at computation in other bases provides insight into computation in base ten. Use of multibase blocks helps in building an addition table for different bases. Table 10 is a base-five addition table.

School Book Page

Subtracting 3-Digit Numbers

Lesson 3-8

Common Core

3.NBT.2 Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction. Also 3.OA.8, 3.NBT.1

Subtracting 3-Digit Numbers

Hands-On place-value blocks

How can you use subtraction to solve problems?

Mike and Linda are playing a game. How many more points does Mike have than Linda?

Find 528 − 341.

Estimate: 530 − 340 = 190



Subtract the ones.

8 ones > 1 one
You do not regroup.

8 ones − 1 one = 7 ones

$$\begin{array}{r} 528 \\ -\ 341 \\ \hline 7 \end{array}$$

Subtract the tens.

Since 2 tens < 4 tens, regroup 1 hundred into 10 tens.

12 tens − 4 tens = 8 tens

$$\begin{array}{r} {}^{4}\cancel{5}\,{}^{12}\cancel{2}8 \\ -\ 341 \\ \hline 87 \end{array}$$

Subtract the hundreds.

4 hundreds − 3 hundreds = 1 hundred

$$\begin{array}{r} {}^{4}\cancel{5}\,{}^{12}\cancel{2}8 \\ -\ 341 \\ \hline 187 \end{array}$$

Mike has 187 more points.

187 is close to the estimate of 190. The answer is reasonable.

Another Example How do you subtract with two regroupings?

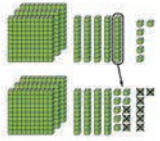
Find 356 − 189.

Estimate: 400 − 200 = 200

Step 1

Subtract the ones. Regroup if needed.

6 ones < 9 ones. So, regroup 1 ten into 10 ones.

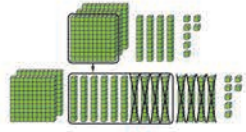


$$\begin{array}{r} {}^{4}3\,{}^{16}\cancel{5}\cancel{6} \\ -\ 189 \\ \hline 7 \end{array}$$

Step 2

Subtract the tens. Regroup if needed.

4 tens < 8 tens. So, regroup 1 hundred into 10 tens.



$$\begin{array}{r} {}^{2}\cancel{3}\,{}^{14}\cancel{5}\,{}^{16}6 \\ -\ 189 \\ \hline 67 \end{array}$$

Step 3

Subtract the hundreds.



$$\begin{array}{r} {}^{2}\cancel{3}\,{}^{14}\cancel{5}\,{}^{16}6 \\ -\ 189 \\ \hline 167 \end{array}$$

The answer 167 is reasonable because it is close to the estimate.

Explain It

1. Why do you need to regroup both a ten and a hundred?
2. **Reason** How is 3 hundreds 5 tens 6 ones the same as 3 hundreds 4 tens 16 ones? How is 3 hundreds 4 tens 16 ones the same as 2 hundreds 14 tens 16 ones?

Guided Practice*

MATHEMATICAL PRACTICES

Do you know HOW?

In **1–6**, subtract. Use place-value blocks, if you wish.

1. $\begin{array}{r} 374 \\ -\ 176 \end{array}$
2. $\begin{array}{r} 431 \\ -\ 145 \end{array}$
3. $\begin{array}{r} 568 \\ -\ 269 \end{array}$
4. $\begin{array}{r} 327 \\ -\ 238 \end{array}$
5. 574 − 86
6. 410 − 257

Do you UNDERSTAND?

7. **Communicate** In the example above, explain how to decide if regrouping is needed.
8. At the end of their game, Lora had 426 points, and Lou had 158 points. How many more points did Lora have than Lou?
 - **a** Write a number sentence.
 - **b** Estimate the answer.
 - **c** Solve the problem.
 - **d** Explain why your answer is reasonable.

Independent Practice

Estimate. Then find each difference. Check answers for reasonableness.

9. $\begin{array}{r} 385 \\ -\ 296 \end{array}$
10. $\begin{array}{r} 276 \\ -\ 97 \end{array}$
11. $\begin{array}{r} 516 \\ -\ 238 \end{array}$
12. $\begin{array}{r} 629 \\ -\ 453 \end{array}$
13. $\begin{array}{r} 948 \\ -\ 569 \end{array}$

eTools www.pearsonsuccessnet.com

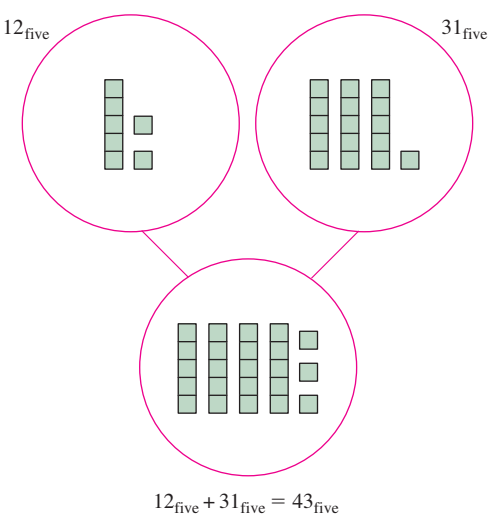
*For another example, see Set D on page 93.

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Table 10 Base-Five Addition Table

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Using the addition facts in Table 10, we apply the algorithms that we developed for base-ten addition to base-five addition. We show the computation using a concrete model in Figure 38(a); in Figure 38(b), we use an expanded algorithm; in Figure 38(c), we use the standard algorithm.



(a) Concrete model

$$\begin{array}{r} 12_{five} \\ +\ 31_{five} \\ \hline 3 \\ +\ 40 \\ \hline 43_{five} \end{array}$$

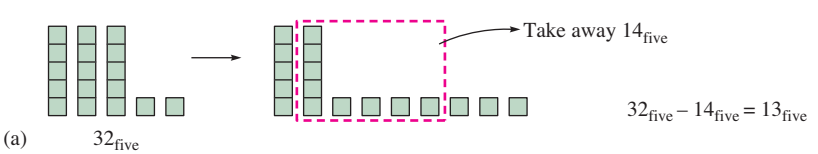
(b) Expanded algorithm

$$\begin{array}{r} 12_{five} \\ +\ 31_{five} \\ \hline 43_{five} \end{array}$$

(c) Standard algorithm

Figure 38

The subtraction facts for base five can also be derived from the addition facts table by using the definition of subtraction. For example, to find $12_{five} - 4_{five}$, recall that $12_{five} - 4_{five} = c$ if, and only if, $c + 4_{five} = 12_{five}$. From Table 10, we see that $c = 3_{five}$. An example of subtraction involving regrouping, $32_{five} - 14_{five}$, is developed in Figure 39.



(a) 32_{five}

	Fives	Ones
	3	2
−	1	4

(b)

	Fives	Ones
	2	12
−	1	4
	1	3

$$\begin{array}{r} {\scriptstyle 2\ 1} \\ 3\!\!\not 2_{five} \\ -\ 14_{five} \\ \hline 13_{five} \end{array}$$

(c)

Figure 39

The equal additions algorithm for subtraction and other algorithms be used in base-ten computations can also be efficiently used in other bases. For example, consider $432_{five} - 43_{five}$:

$$\begin{array}{r} 432_{five} \\ -\ 43_{five} \end{array} \rightarrow \begin{array}{r} 432_{five} + 2 \\ -(43_{five} + 2) \end{array} \rightarrow \begin{array}{r} 434_{five} \\ -100_{five} \\ \hline 334_{five} \end{array}$$

NOW TRY THIS 16

a. Build an addition table for base two.
b. Use the addition table from part (a) to perform: **(i)** $1111_{two} + 111_{two}$ **(ii)** $1101_{two} - 111_{two}$.

Mental Computation and Estimation for Whole-Number Addition and Subtraction

CCSS **According to the grade 4 *Core Curriculum Standards*, students should be able to:**

Assess the reasonableness of answers using mental computation and estimation strategies including rounding. (p. 29)

IMAP Video
Watch Estephania demonstrate the differences between computational strategies and mental mathematics.

Earlier, we focused mainly on paper-and-pencil computational strategies. Now we focus on mental computation and computational estimation. **Mental computation** is the process of producing an answer to a computation without using physical computational aids. **Computational estimation** is the process of forming an *approximate* answer to a numerical problem. Facility with estimation strategies helps to determine whether an answer is reasonable. Proficiency in mental computation can help in everyday estimation skills. It is essential to have these skills even in a time when calculators are readily available. We must be able to judge the reasonableness of calculator answers. Mental computation uses a variety of strategies and properties. We consider several of the most common algorithms for performing operations mentally on whole numbers.

Mental Computation: Addition

1. *Adding from the left*

a.
$$\begin{array}{rll} 67 & 60 + 30 = 90 & \text{(Add the tens.)} \\ +\ 36 & 7 + 6 = 13 & \text{(Add the units.)} \\ & 90 + 13 = 103 & \text{(Add the two sums.)} \end{array}$$

b.
$$\begin{array}{rll} 36 & 30 + 30 = 60 & \text{(Double 30.)} \\ +\ 36 & 6 + 6 = 12 & \text{(Double 6.)} \\ & 60 + 12 = 72 & \text{(Add the doubles.)} \end{array}$$

2. *Breaking up and bridging*

$$\begin{array}{rll} 67 & 67 + 30 = 97 & \text{(Add the first number to the tens in the second number.)} \\ +\ 36 & 97 + 6 = 103 & \text{(Add this sum to the units in the second number.)} \end{array}$$

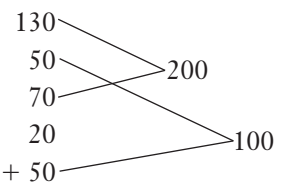
3. *Trading off*

a.
$$\begin{array}{rll} 67 & 67 + 3 = 70 & \text{(Add 3 to make a multiple of 10.)} \\ +\ 36 & 36 - 3 = 33 & \text{(Subtract 3 to compensate for the 3 that was added.)} \\ & 70 + 33 = 103 & \text{(Add the two sums.)} \end{array}$$

b. 67 + 29
67 + 30 = 97 (Add 30 [next multiple of 10 greater than 29].)
97 − 1 = 96 (Subtract 1 to compensate for the 1 that was added.)

The *trading off* algorithm is just the *equal additions* algorithm discussed earlier.

4. *Using compatible numbers*
Compatible numbers are numbers whose sums are easy to calculate mentally.



130 + 70 = 200
50 + 50 = 100
100 + 200 = 300 (add the two sums)
300 + 20 = 320 (add the remaining number)

IMAP Video
Watch Johanna make compatible numbers as she finds the sum of three whole numbers.

5. *Making compatible numbers*

25 + 79
25 + 75 = 100 (Add 25 + 75.)
100 + 4 = 104 (Add 4 more units.)

Mental Computation: Subtraction

1. *Breaking up and bridging*

67 − 36
67 − 30 = 37 (Subtract the tens in the second number from the first number.)
37 − 6 = 31 (Subtract the units in the second number from the difference.)

2. *Trading off*

71 − 39
71 + 1 = 72; 39 + 1 = 40 (Add 1 to both numbers.)
72 − 40 = 32 (Perform the subtraction, which is easier than the original problem.)

Notice that adding 1 to both numbers does not change the answer. (Why?)

3. *Drop the zeros*

8700 − 500
87 − 5 = 82 (Drop these zeros and perform the computation.)
82 → 8200 (Replace the two zeros to obtain proper place value.)

Another mental computation technique for subtraction is called *adding up*. This method is based on the *missing addend* approach and is sometimes referred to as the *cashier's algorithm*. An example follows.

Example 10

Noah owed $11 for his groceries. He used a $50 bill to pay. While handing Noah the change, the cashier said, "11, 12, 13, 14, 15, 20, 30, 50." How much change did Noah receive?

Solution Table 11 shows what the cashier said and how much money Noah received each time. Since $11 plus $1 is $12, Noah must have received $1 when the cashier said $12. The same reasoning follows for $13, $14, and so on. Thus, the total amount of change that Noah received is given by $1 + $1 + $1 + $1 + $5 + $10 + $20 = $39. In other words, $50 − $11 = $39 because $39 + $11 = $50.

Table 11

What the Cashier Said	$11	$12	$13	$14	$15	$20	$30	$50
Amount of Money Noah Received Each Time	0	$1	$1	$1	$1	$5	$10	$20

NOW TRY THIS 17

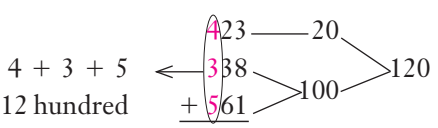
Perform each of the following computations mentally and explain what technique you used to find the answer:

a. 40 + 160 + 29 + 31
b. 3679 − 474
c. 75 + 28
d. 2500 − 700

Computational Estimation

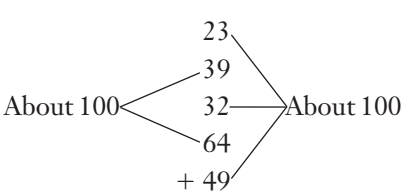
Computational estimation may help determine whether or not an answer is reasonable. This is especially useful when the computation is done on a calculator. Some common estimation strategies for addition are given next.

1. *Front-end with adjustment*
Front-end with adjustment estimation begins by focusing on the lead, or front, digits of the addition. These front, or lead, digits are added and assigned an appropriate place value. At this point we may have an underestimate that needs to be adjusted. The adjustment is made by focusing on the next group of digits. The following example shows how front-end estimation works:



Steps: (1) **Add front-end digits** 4 + 3 + 5 = 12.
(2) **Place value** = 100.
(3) **Adjust** 61 + 38 ≈ 100 and 20 + 100 is 120.
(4) **Adjusted estimate** is 1200 + 120 = 1320.

2. *Grouping to nice numbers*
The strategy used to obtain the adjustment in the preceding example is the *grouping to nice numbers* strategy, which means that numbers that "nicely" fit together are grouped. Another example is given here.



Therefore, the sum is about 100 + 100 = 200.

3. *Clustering*
Clustering is used when a group of numbers cluster around a common value. This strategy is limited to certain kinds of computations. In this example, the numbers seem to cluster around 6000.

6200
5842
6512
5521
+6319

Estimate the "average"—about 6000.

Multiply the average by the number of values to obtain 5 · 6000 = 30,000.

4. *Rounding*

Rounding is a way of changing numbers so they are easier to handle. Rounding enables us to find approximate answers to calculations, as follows:

4724	5000	(Round 4724 to 5000.)
+3192	+3000	(Round 3192 to 3000.)
	8000	(Add the rounded numbers.)
1267	1300	(Round 1267 to 1300.)
− 510	− 500	(Round 510 to 500.)
	800	(Subtract the rounded numbers.)

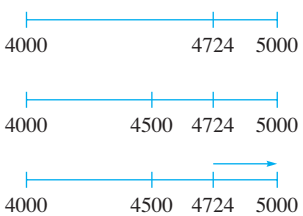


Figure 40

Performing estimations requires a knowledge of place value and rounding techniques. We illustrate a rounding procedure that can be generalized to all rounding situations. For example, suppose we wish to round 4724 to the nearest thousand. We may proceed in four steps (see also Figure 40).

a. Determine between which two consecutive thousands the number lies.
b. Determine the midpoint between the thousands.
c. Determine which thousand the number is closer to by observing whether it is greater than or less than the midpoint. (*Not all texts use the same rule for rounding when a number falls at a midpoint.*)
d. If the number to be rounded is greater than or equal to the midpoint, round the given number to the greater thousand; otherwise, round to the lesser thousand. In this case, 4724 rounds to 5000.

5. *Using the range*

It is often useful to know into what *range* an answer falls. The range is determined by finding a low estimate and a high estimate and reporting that the answer falls in this interval. An example follows:

Problem	Low Estimate	High Estimate
378	300	400
+ 524	+ 500	+ 600
	800	1000

Thus, a range for this problem is from 800 to 1000.



The student page shown on page 143 shows both the *rounding* and *front-end* estimation strategies applied to a problem.

Assessment 3-4A

1. Find the missing digits in each of the following addition problems.

a.
$$\begin{array}{r} _\ _\ 1 \\ +4\ 2\ _ \\ \hline -4\ 0\ 2 \end{array}$$

b.
$$\begin{array}{r} _\ 0\ 2\ 5 \\ 1\ 1\ _\ 6 \\ +3\ 1\ 4\ 8 \\ \hline 6\ _\ 6\ _ \end{array}$$

2. Make a drawing similar to the one in Figure 33 to show the use of base-ten blocks to compute 29 + 37.

3. Place the digits 7, 6, 8, 3, 5, and 2 in the boxes to obtain
a. the greatest sum.
b. the least sum.

$$\begin{array}{r} \square\square\square \\ +\square\square\square \end{array}$$

4. In the following problem, the sum is correct but the order of the digits in each addend has been scrambled. Correct the addends to obtain the correct sum.

$$\begin{array}{r} 2834 \\ +6315 \\ \hline 9059 \end{array} \qquad \begin{array}{r} \square\square\square\square \\ +\square\square\square\square \\ \hline 9059 \end{array}$$

School Book Page

Estimating Sums and Differences

Lesson 1-9

Key Idea
There is more than one way to estimate sums and differences.

Vocabulary
- front-end estimation
- rounding (p. 26)

TEST TALK

Think It Through
I only need an **estimate** because it asks about how many pounds.

Estimating Sums and Differences

WARM UP

Round each number to the place of the underlined digit.

1. 1<u>7</u>.333
2. 56<u>7</u>,642
3. 38.<u>0</u>45

LEARN

How can you estimate sums?

Students at Skyline Elementary collected aluminum cans for recycling. About how many pounds of cans did they collect in all?

Recycling Cans				
Grade	3rd	4th	5th	6th
Pounds Collected	398	257	285	318

You can estimate 398 + 257 + 285 + 318 two ways.

Jon used **rounding**.

I'll round each number to the nearest hundred.

398 → 400
257 → 300
285 → 300
+ 318 → + 300
1,300

About 1,300 pounds



Kylie used **front-end estimation** and adjusted the estimate.

I'll first add the front-end digits.

398 → 300
257 → 200
285 → 200
+ 318 → + 300
1,000

Then I'll adjust to include the remaining numbers.
98 → 100.
85 → 100.
57 + 18 → 100.
Less than 1,300 pounds



28

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5. Use the equal-additions approach to compute each of the following subtraction problems.

a.
$$\begin{array}{r} 93 \\ -37 \\ \hline \end{array}$$

b.
$$\begin{array}{r} 321 \\ -\ 38 \\ \hline \end{array}$$

6. Janet worked her addition problems by placing the partial sums as shown here:

$$\begin{array}{r} 569 \\ +\ 645 \\ \hline 14 \\ 10 \\ 11 \\ \hline 1214 \end{array}$$

a. Use this method to work the following addition problems.

i.
$$\begin{array}{r} 687 \\ +\ 549 \\ \hline \end{array}$$

ii.
$$\begin{array}{r} 359 \\ +\ 673 \\ \hline \end{array}$$

b. Explain why this algorithm works.

7. Analyze the following computations. Explain what is wrong in each case.

a.
$$\begin{array}{r} 28 \\ +75 \\ \hline 913 \end{array}$$

b.
$$\begin{array}{r} {\scriptstyle 3}\,28 \\ +75 \\ \hline 121 \end{array}$$

c.
$$\begin{array}{r} 305 \\ -259 \\ \hline 154 \end{array}$$

d.
$$\begin{array}{r} {\scriptstyle 2\,10}\,\not{3}\not{0}5 \\ -259 \\ \hline 56 \end{array}$$

8. Tom's diet allows only 1500 calories per day. For breakfast, Tom had skim milk (90 calories), a waffle with no syrup (120 calories), and a banana (119 calories). For lunch, he had $\frac{1}{2}$ cup of salad (185 calories) with mayonnaise (110 calories) and tea (0 calories). He had pecan pie (570 calories). Can he have dinner consisting of fish (250 calories), $\frac{1}{2}$ cup of salad with no mayonnaise, and tea?

9. Give reasons for each of the following steps.

$$\begin{aligned} 16 + 31 &= (1 \cdot 10 + 6) + (3 \cdot 10 + 1) \\ &= (1 \cdot 10 + 3 \cdot 10) + (6 + 1) \\ &= (1 + 3)10 + (6 + 1) \\ &= 4 \cdot 10 + 7 \\ &= 47 \end{aligned}$$

10. In each of the following addition problems, justify the standard addition algorithm using place value of the numbers, the commutative and associative properties of addition, and the distributive property of multiplication over addition.

a. $66 + 23$ **b.** $124 + 235$

11. Use the lattice algorithm to perform each of the following addition problems.

a. $4358 + 3864$ **b.** $4923 + 9897$

12. Perform each of the following operations using the bases shown and check your answers using a different algorithm.

a. $43_{\text{five}} + 23_{\text{five}}$ **b.** $43_{\text{five}} - 23_{\text{five}}$
c. $432_{\text{five}} + 23_{\text{five}}$ **d.** $42_{\text{five}} - 23_{\text{five}}$
e. $110_{\text{two}} + 11_{\text{two}}$
f. $10001_{\text{two}} - 111_{\text{two}}$

13. Construct an addition table for base eight and then perform each of the following computations.

a. $573_{\text{eight}} - 77_{\text{eight}}$ **b.** $765_{\text{eight}} - 76_{\text{eight}}$

14. Perform each of the following operations:

a.
$$\begin{array}{r} 3 \text{ hr } 36 \text{ min } 58 \text{ sec} \\ +\ 5 \text{ hr } 56 \text{ min } 27 \text{ sec} \\ \hline \end{array}$$

b.
$$\begin{array}{r} 5 \text{ hr } 36 \text{ min } 38 \text{ sec} \\ -\ 3 \text{ hr } 56 \text{ min } 58 \text{ sec} \\ \hline \end{array}$$

15. The following is a supermagic square.

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

a. Find the sum of each row, the sum of each column, and the sum of each diagonal.
b. Find the sum of the four numbers in the center.
c. Find the sum of the four numbers in the corners.
d. Add 5 to each number in the square. Is the square still a magic square?
e. Subtract 1 from each number in the square. Is the square still a magic square?

16. Use scratch addition to perform the following addition problems.

a.
$$\begin{array}{r} 432 \\ 976 \\ +\ 1418 \\ \hline \end{array}$$

b.
$$\begin{array}{r} 32_{\text{five}} \\ 13_{\text{five}} \\ 22_{\text{five}} \\ 43_{\text{five}} \\ 23_{\text{five}} \\ +\ 12_{\text{five}} \\ \hline \end{array}$$

17. Determine what is wrong with the following addition problem.

$$\begin{array}{r} 22_{\text{five}} \\ +\ 33_{\text{five}} \\ \hline 55_{\text{five}} \end{array}$$

18. Find the number to put in the blank to make each equation true. Do not convert to base ten.

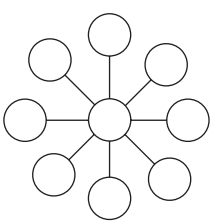
a. $3423_{\text{five}} - _____ = 2132_{\text{five}}$
b. $11011_{\text{two}} + _____ = 100000_{\text{two}}$
c. $TEE_{\text{twelve}} - _____ = 1$
d. $1000_{\text{five}} + _____ = 10000_{\text{five}}$

19. The Hawks played the Elks in a basketball game. Based on the following information, complete the scoreboard showing the number of points scored by each team during each quarter and the final score of the game.

	Quarters				
Teams	**1**	**2**	**3**	**4**	**Final Score**
Hawks					
Elks					

a. The Hawks scored 14 points in the first quarter.
b. The Hawks were behind by 4 points at the end of the first quarter.
c. The Elks scored 5 more points in the second quarter than they did in the first quarter.
d. The Hawks scored 1 fewer point than the Elks in the second quarter.
e. The Elks outscored the Hawks by 6 points in the fourth quarter.
f. The Hawks scored a total of 120 points in the game.
g. The Hawks scored twice as many points in the third quarter as the Elks did in the first quarter.
h. The Elks scored as many points in the third quarter as the Hawks did in the first two quarters combined.

20. **a.** Place the numbers 1 through 9 in the following circles so that the sums are the same along each segment.



b. How many different numbers can be placed in the middle to obtain a solution?

21. A palindrome is any number that reads the same backward as forward, for example, 121 and 2332. Try the following: Begin with any multi-digit number. Is it a palindrome? If not, reverse the digits and add this reversed number to the original number. Is the result a palindrome? If not, repeat the procedure until a palindrome is obtained. For example, start with 78. Because 78 is not a palindrome, we add: $78 + 87 = 165$. Because 165 is not a palindrome, we add: $165 + 561 = 726$. Again, 726 is not a palindrome, so we add $726 + 627$ to obtain 1353. Finally, $1353 + 3531$ yields 4884, which is a palindrome.

Try this method with the following numbers:
a. 93 **b.** 588 **c.** 2003

22. Compute each of the following mentally.
a. $180 + 97 - 23 + 20 - 140 + 26$
b. $87 - 42 + 70 - 38 + 43$

23. Compute each of the following mentally and describe your approach.
a. $475 + 49 + 525$ **b.** $375 - 76$
c. $230 + 60 + 70 + 44 + 6$

24. Use breaking up and bridging to compute each of the following mentally.
a. $567 + 38$ **b.** $418 + 215$

25. Use trading off to compute each of the following mentally.
a. $85 - 49$ **b.** $87 + 33$
c. $143 - 97$ **d.** $58 + 39$

26. Compute each of the following using the *adding up* (cashier's) algorithm.
a. $53 - 28$ **b.** $63 - 47$

27. Round each number to the place value indicated by the digit in bold.
a. 5**2**80 **b.** **1**15,234
c. **11**5,234 **d.** 2,**3**25

28. Estimate each answer by rounding.
a. $878 + 2340$
b. $25{,}201 - 19{,}987$
c. $2215 + 3023 + 5967 + 975$

29. Use front-end estimation with adjustment to estimate each of the following.
a. $2215 + 3023 + 5987 + 975$
b. $234 + 478 + 987 + 319 + 469$

30. **a.** Would the clustering strategy of estimation be a good one to use in each of the following cases? Why or why not?

(i)
$$\begin{array}{r} 474 \\ 1467 \\ 64 \\ +\ 2445 \\ \hline \end{array}$$

(ii)
$$\begin{array}{r} 483 \\ 475 \\ 530 \\ 503 \\ +\ 528 \\ \hline \end{array}$$

b. Estimate each part of (a) using the following strategies:
(i) Front-end
(ii) Grouping to nice numbers
(iii) Rounding

31. Use the range strategy to estimate each of the following. Explain how you arrived at your estimates.
a. $145 + 678$ **b.** $278 + 36$

32. Suppose you had a balance of $3287 in your checking account and you wrote checks for $85, $297, $403, and $523. Estimate your balance and tell what you did.

33. The following is a list of the areas in square miles of Europe's largest countries. Mentally use this information to decide if each of the given statements is true.

France	211,207
Spain	194,896
Sweden	173,731
Finland	130,119
Norway	125,181

a. Sweden is less than 40,000 mi^2 larger than Finland.
b. France is more than twice the size of Norway.
c. France is more than 100,000 mi^2 larger than Norway.
d. Spain is about 21,000 mi^2 larger than Sweden

34. The attendance at a World's Fair for one week follows.

Monday	72,250
Tuesday	63,891
Wednesday	67,490
Thursday	73,180
Friday	74,918
Saturday	68,480

Estimate the week's attendance and tell what strategy you used.

Assessment 3-4B

1. Find the missing digits in each of the following subtraction problems.

a.
$$\begin{array}{r} 3\ _\ _ \\ -\ 1\ 5\ 9 \\ \hline _\ 2\ 4 \end{array}$$

b.
$$\begin{array}{r} 1\ _\ _\ _\ 6 \\ -\ \ 8\ 3\ 0\ 9 \\ \hline \ \ 4\ 9\ 8\ 7 \end{array}$$

2. Make an appropriate drawing like the one in Figure 33 to show the use of base-ten blocks to compute 46 + 38.

3. Place the digits 7, 6, 8, 3, 5, and 2 in the boxes to obtain
a. the greatest difference.
b. the least positive difference.

$$\begin{array}{r} \square\square\square \\ -\square\square\square \\ \hline \end{array}$$

4. In the following problem, the sum is correct but the order of the digits in each addend has been scrambled. Correct the addends to obtain the correct sum.

$$\begin{array}{r} 8\,3\,5\,4 \\ +\,3\,4\,5\,6 \\ \hline 1\,1\,7\,2\,9 \end{array} \qquad \begin{array}{r} \square\square\square\square \\ +\square\square\square\square \\ \hline 1\,1\,7\,2\,9 \end{array}$$

5. Use the equal-additions approach to compute each of the following subtraction problems.

a.
$$\begin{array}{r} 86 \\ -\ 38 \\ \hline \end{array}$$

b.
$$\begin{array}{r} 582 \\ -\ 44 \\ \hline \end{array}$$

6. Janet computed her addition problems by placing the partial sums as shown here:

$$\begin{array}{r} 768 \\ +\ 647 \\ \hline 15 \\ 10 \\ 13 \\ \hline 1415 \end{array}$$

Use this method to work the following addition problems.

a.
$$\begin{array}{r} 987 \\ +\ 356 \\ \hline \end{array}$$

b.
$$\begin{array}{r} 415 \\ +\ 79 \\ \hline \end{array}$$

7. Analyze the following computations. Explain what is wrong in each case.

a.
$$\begin{array}{r} 135 \\ +\ 47 \\ \hline 172 \end{array}$$

b.
$$\begin{array}{r} 87 \\ +\ 25 \\ \hline 1012 \end{array}$$

c.
$$\begin{array}{r} 57 \\ -\ 38 \\ \hline 21 \end{array}$$

d.
$$\begin{array}{r} 56 \\ -\ 18 \\ \hline 48 \end{array}$$

8. George is cooking a meal for Thanksgiving. He can cook only one thing at a time in his microwave oven. His turkey takes 75 min; the pumpkin pie takes 18 min; rolls take 45 sec; and a cup of coffee takes 30 sec to heat. How much time does he need to cook the meal? When does he need to start in order to complete the cooking at 4 P.M.?

9. Give reasons for each of the following steps:

$$\begin{aligned} 123 + 45 &= (1\cdot 10^2 + 2\cdot 10 + 3) + (4\cdot 10 + 5) \\ &= 1\cdot 10^2 + (2\cdot 10 + 4\cdot 10) + (3 + 5) \\ &= 1\cdot 10^2 + (2 + 4)10 + (3 + 5) \\ &= 1\cdot 10^2 + 6\cdot 10 + 8 \\ &= 168 \end{aligned}$$

10. In each of the following addition problems justify the standard addition algorithm using place value of the numbers, the commutative and associative properties of addition, and the distributive property of multiplication over addition:
a. 46 + 32 **b.** 3214 + 783

11. Use the lattice algorithm to perform each of the following addition problems.
a. 2345 + 8888 **b.** 8713 + 4214

12. Perform each of the following operations using the bases shown:
a. $43_{five} - 24_{five}$
b. $143_{five} + 23_{five}$
c. $32_{five} - 23_{five}$
d. $232_{five} + 43_{five}$
e. $110_{two} + 111_{two}$
f. $10001_{two} - 101_{two}$

13. Construct an addition table for base six, and then perform each of the following subtractions, and check your answers by computing an equivalent addition problem.
a. $231_{six} - 144_{six}$ **b.** $342_{six} - 144_{six}$

14. Perform each of the following operations (2 c = 1 pt, 2 pt = 1 qt, 4 qt = 1 gal).

a.
$$\begin{array}{r} 1\text{ qt }1\text{ pt }1\text{ c} \\ +\quad 1\text{ pt }1\text{ c} \\ \hline \end{array}$$

b.
$$\begin{array}{r} 1\text{ qt}\quad 1\text{c} \\ -\quad 1\text{ pt }1\text{c} \\ \hline \end{array}$$

c.
$$\begin{array}{r} 1\text{ gal }3\text{ qt }1\text{ c} \\ -\quad 4\text{ qt }2\text{ c} \\ \hline \end{array}$$

15. The following is a supermagic square taken from an engraving called *Melancholia* by Dürer. Notice 1514 in the bottom row, the year the engraving was made.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

a. Find the sum of each row, the sum of each column, and the sum of each diagonal.
b. Find the sum of the four numbers in the center.
c. Find the sum of the four numbers in the corners.
d. Add 11 to each number in the square. Is the square still a magic square?
e. Subtract 11 from each number in the square. Is the square still a magic square?

16. Use scratch addition to perform the following addition problems.

a.
$$\begin{array}{r} 537 \\ 318 \\ +\,2345 \\ \hline \end{array}$$

b.
$$\begin{array}{r} 41_{six} \\ 32_{six} \\ 22_{six} \\ 43_{six} \\ 22_{six} \\ +54_{six} \\ \hline \end{array}$$

17. Determine what is wrong with the following addition problem.

$$\begin{array}{r} 23_{six} \\ +\ 43_{six} \\ \hline 66_{six} \end{array}$$

18. Find the number to put in the blank to make each equation true. Do not convert to base ten.

a. $342_{five} + _____ = 213_{five}$

b. $1101_{two} + _____ = 1011_{two}$

c. $E08_{twelve} + _____ = 9_{twelve}$

d. $100_{two} + _____ = 10000_{two}$

19. The Hawks played the Elks in a basketball game. Based on the following information, complete the scoreboard showing the number of points scored by each team during each quarter and the final score of the game.

Quarters					
Teams	**1**	**2**	**3**	**4**	**Final Score**
Hawks					
Elks					

a. The Hawks scored 15 points in the first quarter.

b. The Hawks were behind by 5 points at the end of the first quarter.

c. The Elks scored 5 more points in the second quarter than they did in the first quarter.

d. The Hawks scored 7 more points than the Elks in the second quarter.

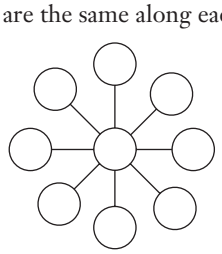
e. The Elks outscored the Hawks by 6 points in the fourth quarter.

f. The Hawks scored a total of 120 points in the game.

g. The Hawks scored twice as many points in the third quarter as the Elks did in the first quarter.

h. The Elks scored as many points in the third quarter as the Hawks did in the first two quarters combined.

20. **a.** Place the numbers 24 through 32 in the following circles so that the sums are the same along each segment.



b. How many different numbers can be placed in the middle to obtain a solution?

21. What is the greatest possible addend that can be placed in the boxes to make this a correct computation?

$$\begin{array}{r} 8\ 7 \\ 7\ 8 \\ +\,\square\square \\ \hline 2\square\square \end{array}$$

22. Compute each of the following mentally.

a. $160 + 92 - 32 + 40 - 18$

b. $36 + 97 - 80 + 44$

23. Supply reasons for each of the first four steps given here.

$$\begin{aligned} (525 + 37) + 75 &= 525 + (37 + 75) \\ &= 525 + (75 + 37) \\ &= (525 + 75) + 37 \\ &= 600 + 37 \\ &= 637 \end{aligned}$$

24. Use breaking and bridging to compute each of the following mentally.

a. $997 - 32$ **b.** $560 + 136$

25. Use trading off to compute each of the following mentally.

a. $75 - 38$ **b.** $57 + 35$

c. $137 - 29$ **d.** $78 + 49$

26. Compute each of the following using the *adding up* (cashier's) algorithm.

a. $74 - 63$ **b.** $73 - 57$

27. Round each number to the place value indicated by the digit in bold.

a. 3**5**87 **b.** **1**48,213

c. 2**3**,785 **d.** 2,3**5**7

28. Estimate each answer by rounding.

a. $937 + 28$

b. $32{,}285 - 18{,}988$

c. $3215 + 3789 + 5987$

29. Use front-end estimation with adjustment to estimate each of the following.

a. $2345 + 5250 + 4210 + 910$

b. $345 + 518 + 655 + 270$

30. **a.** Would the clustering strategy of estimation be a good one to use in each of the following cases? Why or why not?

(i)
$$\begin{array}{r} 318 \\ 2314 \\ 57 \\ +\ 3489 \\ \hline \end{array}$$

(ii)
$$\begin{array}{r} 2350 \\ 1987 \\ 2036 \\ 2103 \\ +\ 1890 \\ \hline \end{array}$$

b. Estimate each part of (a) using the following strategies:

(i) Front-end with adjustment

(ii) Grouping to nice numbers

(iii) Rounding

31. Use the range strategy to estimate each of the following. Explain how you arrived at your estimates.

a. $123 + 780$

b. $482 + 246$

32. Suppose you had a balance of \$1237 in your checking account and you wrote checks for \$65, \$190, \$45, \$212, and \$420. Estimate your balance and tell what you did.

33. In each of the following, answer the question using estimation methods if possible. If estimation is not appropriate, explain why not.

a. Josh has \$380 in his checking account. He wants to write checks for \$39, \$28, \$59, and \$250. Will he have enough money in his account to cover these checks?

b. Gila deposited two checks into her account, one for \$981 and the other for \$1140. Does she have enough money in

her account to cover a check for $2000 if we know she has a positive starting balance to start with?

c. Alberto and Juan are running for city council. They receive votes from two districts. Alberto receives 3473 votes from one district and 5615 votes from the other district. Juan receives 3463 votes from the first district and 5616 from the second. Who gets elected?

34. The attendance at a County Fair for six days follows:

Monday	71,150
Tuesday	64,993
Wednesday	68,490
Thursday	72,980
Friday	84,968
Saturday	69,495

Estimate the total attendance for the six days.

Mathematical Connections 3-4

Reasoning

1. Discuss the merit of the following expanded algorithm for addition where we first add the ones, then the tens, then the hundreds, and then the total.

$$\begin{array}{r} 479 \\ +\ 385 \\ \hline 14 \\ 150 \\ +\ 700 \\ \hline 864 \end{array}$$

2. The following example uses a regrouping approach to subtraction. Discuss the merit of this approach in teaching subtraction.

$$\begin{array}{r} 843 \\ -\ 568 \\ \hline \end{array} \rightarrow \begin{array}{r} 800 + 40 + 3 \\ -\ (500 + 60 + 8) \\ \hline \end{array} \rightarrow$$

$$\begin{array}{r} 800 + 30 + 13 \\ -\ (500 + 60 + \ 8) \\ \hline \end{array} \rightarrow \begin{array}{r} 700 + 130 + 13 \\ -\ (500 + \ 60 + \ 8) \\ \hline 200 + \ 70 + \ 5 \end{array} = 275$$

3. Explain why the scratch addition algorithm works.

4. Consider the following subtraction algorithm.

$$\begin{aligned} 585 - 277 &= 585 - 200 - \ 77 \\ &= 385 - \ 77 \\ &= 385 - 100 + \ 23 \\ &= 285 + \ 23 = 308 \end{aligned}$$

a. Explain why it works.

b. Use this algorithm to find 787 − 398.

5. Explain why in paper-and-pencil addition and subtraction the terms *trading* and *regrouping* are used instead of the terms *carrying* and *borrowing*.

6. Is the front-end estimate for addition before adjustment always less than the exact sum? Explain why or why not.

7. In the new textbooks, there is an emphasis on mental mathematics and estimation. Explain why these topics are important for today's students.

8. Suppose x and y are positive (greater than 0) whole numbers. If x is greater than y and you estimate $x - y$ by rounding x up and y down, will your estimate always be too high or too low or could it be either? Explain.

Open-Ended

9. Search for or develop an algorithm for whole-number addition or subtraction, and write a description of your algorithm so that others can understand and use it.

10. Give several examples from real-world situations where an estimate, rather than an exact answer, is sufficient.

11. **a.** Give a numerical example of when front-end estimation and rounding can produce the same estimate.

b. Give an example of when they can produce a different estimate.

Cooperative Learning

12. In this section you have been exposed to many different algorithms. Discuss in your group whether children should be encouraged to develop and use their own algorithms for whole-number addition and subtraction or whether they should be taught only one algorithm per operation and all students should use only one algorithm.

13. Prepare a grocery list of 10 items with prices, and find the total price using a calculator. Give the list to each member of your group without revealing the total price. Ask them to estimate the total price in one minute. Find who came closest to the total price. Take turns preparing a list.

14. Play the following game with a partner or teams of two. Make a deck of 32 cards to include four each of the numbers 1 through 9. (You could use a playing deck of cards and use the ace for a 1.) Two players draw alternately from the deck and after each draw place the card in either the *tens* column or the *ones* column. The player or team closest to a sum of 100 after 5 draws is the winner. For example, if you drew 6, 1, 5, 3, 5 and placed them as shown, your total would be 101.

tens	ones
6	1
3	5
	5

Connecting Mathematics to the Classroom

15. Tira, a fourth grader, performs addition by adding and subtracting the same number. She added as follows:

$$\begin{array}{r} 39 \\ +\ 84 \\ \hline \end{array} \rightarrow \begin{array}{r} 39 + 1 \\ +\ 84 - 1 \\ \hline \end{array} \rightarrow \begin{array}{r} 40 \\ +\ 83 \\ \hline 123 \end{array}$$

How would you respond if you were her teacher?

16. Cathy found her own algorithm for subtraction. She subtracted as follows:

$$\begin{array}{r} 97 \\ -28 \\ \hline -1 \\ +\ 70 \\ \hline 69 \end{array}$$



How would you respond if you were her teacher?

17. To find $68 - 19$, Joe began by finding $6 - 1$ and then $9 - 8$ and wrote 51 as the answer. How would you respond?

18. Jill subtracted $415 - 212$ by writing $4 - 2 = 2$ and $15 - 12 = 3$ and wrote 23 as the answer. How would you help her?

19. Betsy found $518 - 49 = 469$. She was not sure she was correct, so she tried to check her answer by adding $518 + 49$. How could you help her?

20. A child is asked to compute $7 + 2 + 3 + 8 + 11$ and writes $7 + 2 = 9 + 3 = 12 + 8 = 20 + 11 = 31$. Noticing that the answer is correct, if you were the teacher how would you react?

21. When Richie was asked to round 2568 to the nearest ten, he wrote 2578. How would you respond?

22. When Ashley was asked to round 345 to the nearest 100, she rounded 345 to 350 and then rounded 350 to 400. Billie claimed the answer was 300. How would you respond to these students?

23. Pauli subtracted 549 from 5004 as follows. How would you respond?

$$\begin{array}{r} {}^{499} \\ \cancel{5004} \\ -\ 549 \\ \hline 4455 \end{array}$$

24. Molly computed $261 - 48$ by first subtracting 50 from 261 to obtain 211; then, to make up for adding 2 to 48, she subtracted 2 from 211 to obtain an answer of 209. Is her thinking correct? If not, how could you help her?

25. A student asks why he has to learn about any estimation strategy other than rounding. What is your response?

26. In order to finish her homework quickly, an elementary student does her estimation problems by using a calculator to find the exact answers and then rounds them to get her estimate. What do you tell her?

Review Problems

27. What is the value of MCDX in Hindu-Arabic numerals?

28. Convert each of the following to base ten.

a. $E0T_{\text{twelve}}$ **b.** 1011_{two} **c.** 43_{five}

29. Is the set $\{1, 2, 3\}$ closed under addition? Why?

30. Give an example of the associative property of addition of whole numbers.

31. Find all whole numbers for which the following statements are true.

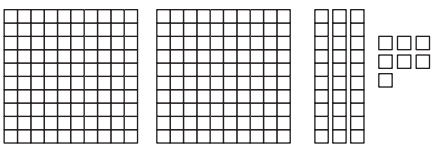
a. $20 - x = x$

b. $20 - x - 6 = 0$

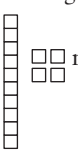
c. $x + 4 = 3 + x + 1$

National Assessments

National Assessment of Educational Progress (NAEP) Questions



The figure above represents 237. Which number is



more than 237?

A. 244 B. 249
C. 251 D. 377

NAEP, Grade 4, 2007

There are 30 people in the music room. There are 74 people in the cafeteria. How many more people are in the cafeteria than in the music room?

A. 40 B. 44
C. 54 D. 104

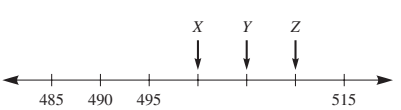
NAEP, Grade 4, 2009

Audrey used only the number tiles with the digits 2, 3, 4, 6, and 9. She placed one tile in each box below so the difference was 921.

Write the numbers in the boxes below to show where Audrey placed the tiles.

$$\begin{array}{r} \square\square\square \\ -\ \ \square\square \\ \hline 9\ 2\ 1 \end{array}$$

NAEP Grade 4, 2005



On the number line above, what is the sum of the numbers to which the arrows *X*, *Y*, and *Z* point?

A. 1,491 B. 1,515
C. 1,530 D. 1,545

Did you use the calculator on this question?

NAEP Grade 4, 2011

3-5 Multiplication and Division Algorithms, Mental Computation, and Estimation

3-5 Objectives

Students will be able to understand and explain

- Properties of exponents and how these can be used to develop multiplication and division algorithms.
- Models to develop algorithms for multiplication and division.
- Multiplication and division algorithms and how to use them to solve problems.
- Bases other than ten to provide insight into base-ten multiplication and division.
- Mental multiplication and division skills and estimation techniques.

CCSS **In the grade 4 *Common Core Standards*, we find the following:**

Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop fluency with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. (p. 27)

In this section we discuss the use of efficient, accurate, and generalizable methods to compute products and quotients. We also discuss appropriate methods to estimate and mentally calculate products and quotients. We begin with a discussion of the properties of exponents.

Properties of Exponents

Some properties of exponents are used in this section and in the following chapters.

Definition of a^n

If a, the base, and n, the exponent, are whole numbers and $n \neq 0$, then

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}.$$

When multiplying powers of 10, the definition of exponents is used; for example, $10^2 \cdot 10^1 = (10 \cdot 10) \cdot 10 = 10^3 = 10^{2+1}$. In general, where a is a whole number and m and n are natural numbers, $a^m \cdot a^n$ is given by the following:

$$a^m \cdot a^n = \underbrace{(a \cdot a \cdot a \cdot \ldots \cdot a)}_{m \text{ factors}} \cdot \underbrace{(a \cdot a \cdot a \cdot \ldots \cdot a)}_{n \text{ factors}}$$

$$= \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m + n \text{ factors}} = a^{m+n}$$

Consequently, $a^m \cdot a^n = a^{m+n}$.

Therefore we have the following theorem.

Theorem 3-9

For every whole number a and natural numbers m and n:

$$a^m \cdot a^n = a^{m+n}$$

The definition of exponents and Theorem 3-9 can be used to rewrite an expression such as $(5^2)^3$ using a single exponent:

$$(5^2)^3 = 5^2 \cdot 5^2 \cdot 5^2 = 5^{2+2+2} = 5^{3 \cdot 2} = 5^6$$

This suggests the following theorem:

Theorem 3-10

For every whole number a, and natural numbers m and n:

$$(a^m)^n = a^{mn}$$

Proof

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot \ldots \cdot a^m}_{n \text{ factors}} = a^{\overbrace{m+m+\ldots+m}^{n \text{ terms}}} = a^{mn}$$

■

The preceding theorems involve exponents and multiplication. However, corresponding properties for exponents and addition do not exist; for example $2^5 + 2^3 \neq 2^{5+3}$.

Sometimes it is convenient to write a product such as $2^3 \cdot 5^3$ with a single exponent:

$$2^3 \cdot 5^3 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 = (2 \cdot 5)(2 \cdot 5)(2 \cdot 5) = (2 \cdot 5)^3 = 10^3$$

In general we have Theorem 3-11.

Theorem 3-11

For whole numbers a and b, and natural numbers n:

$$a^n \cdot b^n = (ab)^n$$

The proof of this theorem is similar to the one in the previous example and is left as an exercise. It is often useful to divide exponential expressions with the same base. Consider writing $2^6 \div 2^2$ with a single exponent. Using the definition of division, $2^6 \div 2^2 = \square$ if, and only if, $\square \cdot 2^2 = 2^6$. Since $\boxed{2^4} \cdot 2^2 = 2^6$, it follows that $2^6 \div 2^2 = 2^4$. This example suggests the following theorem:

Theorem 3-12

If a, m, and n are natural numbers with $m > n$, then

$$a^m \div a^n = a^{m-n}.$$

Proof

By definition of division, $a^m \div a^n = a^x$ if, and only if, $a^x \cdot a^n = a^m$. We want to find x in terms of m and n. By Theorem 3-9, $a^{x+n} = a^m$. Hence $x + n = m$, so $x = m - n$. Thus $a^m \div a^n = a^{m-n}$. ■

If the laws of exponents for natural numbers are to hold for the exponent 0, then $a^0 \cdot a^n = a^{0+n} = a^n$. Because $a^0 \cdot a^n = a^n$ we have $a^0 = 1$. However, 0^0 is considered to be undefined.

Definition of a^0 for natural number a

If a is a natural number, then

$$a^0 = 1.$$

Example 11

Write each of the following with only one exponent.

a. $2^6 \cdot 8^5 \cdot 16^3$ **b.** $(9^4 \cdot 36^5) \div 3^{18}$

Solution

a. $2^6 \cdot 8^5 \cdot 16^3 = 2^6 \cdot (2^3)^5 \cdot (2^4)^3 = 2^6 \cdot 2^{15} \cdot 2^{12}$
$= 2^{6+15+12} = 2^{33}$

b. $9^4 \cdot 36^5 = (3^2)^4 \cdot (2^2 \cdot 3^2)^5 = 3^{2 \cdot 4} \cdot (2^2)^5 \cdot (3^2)^5$
$= 3^8 \cdot 2^{10} \cdot 3^{10}$
$= 3^8 \cdot 3^{10} \cdot 2^{10}$
$= 3^{18} \cdot 2^{10}$

Thus $(9^4 \cdot 36^5) \div 3^{18} = (3^{18} \cdot 2^{10}) \div 3^{18}$
$= 2^{10}$.

IMAP Video

Watch Brooke discuss an algorithm for multiplying whole numbers she invented herself.

Multiplication Algorithms

To develop algorithms for multiplying multi-digit whole numbers, we look at a simple example. Consider $4 \cdot 12$. This computation could be pictured as in Figure 41(a) with 4 rows of 12 blocks, or 48 blocks. These blocks in Figure 41(a) can also be partitioned to show that $4 \cdot 12 = 4(10 + 2) = 4 \cdot 10 + 4 \cdot 2$. The numbers $4 \cdot 10$ and $4 \cdot 2$ are *partial products*. This technique is often referred to as the **partial products algorithm**.

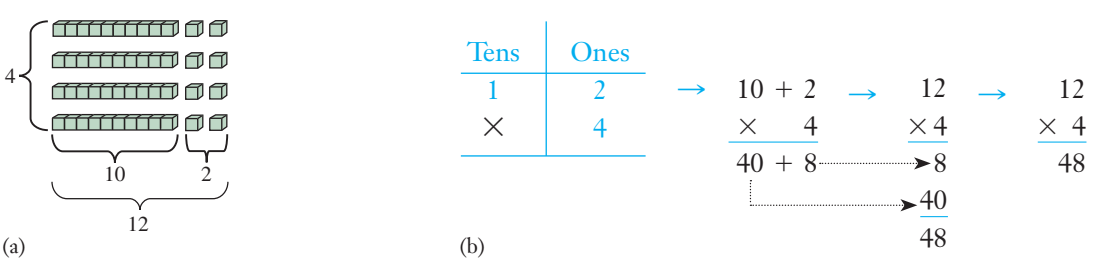


Figure 41

Figure 41(a) illustrates the distributive property of multiplication over addition on the set of whole numbers. The process leading to an algorithm for multiplying $4 \cdot 12$ is seen in Figure 41(b). There is similarity between the multiplication in Figure 41 and the following algebra multiplication:

$$4(x + 2) = 4x + 4 \cdot 2$$
$$= 4x + 8$$

Similarly there is an analogy between the products

$$23 \cdot 14 = (2 \cdot 10 + 3)(1 \cdot 10 + 4) \quad \text{and} \quad (2x + 3)(x + 4).$$

The analogy is continued as shown:

$$\begin{array}{r} 2\cdot 10+3 \\ \times\ (1\cdot 10+4) \\ \hline 12 \\ 8\cdot 10 \\ 3\cdot 10 \\ \hline 2\cdot 10^2 \\ \hline 2\cdot 10^2+11\cdot 10+12 \end{array} \qquad \begin{array}{r} 2x+3 \\ \times\ (x+4) \\ \hline 12 \\ 8x \\ 3x \\ \hline 2x^2 \\ \hline 2x^2+11x+12 \end{array}$$

Multiplication of a three-digit number by a one-digit factor will be explored after discussing multiplication by a power of 10.

Multiplication by 10^n

Consider multiplication by powers of 10. What happens when a given number is multiplied by 10, such as $10\cdot 23$? If we begin with the base-ten block representation of 23, we have 2 longs and 3 units. To multiply by 10, we must replace each piece with a base-ten piece that represents the next higher power of 10. This is shown in Figure 42. Notice that the 3 units in 23 when multiplied by 10 become 3 longs or 3 tens. Therefore, after multiplication by 10 there are no units and hence we have 0 in the units place. In general, if we multiply any natural number by 10, we append a 0 to the right of the original number.

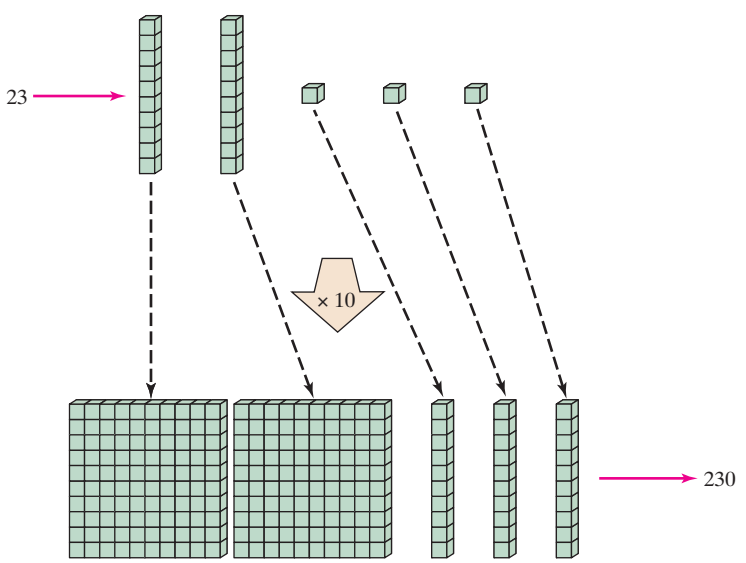


Figure 42

The computation $23\cdot 10$ from Figure 42 can be shown as follows:

$$\begin{aligned} 23\cdot 10 &= (2\cdot 10+3)10 \\ &= (2\cdot 10)10+3\cdot 10 \\ &= 2(10\cdot 10)+3\cdot 10 \\ &= 2\cdot 10^2+3\cdot 10 \\ &= 2\cdot 10^2+3\cdot 10+0\cdot 1 \\ &= 230 \end{aligned}$$

To compute products such as $3 \cdot 200$, proceed as follows:

$$\begin{aligned} 3 \cdot 200 &= 3(2 \cdot 10^2) \\ &= (3 \cdot 2)10^2 \\ &= 6 \cdot 10^2 \\ &= 6 \cdot 10^2 + 0 \cdot 10 + 0 \cdot 1 \\ &= 600 \end{aligned}$$

Multiplying 6 by 10^2 results in appending two zeros to the right of 6. This idea can be generalized to the statement that *multiplication of any natural number by* 10^n *where n is a whole number, results in appending n zeros to the right of the number*.

The appending of n zeros to a natural number when multiplying by 10^n can also be explained as follows. First multiply by 10, resulting in appending one zero (as in $23 \cdot 10 = 230$). Then multiply by another 10, resulting in appending another zero (as in $230 \cdot 10 = 2300$). Since we multiply n times by 10, n zeros are appended to the right of the original natural number. Note that if $n = 0$, then $10^0 = 1$ appending no zeros.

Multiplication by a power of 10 is helpful in calculating the product of a one-digit number and a three-digit number. In the following example, we assume the previously developed algorithm for multiplying a one-digit number times a two-digit number:

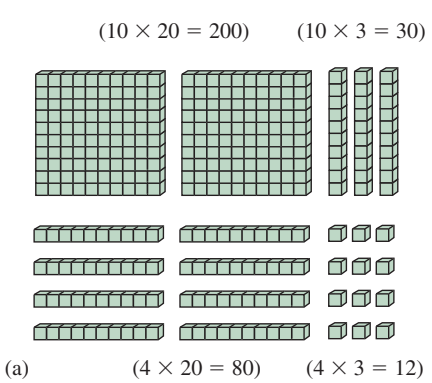
$$\begin{aligned} 4 \cdot 367 &= 4(3 \cdot 10^2 + 6 \cdot 10 + 7) \\ &= 4(3 \cdot 10^2) + 4(6 \cdot 10) + 4 \cdot 7 \\ &= (4 \cdot 3)10^2 + (4 \cdot 6)10 + 4 \cdot 7 \\ &= 12 \cdot 10^2 + 24 \cdot 10 + 28 \\ &= 1200 + 240 + 28 \\ &= 1468 \end{aligned}$$

$\rightarrow$

$$\begin{array}{r} 367 \\ \times\ 4 \\ \hline 28 \\ 240 \\ 1200 \\ \hline 1468 \end{array}$$

Multiplication with Two-Digit Factors

Consider $14 \cdot 23$. Model this computation by first using base-ten blocks, as shown in Figure 43(a), and then showing all the *partial products* and adding, as shown in Figure 43(b).



$$\begin{array}{r} 23 \\ \times\ 14 \\ \hline 12 \\ 80 \\ 30 \\ +\ 200 \\ \hline 322 \end{array}$$

(b)

(4 × 3)
(4 × 20)
(10 × 3)
(10 × 20)
(12 + 80 + 30 + 200)

Figure 43

Figure 43(b) leads to an algorithm for multiplication:

$$\begin{array}{r} 23 \\ \times\ 14 \\ \hline 92 \\ 230 \\ \hline 322 \end{array} \quad \begin{array}{l} \\ \\ (4 \cdot 23) \\ (10 \cdot 23) \\ \\ \end{array} \quad \text{or} \quad \begin{array}{r} 23 \\ \times\ 14 \\ \hline 92 \\ 23 \\ \hline 322 \end{array}$$

It is not uncommon to see the partial product 230 written without the zero, as 23. The placement of 23 with 3 in the tens column obviates having to write the 0 in the units column. We encourage the inclusion of the zero. This promotes better understanding and helps to avoid errors.

The distributive property of multiplication over addition can be used to explain why the algorithm for multiplication works. Again, consider $14 \cdot 23$.

$$\begin{aligned} 14 \cdot 23 &= (10 + 4)23 \\ &= 10 \cdot 23 + 4 \cdot 23 \\ &= 230 + 92 \\ &= 322 \end{aligned}$$

Because algorithms are powerful, there is sometimes a tendency to overapply them or to use paper and pencil for a task that should be done mentally. For example, consider the following example.

$$\begin{array}{r} 213 \\ \times\ 1000 \\ \hline 000 \\ 000 \\ 000 \\ 213 \\ \hline 213000 \end{array}$$

This application is not wrong but is inefficient.

Mental computation and estimation are important skills in learning mathematics and should be practiced in addition to paper-and-pencil computations. Children should be encouraged to *estimate* whether their answers are reasonable. For example, in the computation $14 \cdot 23$, we know that the answer must be between $10 \cdot 20 = 200$ and $20 \cdot 30 = 600$ because $10 < 14 < 20$ and $20 < 23 < 30$. Also, since 14 is close to 10 and 23 is close to 20, the product should be close to 200.

Lattice Multiplication

Lattice multiplication has the advantage of delaying all additions until the single-digit multiplications are complete. Because of this, it is sometimes referred to as a *low-stress algorithm*. Students like this algorithm, perhaps because of the structure provided by the lattice. The lattice multiplication algorithm for multiplying 14 and 23 is shown in Figure 44. (Determining the reasons why lattice multiplication works is left as an exercise.)

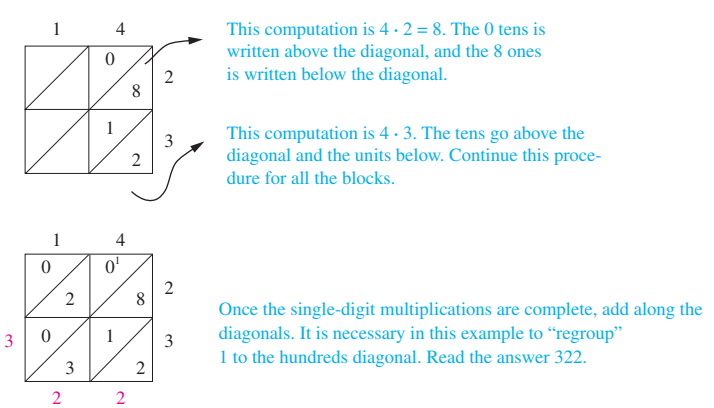


Figure 44

Division Algorithms

Using Repeated Subtraction to Develop the Standard Division Algorithm

As we have seen in Section 3-3, division of whole numbers can be modeled by repeated subtraction. We use this approach in the following question.

> *A shopkeeper is packaging juice in cartons that hold 6 bottles each. She has 726 bottles. How many cartons does she need?*

We reason that if 1 carton holds 6 bottles, then 10 cartons hold 60 bottles and 100 cartons hold 600 bottles. If 100 cartons are filled, there are $726 - 100 \cdot 6$, or 126, bottles remaining. If 10 more cartons are filled, then $126 - 10 \cdot 6$, or 66, bottles remain. Similarly, if 10 more cartons are filled, $66 - 10 \cdot 6$, or 6, bottles remain. Finally, 1 carton will hold the remaining 6 bottles. The total number of cartons necessary is $100 + 10 + 10 + 1$, or 121. This procedure is summarized in Figure 45(a). This method is sometimes called the **partial quotients algorithm**. A more efficient method is shown in Figure 45(b).

6)726	
−600	100 sixes
126	
−60	10 sixes
66	
−60	10 sixes
6	
−6	1 six
0	121 sixes

(a)

6)726	
−600	100 sixes
126	
−120	20 sixes
6	
−6	1 six
0	121 sixes

(b)

Figure 45

NOW TRY THIS 18

Study the following division, and explain what was done in each step.

97 R14	
45)4379	
−2250	50
2129	
−1800	40
329	
−270	6
59	
−45	1
14	97

Historical Note



Lattice multiplication dates back to tenth-century India. This algorithm was imported to Europe and was popular in the fourteenth and fifteenth centuries. Napier's rods (or bones), developed by John Napier in the early 1600s, were modeled on lattice multiplication. ●

Divisions such as the one in Figure 45 are usually shown in elementary school texts in the most efficient form, as in Figure 46(b), in which the numbers in color in Figure 46(a) are omitted. The technique used in Figure 46(a) is often called the **scaffold algorithm** and may be used as a preliminary step to achieving the standard algorithm, as in Figure 46(b). Scaffolding takes the numbers on the right in Figure 45(b), and places them on the top as in Figure 46(a). The scaffolding shows place value. Place value is important to understanding the standard algorithm.

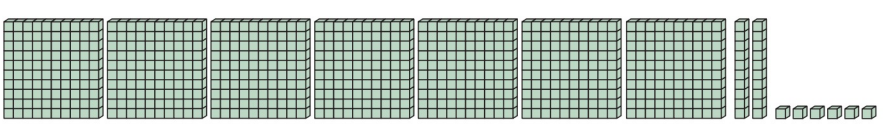
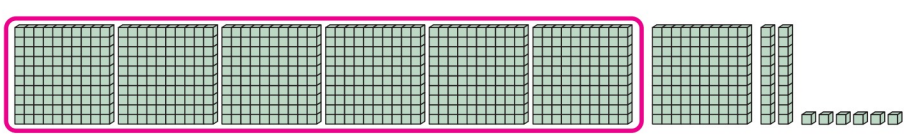
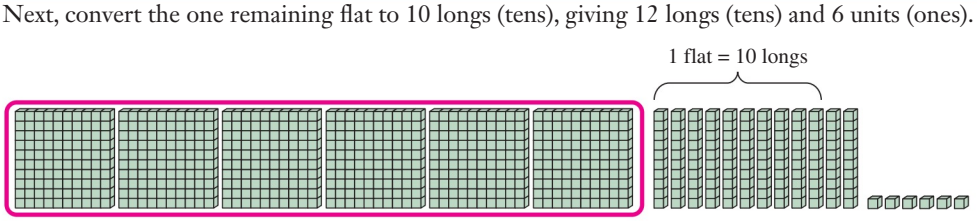
```
  121                 121
    1               6)726
   20               −6
  100                12
6)726               −12
−600                  6
  126                 −6
 −120                  0
    6                 (b)
   −6
    0
   (a)
```

Figure 46

Using Base-ten Blocks to Develop the Standard Division Algorithm

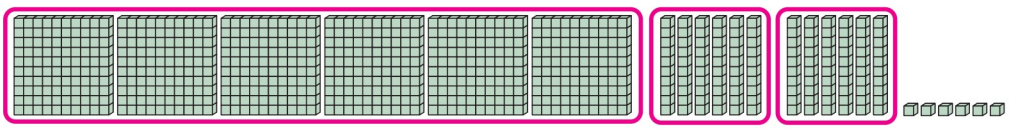
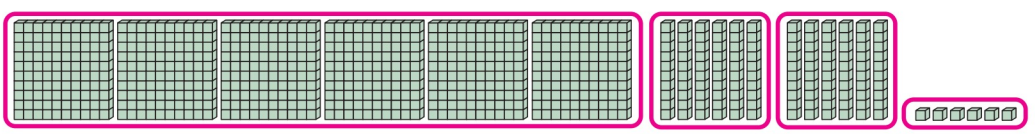
Students need to see why each move in an algorithm is appropriate rather than just knowing which sequence of moves to make. Below we use base-ten blocks to justify why each move in the standard algorithm is appropriate. In Table 12 we perform the division 726 ÷ 6. The base-ten block model is on the left with the corresponding steps in the standard algorithm on the right.

Table 12

Base-ten Blocks	Algorithm
1. First represent 726 with base-ten blocks. 	$6\overline{)726}$
2. Next determine how many sets of 6 flats (hundreds) there are in the representation. There is 1 set of 6 flats with 1 flat, 2 longs (tens), and 6 units (ones) remaining. 	1 set of 6 flats $\begin{array}{r} 1 \\ 6\overline{)726} \\ -\,6 \\ \hline 1 \end{array}$ 1 flat 2 longs 6 units remaining
3. Next, convert the one remaining flat to 10 longs (tens), giving 12 longs (tens) and 6 units (ones). 	1 set of 6 flats $\begin{array}{r} 1 \\ 6\overline{)726} \\ -\,6 \\ \hline 12 \end{array}$ 12 longs 6 units remaining

(*continued*)

Table 12 *(continued)*

Base-ten Blocks	Algorithm
4. Then determine how many sets of 6 longs (tens) there are in 12 longs and 6 units, giving 2 sets of 6 longs and 6 units left over. 	1 set of 6 flats 2 sets of 6 longs $\begin{array}{r} 12 \\ 6\overline{)726} \\ -6 \\ \hline 12 \\ -12 \\ \hline 6 \end{array}$ 6 units left over
5. Finally determine how many sets of 6 units (ones) there are in the 6 remaining units. There is 1 set of 6 units with no units left over (the remainder is 0). 	1 set of 6 flats 2 sets of 6 longs 1 set of 6 units $\begin{array}{r} 121 \\ 6\overline{)726} \\ -6 \\ \hline 12 \\ -12 \\ \hline 6 \\ -6 \\ \hline 0 \end{array}$ 0 remainder

Therefore, in the base-ten block representation of 726, there is 1 group of 6 flats (hundreds), 2 groups of 6 longs (tens), and 1 group of 6 units (ones) with none left over. Hence, the quotient is 121 with a remainder of 0. The steps in the algorithm are shown alongside the work with the base-ten blocks.

Using an Area Model to Develop the Standard Division Algorithm

The division $726 \div 6$ can be viewed using an area model. We can think of this as finding the unknown side length of a rectangular region with area 726 square units and side length 6 units as shown in Figure 47. The amount of hundreds is found, then tens, then ones. This yields three regions of dimensions 6 by 100, 6 by 20, and 6 by 1 as seen in Figure 47(b). It can be connected with the decomposition of 726 as $6 \cdot 100$, $6 \cdot 20$, and $6 \cdot 1$. By the distributive property, this is $6(100 + 20 + 1) = 6 \cdot 121 = 726$, so the unknown side length is 121. This can be recorded as in Figure 46(a).

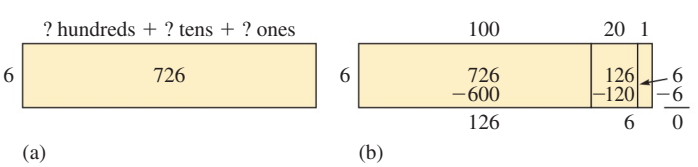


Figure 47

NOW TRY THIS 19

Using base-ten blocks, perform the division $726 \div 6$ by dividing the blocks into 6 equal piles. Compare your answer to that in Table 12. Which technique do you prefer? Why?

Short Division

The process used in Table 12 is usually referred to as "long" division. Another technique, called the **short division algorithm**, can be used when the divisor is a one-digit number and most of the work is done mentally. An example of the short division algorithm is given in Figure 48.

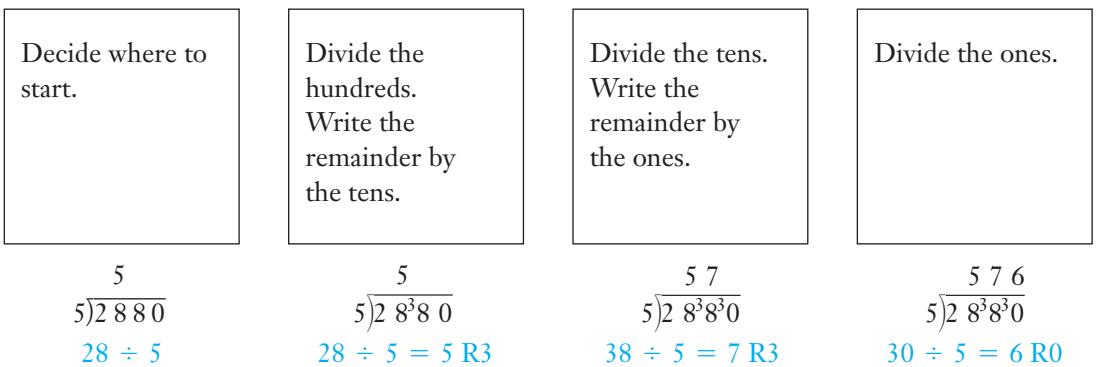


Figure 48

Division by a Two-digit Divisor

An example of division by a divisor of more than one digit is given next. Consider $32\overline{)2618}$.

1. Estimate the quotient in $32\overline{)2618}$. Because $1 \cdot 32 = 32$, $10 \cdot 32 = 320$, and $100 \cdot 32 = 3200$, the quotient is between 10 and 100.
2. Find the number of tens in the quotient. Because $26 \div 3$ is approximately 8, 26 hundreds divided by 3 tens is approximately 8 tens. Then write the 8 in the tens place, as shown:

$$\begin{array}{r} 80 \\ 32\overline{)2618} \\ -2560 \quad (32 \cdot 80) \\ \hline 58 \end{array}$$

3. Find the number of units in the quotient. Because $5 \div 3$ is approximately 1, 5 tens divided by 3 tens is approximately 1. This is shown on the left, with the standard algorithm shown on the right.

$$\begin{array}{r} 81 \\ 1 \\ 80 \\ 32\overline{)2618} \\ -2560 \\ \hline 58 \\ -32 \quad (32 \cdot 1) \\ \hline 26 \end{array} \quad \rightarrow \quad \begin{array}{r} 81 \text{ R}26 \\ 32\overline{)2618} \\ -256 \\ \hline 58 \\ -32 \\ \hline 26 \end{array}$$

 Normally grade-school books show the format on the right, which places the remainder beside the quotient.
4. Check: $32 \cdot 81 + 26 = 2618$.

Multiplication and Division in Different Bases

In multiplication, as with addition and subtraction, we identify the basic facts of single-digit multiplication before developing any algorithms. The multiplication facts for base five are given in Table 13. These facts can be derived by using repeated addition.

Table 13 Base Five Multiplication Table

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

There are various ways to compute the product $21_{five} \cdot 3_{five}$.

Fives	Ones
2	1
×	3

$(20 + 1)_{five}$
$\times \quad 3_{five}$
$(110 + 3)_{five}$

21_{five}
$\times 3_{five}$
3
110
113_{five}

21_{five}
$\times \ 3_{five}$
113_{five}

The multiplication of a two-digit number by a two-digit number is developed next:

23_{five}
$\times 14_{five}$
22
130
30
200
432_{five}

$(4 \cdot 3)_{five}$
$(4 \cdot 20)_{five}$
$(10 \cdot 3)_{five}$
$(10 \cdot 20)_{five}$

23_{five}
$\times 14_{five}$
202
230
432_{five}

Lattice multiplication can also be used to multiply numbers in various number bases. This is explored in Assessments 3-5A and B.

Division in different bases can be performed using the multiplication facts and the definition of division. For example, $22_{five} \div 3_{five} = c$ if, and only if, $c \cdot 3_{five} = 22_{five}$. From Table 13, we see that $c = 4_{five}$. As in base ten, computing multi-digit divisions efficiently in different bases requires practice. The ideas behind the algorithms for division can be developed by using subtraction. For example, $3241_{five} \div 43_{five}$ is computed by means of the subtraction technique in Figure 49(a) and by means of the standard algorithm in Figure 49(b). Thus, $3241_{five} \div 43_{five}$ results in quotient 34_{five} and remainder 14_{five}. Using the **division algorithm**, this can be written as $3241_{five} = 34_{five} \cdot 43_{five} + 14_{five}$.

$43_{five})\overline{3241_{five}}$
-430 $(10 \cdot 43)_{five}$
2311
-430 $(10 \cdot 43)_{five}$
1331
-430 $(10 \cdot 43)_{five}$
401
-141 $(2 \cdot 43)_{five}$
-210
-141 $(2 \cdot 43)_{five}$
14 $(34 \cdot 43)_{five}$

(a)

$34_{five}\text{R}14_{five}$
$43_{five})\overline{3241_{five}}$
-234
401
-332
14_{five}

(b)

Figure 49

Computations involving base two are demonstrated in Example 12.

Example 12

a. Multiply:

$$\begin{array}{r} 101_{\text{two}} \\ \times\ 11_{\text{two}} \\ \hline \end{array}$$

b. Divide:

$$101_{\text{two}}\overline{)110110_{\text{two}}}$$

Solution

a.
$$\begin{array}{r} 101_{\text{two}} \\ \times 11_{\text{two}} \\ \hline 101 \\ 101 \\ \hline 1111_{\text{two}} \end{array}$$

b.
$$\begin{array}{r} 1010_{\text{two}}\text{R}100_{\text{two}} \\ 101_{\text{two}}\overline{)110110_{\text{two}}} \\ -101 \\ \hline 111 \\ -101 \\ \hline 100_{\text{two}} \end{array}$$

IMAP Video

Watch Gilberto find the product of whole numbers using the Front-end multiplying strategy.

Mental Computation: Multiplication

As with addition and subtraction, mental computation is useful for multiplication. For example, consider $8 \cdot 26$. Students may think of this computation in a variety of ways, as shown here.

$26 = 20 + 6$	$26 = 25 + 1$	$26 = 30 - 4$
$8 \cdot 20$ is 160;	$8 \cdot 25$ is 200;	$8 \cdot 30$ is 240;
$8 \cdot 6$ is 48, so	$8 \cdot 1$ is 8, so	subtract $8 \cdot 4 = 32$.
$8 \cdot 26$ is $160 + 48 = 208$	$8 \cdot 26$ is $200 + 8 = 208$	$8 \cdot 26 = 240 - 32 = 208$

Next we consider several of the most common strategies for performing mental computation using multiplication.

1. *Front-end multiplying*

$$\begin{array}{r} 64 \\ \times\ 5 \\ \hline \end{array} \qquad \begin{array}{r} 60 \cdot 5 = 300 \\ 4 \cdot 5 = 20 \\ 300 + 20 = 320 \end{array}$$

(Multiply the number of tens in the first number by 5.)
(Multiply the number of units in the first number by 5.)
(Add the two products.)

2. *Using compatible numbers*

$2 \cdot 9 \cdot 5 \cdot 20 \cdot 5$ Rearrange as $9 \cdot (2 \cdot 5) \cdot (20 \cdot 5) = 9 \cdot 10 \cdot 100 = 9000$.

3. *Thinking money*

a.
$$\begin{array}{r} 64 \\ \times\ 5 \\ \hline \end{array}$$
(Think of the product as 64 nickels, which can be thought of as 32 dimes, which is $32 \times 10 = 320$ cents.)

b.
$$\begin{array}{r} 64 \\ \times 50 \\ \hline \end{array}$$
(Think of the product as 64 half-dollars, which is 32 dollars, or 3200 cents.)

c.
$$\begin{array}{r} 64 \\ \times 25 \\ \hline \end{array}$$
(Think of the product as 64 quarters, which is 32 half-dollars, or 16 dollars. Thus, we have 1600 cents.)

Mental Computation: Division

1. *Breaking up the dividend*

$7\overline{)4256}$ $7\overline{)42 \mid 56}$ (Break up the dividend into parts.)

$$\begin{array}{r} 600 + 8 \\ 7\overline{)4200 + 56} \end{array}$$ (Divide both parts by 7.)

$600 + 8 = 608$ (Add the answers together.)

2. *Using compatible numbers*

a. $3\overline{)105}$ $105 = 90 + 15$ (Look for numbers that you recognize as divisible by 3 and having a sum of 105.)

$$\begin{array}{r} 30 + 5 \\ 3\overline{)90 + 15} \end{array}$$ Thus $105 \div 3 = 35$. (Divide both parts and add the answers.)

b. $8\overline{)232}$ $232 = 240 - 8$ (Look for numbers that are easily divisible by 8 and whose difference is 232.)

$$\begin{array}{r} 30 - 1 \\ 8\overline{)240 - 8} \end{array}$$ Thus $232 \div 8 = 29$. (Divide both parts and take the difference.)

Computational Estimation

Computational estimation may help determine whether an answer is reasonable or not. This is especially useful when the computation is done on a calculator. Some common estimation strategies for multiplication and division are given next.

1. *Front-end*

$$\begin{array}{r} 524 \\ \times 8 \end{array}$$ $500 \cdot 8 = 4000$ (Start multiplying at the front to obtain a first estimate.)

$20 \cdot 8 = 160$ (Multiply the next important digit by 8.)

$4000 + 160 = 4160$ (Adjust the first estimate by adding the two numbers.)

2. *Compatible numbers*

$5\overline{)4163}$ $5\overline{)4000}$ (Change 4163 to a number close to it that you know is divisible by 5.)

$$\begin{array}{r} 800 \\ 5\overline{)4000} \end{array}$$ (Carry out the division and obtain the first estimate of 800. Various techniques can be used to adjust the first estimate.)

NOW TRY THIS 20

Estimate each of the following mentally, and explain what technique you used to find the answer.

a. A sold-out concert was held in a theater with a capacity of 4525 people. Tickets were sold for $9 each. Approximately how much money was collected?

b. Fliers are to be delivered to 3625 houses, and there are 42 people who will be doing the distribution. If distributed equally, approximately how many houses will each person visit?

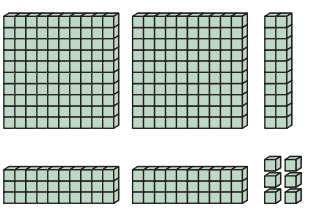
Assessment 3-5A

1. Simplify each of the following multiplications using properties of exponents. Leave answers as powers.
 a. $5^7 \cdot 5^{12}$ **b.** $6^{10} \cdot 6^2 \cdot 6^3$
 c. $10^{296} \cdot 10^{17}$ **d.** $2^7 \cdot 10^5 \cdot 5^7$

2. **a.** Which is greater, $2^{80} + 2^{80}$ or 2^{100}? Why?
 b. Which is greatest, $4 \cdot 2^{99}$, $3 \cdot 2^{100}$, or 2^{102}? Why?

3. Illustrate how to use base-ten blocks to compute $3 \cdot 42$.

4. The following model illustrates $22 \cdot 13$.



 a. Explain how the partial products are shown in the figure.
 b. Draw a similar model for $15 \cdot 21$.
 c. Draw a similar base-five model for the product $43_{five} \cdot 23_{five}$. Explain how the model can be used to find the answer in base five.

5. Fill in the missing numbers in each of the following computations.

a.
$$\begin{array}{r} 4_6 \\ \times\ 783 \\ \hline 1_78 \\ 3408 \\ _982_ \\ \hline 3335_8 \end{array}$$

b.
$$\begin{array}{r} 327 \\ \times\ 9_1 \\ \hline 327 \\ 1_08 \\ _9_3_ \\ \hline 30_\ _07 \end{array}$$

6. Perform the following multiplications using the lattice multiplication algorithm:
 a. 728×94 **b.** 306×24

7. The following chart displays the average daily water use per person for various countries; an approximate population for the countries is also given. Compute the daily total water use in each country.

Country/population	Daily Water Use per Person in Liters
India/1,200,000,000	140
Brazil/200,000,000	200
Nigeria/160,000,000	40
Japan/130,000,000	375
Australia/23,000,000	500
Kuwait/3,000,000	500
Bahrain/800,000	450

8. **a.** Use the distributive property of multiplication over addition to explain why multiplication of a natural number in base two by 10_{two} results in appending 0 to the number.
 b. Use part (b) to explain why multiplication in base two by 100_{two} results in appending 00 to the number.
 c. Use the distributive property of multiplication over addition and parts (a) and (b) to compute $110_{two} \cdot 11_{two}$.

9. The Russian peasant algorithm for multiplying $27 \cdot 68$ follows. (Disregard remainders when halving.)

Halves			Doubles	
	→	27 ×	(68)	
Halve 27	→	13	(136)	Double 68
Halve 13	→	6	272	Double 136
Halve 6	→	3	(544)	Double 272
Halve 3	→	1	(1088)	Double 544

In the "Halves" column, choose the odd numbers marked by an arrow. In the "Doubles" column, circle the numbers paired with the odds from the "Halves" column. Add the circled numbers.

$$\begin{array}{r} 68 \\ 136 \\ 544 \\ \underline{1088} \\ 1836 \end{array}$$ This is the product of $27 \cdot 68$.

Try the Russian peasant algorithm for $17 \cdot 63$.

10. Answer the following questions based on the activity chart provided:

Activity	Calories Burned per Hour
Playing tennis	462
Snowshoeing	708
Cross-country skiing	444
Playing volleyball	198

 a. How many calories are burned during 3 hours of cross-country skiing?
 b. Jane played tennis for 2 hours while Carolyn played volleyball for 3 hours. Who burned more calories, and how many more?
 c. Lyle went snowshoeing for 3 hours and Maurice went cross-country skiing for 5 hours. Who burned more calories, and how many more?

11. On a 14-day vacation, Glenn increased his caloric intake by 1500 calories per day. He also worked out more than usual by swimming 2 hours a day. Swimming burns 666 calories per hour, and a net gain of 3500 calories adds 1 lb of weight. Did Glenn gain at least 1 lb during his vacation?

12. Dave purchased a $50,000 life insurance policy at the price of $30 for each $1000 of coverage. If he pays the annual premium quarterly, how much is each installment?

13. Perform each of the following divisions using both the repeated-subtraction and standard algorithms.
a. $8\overline{)623}$ **b.** $36\overline{)298}$
c. $391\overline{)4001}$

14. Place the digits 4, 5, 7, and 3 in the boxes $\square\overline{)\square\square\square}$ to obtain
a. the greatest quotient.
b. the least quotient.

15. Using a calculator, Ralph multiplied by 10 when he should have divided by 10. The display read 300. What should the correct answer be?

16. Consider the following multiplications. Notice that when the digits in the factors are reversed, the products are the same.

$$\begin{array}{r} 36 \\ \times\ 42 \\ \hline 1512 \end{array} \qquad \begin{array}{r} 63 \\ \times\ 24 \\ \hline 1512 \end{array}$$

a. Find other multiplications where this procedure works.
b. Find a pattern for the numbers that work in this way.

17. Dan has 4520 pennies in three boxes. He says that there are 3 times as many pennies in the first box as in the third and twice as many in the second box as in the first. How much does he have in each box?

18. Gina buys apples from an orchard and then sells them at a country fair in bags 3 for $1. She bought 50 boxes of apples, 36 apples in a box, and paid $452. If she sold all but 18 apples, what was her total profit?

19. Discuss possible error patterns in each of the following computations.

a.
$$\begin{array}{r} 35 \\ \times\ 26 \\ \hline 90 \end{array}$$

b.
$$\begin{array}{r} 53 \\ 5\overline{)2515} \\ -25 \\ \hline 15 \\ -15 \\ \hline 0 \end{array}$$

20. Give reasons for each of the following steps.

$$\begin{aligned} 56 \cdot 10 &= (5 \cdot 10 + 6) \cdot 10 \\ &= (5 \cdot 10) \cdot 10 + 6 \cdot 10 \\ &= 5 \cdot (10 \cdot 10) + 6 \cdot 10 \\ &= 5 \cdot 10^2 + 6 \cdot 10 \\ &= 5 \cdot 10^2 + 6 \cdot 10 + 0 \cdot 1 \\ &= 560 \end{aligned}$$

21. **a.** Find all whole numbers that leave remainder 3 upon division by 4. Write your answer using set-builder notation.
b. Write the numbers from part (a) in a sequence in increasing order.
c. What kind of sequence is this? Why?

22. For what possible bases are each of the following computations correct?

a.
$$\begin{array}{r} 213 \\ +\ 308 \\ \hline 522 \end{array}$$

b.
$$\begin{array}{r} 213 \\ \times\ 32 \\ \hline 430 \\ 1043 \\ \hline 11300 \end{array}$$

23. Use lattice multiplication to compute $323_{\text{five}} \cdot 42_{\text{five}}$.

24. Find the least values of a and b such that $32_a = 23_b$. Explain your reasoning.

25. Place the digits 7, 6, 8, and 3 in the boxes to obtain

$$\begin{array}{r} \square\square\square \\ \times\ \square \\ \hline \end{array}$$

a. the greatest product.
b. the least product.

26. Perform each of these operations using the bases shown.
a. $32_{\text{five}} \cdot 4_{\text{five}}$
b. $32_{\text{five}} \div 4_{\text{five}}$
c. $43_{\text{six}} \cdot 23_{\text{six}}$
d. $143_{\text{five}} \div 3_{\text{five}}$
e. $10010_{\text{two}} \div 11_{\text{two}}$
f. $10110_{\text{two}} \cdot 101_{\text{two}}$

27. A car trip took 8 hr of driving at an average of 62 mph. Mentally compute the total number of miles traveled.

28. A theater has 38 rows with 23 seats in each row. Estimate the number of seats in the theater and tell how you arrived at your estimate.

29. Without computing, tell which of the following have the same answer. Describe your reasoning.
a. $44 \cdot 22$ and $22 \cdot 11$
b. $22 \cdot 32$ and $11 \cdot 64$
c. $13 \cdot 33$ and $39 \cdot 11$

30. In each of the following, determine whether the estimate given in parentheses is high (higher than the actual answer) or low (lower than the actual answer). Justify your answers without computing the exact values.
a. $299 \cdot 300$ (90,000)
b. $6001 \div 299$ (20)
c. $6000 \div 299$ (20)
d. $999 \div 99$ (10)

31. Use your calculator to calculate 25^2, 35^2, 45^2, and 55^2, and then see if you can find a pattern that will let you find 65^2 and 75^2 mentally.

32. A student uses front-end estimation to estimate the product of two numbers as 42,000. List a pair of possible factors.

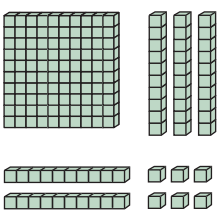
Assessment 3-5B

1. Simplify each of the following multiplications using properties of exponents. Leave answers as powers.
a. $3^8 \cdot 3^4$ **b.** $5^2 \cdot 5^4 \cdot 5^2$
c. $6^2 \cdot 2^2 \cdot 3^2$ **d.** $4^8 \cdot 8^4 \cdot 32^5$

2. **a.** Which is greater, $2^{20} + 2^{20}$ or 2^{21}? Why?
b. Which is greatest, 3^{31}, $9 \cdot 3^{30}$, or $3 \cdot 3^{32}$? Why?

3. Illustrate how to use base-ten blocks to compute $4 \cdot 13$.

4. The following model illustrates $13 \cdot 12$.



a. Explain how the partial products are shown in the figure.
b. Draw a similar model for $12 \cdot 22$.
c. Draw a similar base-five model for the product $14_{\text{five}} \cdot 23_{\text{five}}$. Explain how the model can be used to find the answer in base five.

5. Fill in the missing numbers in the following.

$$\begin{array}{r} 4_4 \\ \times\ 3\ 2\ 7 \\ \hline 3_8\ 8 \\ 9\ 6\ 8 \\ _4\ 5\ 2 \\ \hline 1\ 5\ 8\ 2_8 \end{array}$$

6. Perform the following multiplications using the lattice multiplication algorithm.
a. $\begin{array}{r} 327 \\ \times\ 43 \\ \hline \end{array}$ **b.** $\begin{array}{r} 2618 \\ \times\ 137 \\ \hline \end{array}$

7. The following chart gives average water usage for various activities for one person for one day.

Use	Average Amount
Taking bath	110 L (liters)
Taking shower	75 L
Flushing toilet	22 L
Washing hands, face	7 L
Getting a drink	1 L
Brushing teeth	1 L
Doing dishes (one meal)	30 L
Cooking (one meal)	18 L

a. Use the chart to calculate how much water you use each day.
b. An average American uses approximately 200 L of water per day and there are approximately 310,000,000 people in the United States. On average approximately how much water is used in the United States per day?

8. **a.** Use the distributive property of multiplication over addition to explain why multiplication of a natural number in base five by 10_{five} results in appending 0 to the number.
b. Use part (b) to explain why multiplication of a natural number in base five by 100_{five} results in appending two 0s to the number.
c. Use the distributive property of multiplication over addition and part (b) to compute $14_{\text{five}} \cdot 23_{\text{five}}$.

9. Use the Russian peasant algorithm from exercise 9 in Assessment 3-5A to find $31 \cdot 69$.

10. Answer the following questions based on the activity chart.

Activity	Calories Burned per Hour
Playing tennis	462
Snowshoeing	708
Cross-country skiing	444
Playing volleyball	198

a. How many calories are burned during 4 hours of cross-country skiing?
b. Jane played tennis for 3 hours while Carolyn played volleyball for 4 hours. Who burned more calories, and how many more?
c. Lyle went snowshoeing for 4 hours and Maurice went cross-country skiing for 5 hours. Who burned more calories, and how many more?

11. On a 14-day vacation, Glenn increased his calorie intake by 1800 calories per day. He also worked out more than usual by swimming 3 hours a day. Swimming burns 666 calories per hour, and a net gain of 3500 calories adds 1 lb of weight. Did Glenn gain at least 1 lb during his vacation?

12. Sue purchased a $30,000 life-insurance policy at the price of $24 for each $1000 of coverage. If she pays the premium in 12 monthly installments, how much is each installment?

13. Perform each of the following divisions using both the repeated-subtraction and the standard algorithms.
a. $7\overline{)392}$ **b.** $37\overline{)925}$
c. $423\overline{)5002}$

14. Place the digits 7, 6, 8, and 3 in the boxes $\square\overline{)\square\square\square}$ to obtain
a. the greatest quotient.
b. the least quotient.

15. Using a calculator, Jody multiplied by 5 when she should have divided by 5. The display read 250. What should the correct answer be?

16. A student wrote the following addition problem to his parents.

$$\begin{array}{r} \text{SEND} \\ +\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

If each letter represents a different digit how much money did the student request?

17. Debbie has 340 dimes in three boxes. She says that there are 4 times as many dimes in the first box as in the second and 3 times as many in the third box as in the first. How much money in dollars does she have in each box?

18. Xuan saved \$5340 in 3 years. If he saved \$95 per month in the first year and a fixed amount per month for the next 2 years, how much did he save per month during the last 2 years?

19. Discuss possible error patterns in each of the following multiplications

a.
$$\begin{array}{r} 34 \\ \times\ 8 \\ \hline 2432 \end{array}$$

b.
$$\begin{array}{r} 34 \\ \times\ 6 \\ \hline 114 \end{array}$$

20. Give reasons for each of the following steps.

$$\begin{aligned} 35 \cdot 100 &= (3 \cdot 10 + 5)100 \\ &= (3 \cdot 10 + 5)10^2 \\ &= (3 \cdot 10)10^2 + 5 \cdot 10^2 \\ &= 3(10 \cdot 10^2) + 5 \cdot 10^2 \\ &= 3 \cdot 10^3 + 5 \cdot 10^2 \\ &= 3 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 0 \cdot 1 \\ &= 3500 \end{aligned}$$

21. **a.** Find all the whole numbers that leave remainder 1 upon division by 4. Write your answer using set-builder notation.

b. Write the numbers from part (a) in a sequence in increasing order.

c. What kind of sequence is this?

22. For what possible bases are each of the following computations correct?

a.
$$\begin{array}{r} 322 \\ -\ 233 \\ \hline 23 \end{array}$$

b.
$$\begin{array}{r} 101 \\ 11\overline{)1111} \\ -\ 11 \\ \hline 11 \\ -\ 11 \\ \hline 0 \end{array}$$

23. Use lattice multiplication to compute $423_{\text{five}} \cdot 23_{\text{five}}$.

24. Find the least values of a and b such that $41_a = 14_b$.

25. Place the digits 7, 6, 8, 3, and 2 in the boxes to obtain

$$\begin{array}{r} \square\square\square \\ \times\ \ \square\square \end{array}$$

a. the greatest product. **b.** the least product.

26. Perform each of these operations using the bases shown.

a. $42_{\text{five}} \cdot 3_{\text{five}}$ **b.** $22_{\text{five}} \div 4_{\text{five}}$

c. $32_{\text{five}} \cdot 42_{\text{five}}$ **d.** $1313_{\text{five}} \div 23_{\text{five}}$

e. $101_{\text{two}} \cdot 101_{\text{two}}$ **f.** $1001_{\text{two}} \div 11_{\text{two}}$

27. A car trip took 6 hours at an average of 65 mph. Mentally compute the total number of miles traveled.

28. About 3540 calories must be burned to lose 1 lb of body weight. Estimate how many calories must be burned to lose 6 lb.

29. Without computing, tell which of the following have the same answer. Describe your reasoning.

a. $88 \cdot 44$ and $44 \cdot 22$ **b.** $93 \cdot 15$ and $31 \cdot 45$

c. $12 \cdot 18$ and $20 \cdot 17$

30. In each of the following, determine if the estimate given in parentheses is high (higher than the actual answer) or low (lower than the actual answer). Justify your answers without computing the exact values.

a. $398 \cdot 500$ (200,000) **b.** $8001 \div 398$ (20)

c. $10{,}000 \div 999$ (10) **d.** $1999 \div 201$ (10)

31. Use your calculator to multiply several two-digit numbers by 99. Then see if you can find a pattern that will let you find the product of any two-digit number and 99 mentally.

32. A student uses front-end estimation to estimate the product of two numbers as 2400. List a pair of possible factors.

Mathematical Connections 3-5

Reasoning

1. How would you explain to students how to multiply $345 \cdot 678$, assuming that they know and understand multiplication by a single digit and multiplication by a power of 10?

2. What happens when you multiply any two-digit number by 101? Explain why this happens.

3. Pick a number. Double it. Multiply the result by 3. Add 24. Divide by 6. Subtract your original number. Is the result always the same? Write a convincing argument for what happens.

4. Do you think it is valuable for students to see more than one method of doing computation problems? Why or why not?

5. Tom claims that long division should receive reduced attention in elementary classrooms. Do you agree or disagree? Defend your answer. (Check the web for related research.)

6. For the given computation, answer the following questions.

$$\begin{array}{r} 122\text{R}37 \\ 43\overline{)5283} \\ -\ 43 \\ \hline 98 \\ -\ 86 \\ \hline 123 \\ -\ 86 \\ \hline 37 \end{array}$$

a. Why is the 1 in the quotient above the 2 in the dividend?

b. Why can we just bring down the 8 to obtain 98?

c. Why can we just bring down the 3 to obtain 123?

d. What does R37 mean?

7. For the base-five computation below, answer the following questions.

$$\begin{array}{r} 1 \\ 31_{\text{five}}\overline{)\,4021_{\text{five}}} \\ \underline{-31} \\ 4 \end{array}$$

a. What does the 4 in the subtraction represent?
b. Complete the division.

Open-Ended

8. If a student presented a new "algorithm" for computing with whole numbers, describe the process you would recommend to the student to determine whether the algorithm would always work.

9. Dina calculated each of the following mentally:

$$49 \cdot 51 + 49 \cdot 49 = 4900$$
$$98 \cdot 37 + 2 \cdot 37 = 3700$$
$$99 \cdot 37 + 37 = 3700$$

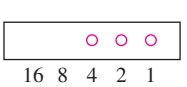
Explain how she performed the calculations and why her method works. Then give three similar calculations based on the same approach and find the answers mentally.

Cooperative Learning

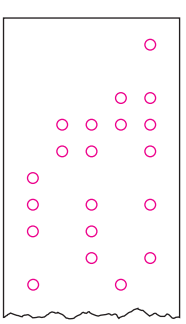
10. The traditional sequence for teaching operations in the elementary school is first addition, then subtraction, followed by multiplication, and finally division. Some educators advocate teaching addition followed by multiplication, then subtraction followed by division. Within your group, prepare arguments for teaching the operations in either order listed.

11. **a.** Without actually finding the answers determine which is greater: $19{,}876 \cdot 43$ or $19{,}875 \cdot 44$. Explain your approach. Compare the explanations with other members in your group.
b. Come up with three similar pairs of products and determine which is greater. Justify your answers. Compare your answers with the other students in your group.

12. Messages can be coded on paper tape in base two. A hole in the tape represents 1, whereas the absence of a hole represents 0. The value of each hole depends on its position; from left to right, the values are 16, 8, 4, 2, 1 (all powers of 2). Letters of the alphabet may be coded in base two according to their position in the alphabet. For example, G is the seventh letter. Since $7 = 1 \cdot 4 + 1 \cdot 2 + 1$, the holes appear as they do below.



a. Decode the message below.



b. Write your name on a tape using base two.
c. Consider the cards, which are modeled on base-two representations.

Card E

16	24
17	25
18	26
19	27
20	28
21	29
22	30
23	31

Card D

8	24
9	25
10	26
11	27
12	28
13	29
14	30
15	31

Card C

4	20
5	21
6	22
7	23
12	28
13	29
14	30
15	31

Card B

2	18
3	19
6	22
7	23
10	26
11	27
14	30
15	31

Card A

1	17
3	19
5	21
7	23
9	25
11	27
13	29
15	31

(i) Suppose a person in your group under 32 tells you that his age appears on cards E, C, and B. How can you use that information to find the person's age? Explain how the system works in general.
(ii) Design card F and redesign cards A to E so that the numbers 1 through 63 appear on the cards.

Connecting Mathematics to the Classroom

13. A student asks why she should learn the standard long division algorithm if she can get a correct answer using repeated subtraction. How do you respond?

14. A student divides as follows. How do you help?

$$\begin{array}{r} 15 \\ 6\overline{)36} \\ \underline{-\,6} \\ 30 \\ \underline{30} \end{array}$$

15. A student asks how you can find the quotient and the remainder in a division problem like $593 \div 36$ using a calculator that can perform only the four basic operations. How do you respond?

16. A student claims that to divide a number with the units digit 0 by 10, she just crosses out the 0 to get the answer. She wants to know if this is always true and if the 0 has to be the units digit. How do you respond?

17. **a.** A student notices that $39 + 41 = 40 + 40$ and wonders if it is also true that $39 \cdot 41 = 40 \cdot 40$. How do you reply?
b. Another student says that he knows that $39 \cdot 41 \neq 40 \cdot 40$ but he found that $39 \cdot 41 = 40 \cdot 40 - 1$. He also found that $49 \cdot 51 = 50 \cdot 50 - 1$. He wants to know if this pattern continues. How would you respond?

18. Howard is having trouble finding a number to fill in the box below. What might be his problem, and how do you help him?

$$\square \div 5 = 6$$

19. What errors do you think the student is making in each of the following? How would you help these students?

a.
$$\begin{array}{r} 120 \\ 4\overline{)682} \end{array}$$

b.
$$\begin{array}{r} 19\text{R}1 \\ 5\overline{)906} \\ -\,5 \\ \hline 4\;6 \\ -\,4\;5 \\ \hline 1 \end{array}$$

20. A student performs the following computation.

$$\begin{array}{r} {}^{3}{}^{4} \\ 6\,8 \\ \times\;5 \\ \hline 3\,4\,0 \end{array}$$

a. Explain the meaning of the 4 above the 6.
b. Explain the meaning of the 3 in the hundreds place.

21. A student performs the following computation in base five.

$$\begin{array}{r} {}^{3} \\ 3\,4_{\text{five}} \\ \times\;4_{\text{five}} \\ \hline 3\,0\,1_{\text{five}} \end{array}$$

Explain the meaning of the 3 above the 3.

Review Problems

22. Illustrate the identity property of addition for whole numbers.

23. Rename each of the following using the distributive property of multiplication over addition.

a. $ax + bx + 2x$
b. $3(a + b) + x(a + b)$

24. At the beginning of a trip, the mileage odometer registered 52,281. At the end of the trip, the odometer registered 59,260. How many miles were traveled on this trip?

25. Write each of the following division problems as a multiplication problem.

a. $36 \div 4 = 9$
b. $112 \div 2 = x$
c. $48 \div x = 6$
d. $x \div 7 = 17$

National Assessments

National Assessment of Educational Progress (NAEP) Questions

There will be 58 people at a breakfast and each person will eat 2 eggs. There are 12 eggs in each carton. How many cartons of eggs will be needed for the breakfast?
A. 9
B. 10
C. 72
D. 116

NAEP, Grade 4, 2007

Which of these would be easiest to solve by using mental math?
A. $65.12 − $28.19
B. 358 × 2
C. 1,625 ÷ 3
D. $100.00 + $10.00

NAEP, Grade 4, 2007

A student had to multiply 328 × 41. The student's answer was 4,598. Use estimation to explain why this answer is not reasonable.

NAEP, Grade 4, 2007

Hint for Solving the Preliminary Problem

Look at the columns and rows that have only two symbols. Compare these columns and rows. Look for a pattern.

Chapter 3 Summary

KEY CONCEPTS	DEFINITIONS, DESCRIPTIONS, AND THEOREMS
Section 3-1	
Numeration system (p. 85)	A collection of properties and symbols agreed upon to represent numbers.
Hindu-Arabic numeration system (p. 86)	A base-ten system constructed from the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. *Numerals*—symbols representing numbers.
Place value (p. 86)	The value assigned to a digit depending on its placement in a numeral.
Face value (p. 86)	The value of the digit.
Exponent (p. 86)	If a is any number and n is any natural number, then $a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$, where a is the *base* and n is the *exponent*. If $a \neq 0$, then $a^0 = 1$.
Different numeration systems (p. 87)	*Tally numeration system*—single strokes or tally marks used to represent numbers. *Egyptian numeration system*—a grouping system, consisting of seven basic symbols used to represent certain sets of numbers. *Babylonian numeration system*—a place-value system based on 60 using two basic symbols to represent numbers. *Mayan numeration system*—a place-value system based primarily on 20 using three basic symbols and vertical groupings. *Roman numeration system*—a system, consisting of seven symbols, that used *additive*, *subtractive*, and *multiplicative* properties. *Other number-base system*—a place-value system based on a number other than ten, which is studied to understand the base-ten system. Examples of such systems are *base two*, *base five*, and *base twelve*.
Section 3-2	
Addition of whole numbers (p. 101)	Let A and B be two disjoint finite sets. If $n(A) = a$ and $n(B) = b$, then $a + b = n(A \cup B)$.
Models for addition (p. 101)	• *Set* • *Number line*
Ordering whole numbers (p. 102)	*Less than*—for any whole numbers a and b, a *is less than* b, written $a < b$, if, and only if, there exists a natural number k such that $a + k = b$. *Greater than*—for any whole numbers a and b, a *is greater than* b, written $a > b$, if, and only if, $b < a$.
Addition properties for whole numbers (p. 103)	• *Closure*—if a and b are any whole numbers, then $a + b$ is a unique whole number. • *Commutative*—if a and b are any whole numbers, then $a + b = b + a$. • *Associative*—if a, b, and c are any whole numbers, then $(a + b) + c = a + (b + c)$. • *Identity*—there is a unique whole number 0, the *additive identity*, such that for any whole number a, $a + 0 = a = 0 + a$.
Subtraction of whole numbers (p. 106)	For any whole numbers a and b, such that $a \geq b$, $a - b$ is the unique whole number c such that $b + c = a$.
Models for subtraction (p. 106)	• *Take-away* • *Comparison* • *Missing addend* • *Number line*

Section 3-3

Multiplication of whole numbers (p. 116)	*Repeated addition approach*—for any whole number a and $n \neq 0$, $n \cdot a = \underbrace{a + a + a + \ldots + a}_{n \text{ terms}} = na$. If $n = 0$, then $0 \cdot a = 0$. $a \cdot b$, or ab, is the *product*, and a and b are *factors*. *Cartesian product approach*—for finite sets A and B, if $n(A) = a$ and $n(B) = b$, then $a \cdot b = n(A \times B)$.
Models for multiplication	• *Repeated addition* • *Number line* • *Cartesian product* • *Rectangular array* • *Area*
Multiplication properties for whole numbers (p. 118)	• *Closure*—if a and b are any whole numbers, then $a \cdot b$ is a unique whole number. • *Commutative*—if a and b are any whole numbers, then $a \cdot b = b \cdot a$. • *Associative*—if a, b, and c are any whole numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. • *Identity*—there is a unique whole number 1, the *multiplicative identity*, such that for any whole number a, $a \cdot 1 = a = 1 \cdot a$. • *Property of* 0—if a is any whole number, then $a \cdot 0 = 0 = 0 \cdot a$.
Distributive property of multiplication over addition or subtraction (p. 119)	If a, b, and c are any whole numbers, then $a(b + c) = ab + ac$. If a, b, and c are any whole numbers with $b \geq c$, then $a(b - c) = ab - ac$.
Division of whole numbers (p. 122)	*Missing-factor approach*—for any whole numbers a and b, with $b \neq 0$, $a \div b = c$ if, and only if, c is the unique whole number such that $b \cdot c = a$. The number a is the *dividend*, b is the *divisor*, and c is the *quotient*.
Models for division (p. 121)	• *Set (partition)* • *Missing-factor* • *Repeated subtraction*
Division Algorithm (p. 122)	For any whole numbers a and b with $b \neq 0$, there exist unique whole numbers q (quotient) and r (remainder) such that $a = bq + r$, with $0 \leq r < b$.
Relating multiplication and division (p. 124)	Whole number multiplication and division with remainder 0 are inverses of each other.
Division by 0 and 1 (p. 124)	• For any nonzero whole number n, $n \div 0$ is undefined and $0 \div n = 0$. • $0 \div 0$ is undefined. • For any whole number n, $n \div 1 = n$.
Order of operations (p. 126)	Multiplications and divisions are performed in the order they occur from left to right, and then the additions and subtractions are performed in the order they occur from left to right. Computations in parentheses are done first.

Section 3-4

Addition algorithms (p. 131)	• *Concrete model* • *Left-to-right* • *Lattice* • *Scratch*
Subtraction algorithms (p. 135)	• *Concrete model* • *Equal additions*
Addition and subtraction in bases other than ten	Computations involving a number-base system other than base ten are studied to better understand base-ten system algorithms.
Mental computation algorithms for addition (p. 139)	• *Adding from the left* • *Breaking and bridging* • *Trading off* • *Using compatible numbers* • *Making compatible numbers*

Section 3-4		
Mental computation algorithms for subtraction (p. 140)	• *Breaking and bridging* • *Trading off*	• *Drop the zeros* • *Making compatible numbers*
Estimation algorithms for addition and subtraction (p. 141)	• *Front-end with adjustment* • *Grouping to nice numbers* • *Clustering*	• *Rounding* • *Using the range*
Section 3-5		
Definition of a^n	If a and n are whole numbers and $n \neq 0$, then $a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$.	
Properties of whole number exponents (p. 150)	$a^1 = a$ If $a \neq 0$, then $a^0 = 1$. For whole numbers a, and natural numbers m, and n, $a^m \cdot a^n = a^{m+n}$. For whole numbers a, and natural numbers m, and n, $(a^m)^n = a^{mn}$. For whole numbers a and b, natural numbers n, $a^n \cdot b^n = (ab)^n$. For natural numbers a, m, and n with $m \geq n$, $a^m \div a^n = a^{m-n}$.	
Multiplication algorithms (p. 152)	• *Multiplication by 10^n*—results in appending n zeros • *Multiplication with two-digit factors* • *Lattice multiplication*	
Division algorithms (p. 156)	• *Using repeated subtraction* • *Using base-ten blocks, developing the standard division algorithm* • *Short division* • *Division by a two-digit divisor*	
Multiplication and division in bases other than ten	Computations involving a number-base system other than base ten are studied to understand better base-ten system algorithms.	
Mental computation algorithms for multiplication (p. 161)	• *Front-end multiplying* • *Using compatible numbers*	• *Thinking money*
Mental computation algorithms for division (p. 161)	• *Breaking up the dividend*	• *Using compatible numbers*
Estimation algorithms for multiplication and division (p. 162)	• *Front-end multiplication*	• *Compatible number division*

Chapter 3 Review

1. For each of the following base-ten numerals, tell the place value for the underlined digits.
 a. $4\underline{3}2$
 b. $\underline{3}432$
 c. $\underline{1}9324$

2. Convert each of the following to a base-ten number.
 a. $\overline{\text{CD}}\text{XLIV}$
 b. 432_{five}
 c. $ET0_{\text{twelve}}$
 d. 1011_{two}
 e. 4136_{seven}

3. Convert each of the following to a number in the indicated system.
 a. 999 to Roman
 b. 86 to Egyptian
 c. 123 to Mayan
 d. 346_{ten} to base five
 e. 27 to base two

4. Simplify each of the following products, if possible. Write your answers in exponential form, a^b.
 a. $3^4 \cdot 3^7 \cdot 3^6$
 b. $2^{10} \cdot 2^{11}$

5. Write the base-three numeral for the base-three blocks shown.



6. What is the fewest number of base-three blocks needed to represent 51?

7. Draw multibase blocks to represent each of the following.
- **a.** 123_{four}
- **b.** 24_{five}

8. **a.** The first digit from the left (the lead digit) of a base-ten numeral is 4 followed by 10 zeros. What is the place value of 4?
- **b.** A number in base five has 10 digits. What is the place value of the second digit from the left?
- **c.** A number in base two has lead digit 1 followed by 30 zeros and units digit 1. What is the place value of the lead digit?

9. Write the following base-ten numerals in the indicated base without performing any multiplications.
- **a.** $10^{10} + 23$ in base ten
- **b.** $2^{10} + 1$ in base two
- **c.** $5^{10} + 1$ in base five
- **d.** $10^{10} - 1$ in base ten
- **e.** $2^{10} - 1$ in base two
- **f.** $12^5 - 1$ in base twelve
- **g.** $7 \cdot 5^6 + 3 \cdot 5^4 + 11$ in base five.

10. Give an example of a base other than ten used in a real-life situation. How is it used?

11. Describe the important characteristics of each of the following systems.
- **a.** Egyptian
- **b.** Babylonian
- **c.** Roman
- **d.** Hindu-Arabic

12. Write 128 in each of the following bases.
- **a.** Five
- **b.** Two
- **c.** Twelve

13. Write each of the following numerals in the indicated bases without multiplying the various powers.
- **a.** $2^{10} + 2^3$ in base two
- **b.** $11 \cdot 12^5 + 10 \cdot 12^3 + 20$ in base twelve

14. If $123_b = 83$, solve for b.

15. For each of the following equations where $a, b \in W$, identify the properties of the operation(s) for whole numbers illustrated.
- **a.** $3(a + b) = 3a + 3b$
- **b.** $2 + a = a + 2$
- **c.** $16 \cdot 1 = 1 \cdot 16 = 16$
- **d.** $6(12 + 3) = 6 \cdot 12 + 6 \cdot 3$
- **e.** $3(a \cdot 2) = 3(2a)$
- **f.** $3(2a) = (3 \cdot 2)a$

16. Using the definitions of less than or greater than given in this chapter, prove that each of the following inequalities is true.
- **a.** $3 < 13$
- **b.** $12 > 9$

17. For each of the following equations, find all possible replacements to make the statement true for whole numbers.
- **a.** $4 \cdot \square - 37 < 27$
- **b.** $398 = \square \cdot 37 + 28$
- **c.** $\square \cdot (3 + 4) = \square \cdot 3 + \square \cdot 4$
- **d.** $42 - \square \geq 16$

18. Use the distributive property of multiplication over addition, other multiplication properties, and addition facts, if possible, to rename each of the following where $a, b, x, y \in W$.
- **a.** $3a + 7a + 5a$
- **b.** $3x^2 + 7x^2 - 5x^2$
- **c.** $x(a + b + y)$
- **d.** $(x + 5)3 + (x + 5)y$
- **e.** $3x^2 + x$
- **f.** $2x^5 + x^3$

19. How many 12-oz cans of juice would it take to give 60 people one 8-oz serving each?

20. Heidi has a brown pair and a gray pair of slacks; a brown blouse, a yellow blouse, and a white blouse; and a blue sweater and a white sweater. How many different outfits does she have if each outfit she wears consists of slacks, a blouse, and a sweater?

21. I am thinking of a whole number. If I divide it by 13, then multiply the answer by 12, then subtract 20, and then add 89, the result is 93. What was my original number?

22. A ski resort offers a weekend ski package for \$80 per person or \$6000 for a group of 80 people. Which would be the less expensive option for a group of 80?

23. Josi has a job in which she works 30 hr/wk and gets paid \$5/hr. If she works more than 30 hr in a week, she receives \$8/hr for each hour over 30 hr. If she worked 38 hr this week, how much did she earn?

24. In a television game show, there are five questions to answer. Each question is worth twice as much as the previous question. If the last question was worth \$6400, what was the first question worth?

25. **a.** Think of a number.
Add 17.
Double the result.
Subtract 4.
Double the result.
Add 20.
Divide by 4.
Subtract 20.
Your answer will be your original number. Explain how this trick works.
- **b.** Fill in two more steps that will take you back to your original number.
Think of a number.
Add 18.
Multiply by 4.
Subtract 7.
.
.
.
- **c.** Make up a series of instructions such that you will always get back to your original number.

26. Use both the scratch and the traditional algorithms to compute the following sum.

$$\begin{array}{r} 316 \\ 712 \\ +\ 91 \\ \hline \end{array}$$

27. Use both the traditional and the lattice multiplication algorithms to compute the following multiplication.

$$\begin{array}{r} 613 \\ \times\ 98 \\ \hline \end{array}$$

28. Use both the repeated-subtraction and the conventional algorithms to perform the following divisions.
- **a.** $912\overline{)4803}$
- **b.** $11\overline{)1011}$
- **c.** $23_{\text{five}}\overline{)3312_{\text{five}}}$
- **d.** $11_{\text{two}}\overline{)1011_{\text{two}}}$

29. Use the Division Algorithm to check your answers in exercise 28.

30. In some calculations a combination of mental math and a calculator is most appropriate. For example, because

$$\begin{aligned} 200 \cdot 97 \cdot 146 \cdot 5 &= 97 \cdot 146 (200 \cdot 5) \\ &= 97 \cdot 146 \cdot 1000, \end{aligned}$$

we can calculate $97 \cdot 146$ on a calculator and then mentally multiply by 1000. Show how to calculate each of the following products using a combination of mental math and a calculator.
- **a.** $19 \cdot 5 \cdot 194 \cdot 2$
- **b.** $379 \cdot 4 \cdot 193 \cdot 25$
- **c.** $8 \cdot 481 \cdot 73 \cdot 125$
- **d.** $374 \cdot 200 \cdot 893 \cdot 50$

31. Jim was paid \$320 a month for 6 months and \$410 a month for 6 month. What were his total earnings for the year?

32. A soft-drink manufacturer produces 15,600 cans of product each hour. Cans are packed 24 to a case. How many cases could be filled with the cans produced in 4 hours?

33. Apples normally sell for 32¢ each. They go on sale for 3 for 69¢. How much money is saved if you purchase 2 dozen apples while they are on sale?

34. The owner of a bicycle shop reported his inventory of bicycles and tricycles in an unusual way. He said he counted 126 wheels and 108 pedals. How many bikes and how many trikes did he have?

35. Perform each of the following computations.
- **a.** $\begin{array}{r} 123_{\text{five}} \\ +\ 34_{\text{five}} \\ \hline \end{array}$
- **b.** $\begin{array}{r} 1010_{\text{two}} \\ -101_{\text{two}} \\ \hline \end{array}$
- **c.** $\begin{array}{r} 23_{\text{five}} \\ \times\ 34_{\text{five}} \\ \hline \end{array}$
- **d.** $\begin{array}{r} 1001_{\text{two}} \\ \times\ 101_{\text{two}} \\ \hline \end{array}$

36. Use the distributive property of multiplication over addition to compute $44_{\text{five}} \cdot 34_{\text{five}}$.

37. Use repeated subtraction to find $434_{\text{five}} \div 4_{\text{five}}$ without first converting to base ten. Write the answer using the Division Algorithm.

38. Tell how to use compatible numbers mentally to perform each of the following operations.
- **a.** $26 + 37 + 24 - 7$
- **b.** $4 \cdot 7 \cdot 9 \cdot 25$

39. Compute each of the following operations mentally. Name the strategy you used to perform your mental math.
- **a.** $63 \cdot 7$
- **b.** $85 - 49$
- **c.** $(18 \cdot 5)2$
- **d.** $2436 \div 6$

40. Estimate the following addition using **(a)** front-end estimation with adjustment and **(b)** rounding.

$$\begin{array}{r} 543 \\ 398 \\ 255 \\ 408 \\ +\ 998 \\ \hline \end{array}$$

41. Using clustering, estimate the sum $2345 + 2854 + 2234 + 2203$.

42. In some cases, the distributive property of multiplication over addition or distributive property of multiplication over subtraction can be used to obtain an answer quickly. Compute each of the following.
- **a.** $999 \cdot 47 + 47$
- **b.** $43 \cdot 59 + 41 \cdot 43$
- **c.** $1003 \cdot 79 - 3 \cdot 79$
- **d.** $1001 \cdot 113 - 113$
- **e.** $101 \cdot 35$
- **f.** $98 \cdot 35$

43. Recall that addition problems like $3478 + 521$ can be written and computed using expanded notation as shown here, and answer the questions that follow.

$$\begin{array}{r} 3 \cdot 10^3 + 4 \cdot 10^2 + 7 \cdot 10 + 8 \\ +\qquad 5 \cdot 10^2 + 2 \cdot 10 + 1 \\ \hline 3 \cdot 10^3 + 9 \cdot 10^2 + 9 \cdot 10 + 9 \end{array}$$

- **a.** Write a corresponding addition algebra problem (use x for 10) and find the answer.
- **b.** Write a subtraction problem and the corresponding algebra problem and find the answer.
- **c.** Write a multiplication problem and the corresponding algebra problem and compute the answer.

44. Prove that $B = \{5n \mid n \in W\}$ is closed under addition.

45. The Washington School PTA set up a phone tree in order to reach all of its members. Each person's responsibility, after receiving a call, is to call two other assigned members until all members have been called. Assume that everyone is home and answers the phone and that each phone call takes 30 seconds. If one of the 85 members, the PTA president, makes the first phone call and starts the clock, what is the least amount of time necessary to reach all 85 members of the group?

46. Abby baked 124 cookies. While they were cooling, she stepped outside. She saw a group of children run into the kitchen and run back out with cookies in their hands. When she returned to the kitchen, she found only 7 cookies left on the cooking racks. If each child took the same number of cookies, and the number of children who took cookies was more than 10 and fewer than 30, how many were in the group?

47. The number on a license plate consists of five digits. When the license plate is looked at upside down, you can still read it, but the value of the upside-down number is 78,633 greater than the real license number. What is the license number?

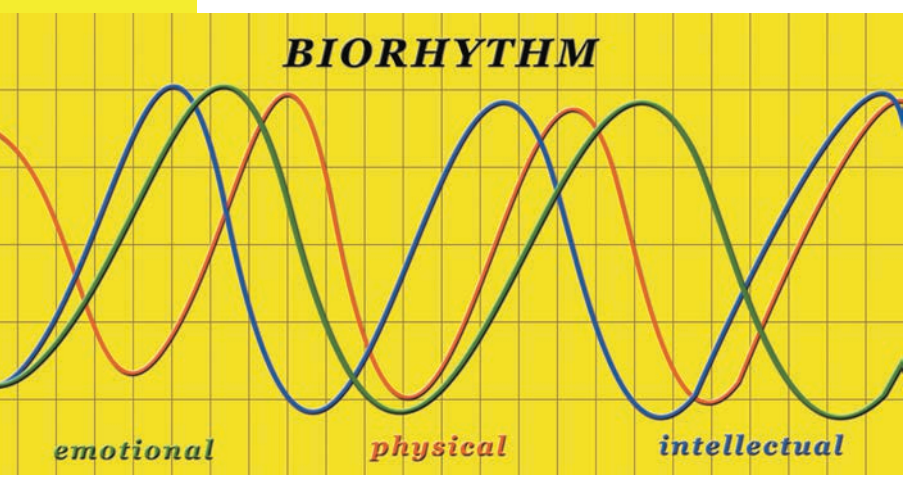
Chapter 4

Number Theory

4-1 Divisibility

4-2 Prime and Composite Numbers

4-3 Greatest Common Divisor and Least Common Multiple



Preliminary Problem

The theory of biorhythms states that there are three "cycles" to life, and each peaks at different times: the physical cycle peaks every 23 days, the emotional cycle peaks every 28 days, and the intellectual cycle peaks every 33 days.

Suppose that today the three cycles are all at their peak. In how many days will they all be at their peak again?

If needed, see Hint on page 215.

Number theory started to flourish in the seventeenth century with the work of Pierre de Fermat (1601–1665). Topics in number theory that occur in the elementary school curriculum include factors, multiples, divisibility tests, prime numbers, prime factorizations, greatest common divisors, and least common multiples. The topic of congruences, introduced by Carl Gauss (1777–1855), is also incorporated into the elementary curriculum through clock arithmetic and modular arithmetic. Clock and modular arithmetic (available online) give students a look at mathematical systems.

4-1 Divisibility

4-1 Objectives

Students will be able to understand and explain

- Divisibility, factors, and multiples.
- Divisibility tests for 2, 3, 4, 5, 6, 8, 9, 10, and 11.
- That the set of factors is finite, and the set of multiples is infinite for any given natural number.

CCSS **The grade 2 *Common Core Standards* state that students should be able to:**

Determine whether a group of objects (up to 20) has an odd or even number of members, for example, by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends. (p. 19)

The concepts *even* and *odd* for whole numbers are commonly used. For example, during summer water shortages in some parts of the country, houses with even-number addresses water on even-numbered days of the month, and houses with odd-number addresses water on odd-numbered days. An **even** whole number is a whole number that has remainder 0 when divided by 2. An **odd** whole number is a whole number that has remainder 1 when divided by 2. From the Division Algorithm if n is even then $n = 2q$ for some whole number q (the remainder is 0). If n is odd, then $n = 2q + 1$ (why?). In the Division Algorithm if the remainder is 0 when m is divided by n, m is **divisible by** n.

The fact that 2 divides 12 can be stated in the following equivalent statements in the left column.

Example	General Statement
12 is divisible by 2.	a is divisible by b.
2 is a divisor of 12.	b is a divisor of a.
12 is a multiple of 2.	a is a multiple of b.
2 is a factor of 12.	b is a factor of a.
2 divides 12.	b divides a.

The statement that "2 divides 12" is written with a vertical segment, as in $2 \mid 12$, where the vertical segment means **divides**. Likewise for $a, b \in W$, "b divides a" is written $b \mid a$. Each statement in the right column above can be written $b \mid a$. We write $5 \nmid 12$ to symbolize that 5 does not divide 12 or that 12 is not divisible by 5. The notation $5 \nmid 12$ also implies that 12 is not a multiple of 5 and 5 is not a factor of 12.

In general, if a is a whole number and b is a nonzero whole number, then a is divisible by b or equivalently b divides a if, and only if, the remainder when a is divided by b is 0. Using the Division Algorithm, this means that there is a unique whole number q (quotient) such that $a = bq$. Thus we have the following definition.

Definition of Divides

For any whole numbers a and b, where $b \neq 0$, b divides a, if, and only if, there is a unique whole number q such that $a = bq$.

If $b \mid a$, then b is a **factor**, or a **divisor**, of a, and a is a **multiple** of b.

Do not confuse $b|a$ with b/a, which means b divided by a (also written as $b \div a$). The former expresses a relation and is either true or false. The latter, b/a, expresses an operation on whole numbers and has a numerical value if $a \neq 0$. Note that if a/b is a whole number, then $b|a$. Also note that for whole numbers $b \nmid a$ is equivalent to saying that the remainder when a is divided by b is not 0.

Example 1

Classify each of the following as true or false. Explain your answers.

a. $3|12$
b. $0|2$
c. 0 is even.
d. $8 \nmid 2$
e. For all whole numbers a, $1|a$.
f. For all nonzero whole numbers a, $a^2|a^5$.
g. $3|6n$ for all whole numbers n.
h. $3 \nmid (5 \cdot 7 \cdot 9 \cdot 11 + 1)$
i. $0|0$

Solution

a. $3|12$ is true because $12 = 4 \cdot 3$.
b. $0|2$ is false because there is no whole number c such that $2 = c \cdot 0$.
c. 0 is even is true because $0 = 0 \cdot 2$.
d. $8 \nmid 2$ is true because there is no whole number c such that $2 = c \cdot 8$.
e. $1|a$ is true for all whole numbers a because $a = a \cdot 1$.
f. True, because $a^5 = a^2 \cdot a^3$.
g. $3|6n$ is true. Because $6n = 3(2n)$, $6n$ is a multiple of 3 and hence $3|6n$.
h. True, because $5 \cdot 7 \cdot 9 \cdot 11 + 1 = 3(5 \cdot 7 \cdot 3 \cdot 11) + 1$, so $5 \cdot 7 \cdot 9 \cdot 11 + 1$ leaves remainder 1 when divided by 3.
i. $0|0$ is false because $0 = 0 \cdot q$ for all whole numbers q, so q is not unique.

Suppose we have one pack of gum and we know that the number of pieces, a, is divisible by 5. Then if we had two packs, the total number of pieces of gum is still divisible by 5. The same is true if we had 10 packs, or 100 packs, or in general n packs where n is any whole number. We could record this observation as follows:

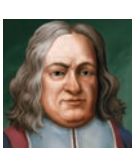
If $5|a$, then $5|na$, where a and n are whole numbers.

The above statement is generalized in the following theorem.

Theorem 4-1

For any whole numbers a and d, if $d|a$ and n is any whole number, then $d|na$.

Historical Note



Pierre de Fermat (1601–1665) was a French lawyer who devoted his leisure time to mathematics—a subject in which he had no formal training. After his death, his son published an edition of Diophantus's *Arithmetica* with Fermat's notes. One of the margin notes asserted that the equation $x^n + y^n = z^n$ has no natural number solutions if n is greater than 2 and commented, "I have found an admirable proof of this, but the margin is too narrow to contain it." Only in 1995 did Andrew Wiles, a Princeton University mathematician, prove the theorem.

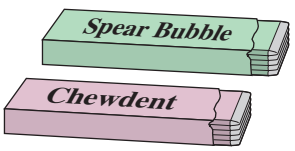


Figure 1

The theorem can be stated in an equivalent form:

If d is a factor of a, then d is a factor of any multiple of a.

Next consider two different brands of chewing gum each having five pieces, as in Figure 1. We can divide each pack of gum equally among five students. In addition, if we opened both packs and put all of the pieces in a bag, we could still divide the number of pieces of gum equally among the five students. To generalize this notion, if we buy gum in larger packages with a pieces in one package and b pieces in a second package with both a and b divisible by 5, we record the preceding discussion as follows:

$$\text{If } 5 \mid a \text{ and } 5 \mid b, \text{ then } 5 \mid (a + b).$$

If the number a of pieces of gum in one package is divisible by 5, but the number b of pieces in the other package is not, then the total $a + b$ cannot be divided equally among the five students. This is recorded as

$$\text{If } 5 \mid a \text{ and } 5 \nmid b, \text{ then } 5 \nmid (a + b).$$

NOW TRY THIS 1

For any whole numbers a and b, determine the truth of the statement: if $5 \nmid a$ and $5 \nmid b$, then $5 \nmid (a + b)$.

Since subtraction can be defined in terms of addition, results similar to those for addition hold for subtraction. These ideas are generalized in Theorem 4-2.

Theorem 4-2

For any whole numbers a, b, and $d \neq 0$:

a. If $d \mid a$ and $d \mid b$, then $d \mid (a + b)$.
b. If $d \mid a$ and $d \nmid b$, then $d \nmid (a + b)$.
c. If $d \mid a$, $d \mid b$, and $a > b$, then $d \mid (a - b)$.
d. If $d \mid a$, $d \nmid b$, and $a > b$, then $d \nmid (a - b)$.
e. If $d \nmid a$, $d \mid b$, and $a > b$, then $d \nmid (a - b)$.

Theorem 4-2 can be extended. For example, if a, b, c, and d are whole numbers, with $d \neq 0$, then,

$$\text{If } d \mid a, d \mid b, \text{ and } d \mid c, \text{ then } d \mid (a + b + c).$$

The proofs of several theorems in this section are left as exercises, but the proofs of Theorem 4-2(a) and (b) are given as illustrations.

Proof

Theorem 4-2(a) is equivalent to the following.

$$\text{If } a \text{ is a multiple of } d \text{ and } b \text{ is a multiple of } d, \text{ then } a + b \text{ is a multiple of } d.$$

Notice that "a is a multiple of d" means $a = md$, for some whole number m. Similarly "b is a multiple of d" means $b = nd$, for some whole number n. To show that $a + b$ is a multiple of d, we add as follows:

$$a + b = md + nd$$

Is $md + nd$ a multiple of d? Because $md + nd = (m + n)d$, then $a + b = (m + n)d$. By the closure property of whole number addition, $m + n$ is a whole number. Consequently, $a + b$ is a multiple of d, and therefore $d \mid (a + b)$. ■

To prove Theorem 4-2(b), we proceed as follows:

Because $d \mid a$, $a = md$ for some $m \in W$. Because $d \nmid b$, when b is divided by d the remainder is not 0; that is, $b = qd + r$, where $q, r \in W$ and $0 < r < d$. Thus,

$$\begin{aligned} a + b &= md + qd + r \\ &= (m + q)d + r. \end{aligned}$$

Consequently when $a + b$ is divided by d, the quotient is $m + q$ and the remainder is r. Because $r \neq 0$, $d \nmid (a + b)$. ■

Example 2

Classify each of the following as true or false, where x, y, and z are whole numbers. If a statement is true, prove it. If a statement is false, provide a counterexample.

a. If $3 \mid x$ and $3 \mid y$, then $3 \mid xy$.
b. If $3 \mid (x + y)$, then $3 \mid x$ and $3 \mid y$.
c. If $9 \nmid a$, then $3 \nmid a$.

Solution

a. True; by Theorem 4-1, if $3 \mid x$, then, for any whole number y, $3 \mid yx$ or $3 \mid xy$.
b. False; for example, $3 \mid (7 + 2)$, but $3 \nmid 7$ and $3 \nmid 2$.
c. False; for example, $9 \nmid 21$, but $3 \mid 21$.

NOW TRY THIS 2

If $x, y \in W$, and $3 \mid x$, is it true that $3 \mid xy$ regardless of whether $3 \mid y$ or $3 \nmid y$? Why?

Example 3



Figure 2

Coach Edgerton bought nine pairs of basketball shoes for team players. She gave the bag of shoes to the student manager with the instructions, "Make sure each player pays for the shoes. Each pair costs the same, and the receipt is in the bag, so you'll be able to figure out how much to charge each player."

However, when the student manager pulled out the receipt from the bag, she realized it had gotten wet, and the last digit had been erased. The cost said \$53__. Despite this, the manager figured out what to charge each player for the shoes. How did she do it? What is the missing digit?

Solution

We know there are 9 pairs of basketball shoes for \$53__. We can divide 9 into 530 and see that it would be 58 with 8 left over. We reason that we need to add 1 to 8 is to have a whole 9 which is divisible by 9. So $(9 \cdot 58) + 9 = 531$. The missing digit is 1.

Divisibility Rules

As shown in Example 3, sometimes it is convenient to know whether one number is divisible by another. We used division and remainders to find the answer, but it would be convenient to have a rule that would allow us to know if one number is divisible by another just by looking at the number or by performing a simple test. Following are some of these divisibility rules.

It is possible to determine whether 1734 is divisible by 17, either by using pencil and paper or a calculator. To avoid decimals, we can use a calculator with an [INT÷] button. On such a calculator, division can be performed using the following sequence of buttons:



[1] [7] [3] [4] [INT÷] [1] [7] [=]

to obtain the display 102 (Q) 0. (R)

This implies $1734/17 = 102$ with a remainder of 0, which, in turn, implies $17 \mid 1734$.

We could reach the conclusion mentally by considering the following:

$$1734 = 1700 + 34$$

Because $17 \mid 1700$ and $17 \mid 34$, by Theorem 4-2(a), we have $17 \mid (1700 + 34)$, or $17 \mid 1734$. Similarly, we could determine mentally that $17 \nmid 1735$.

Divisibility Tests for 2, 5, and 10

To determine whether a given whole number n is divisible by a whole number d without the use of a calculator, we express n as a sum or difference of two whole numbers, of which at least one is divisible by d. We use a concrete example to get an idea of how this works. Consider the number 1362. This number can be represented as in Figure 3. Because 10 and every power of 10 has a factor of two, 2 divides each part of the figure (see dashed segments). Since 2 divides each part of the figure, by the extension of Theorem 4-2, 2 divides the sum of all of the parts and hence $2 \mid 1362$.

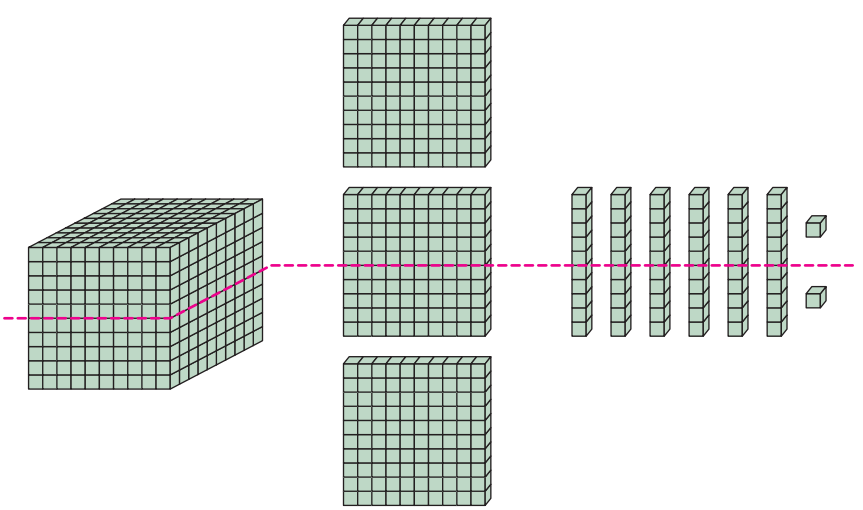


Figure 3

If the original number were 1363, then $2 \mid 1362$ and $2 \nmid 1$, so $2 \nmid 1363$. All we have to do is to determine whether the units digit is divisible by 2 in order to determine whether the number is divisible by 2. Similar tests for divisibility by 5 and 10 hold.

Theorem 4-3: Divisibility Test for 2

A whole number is divisible by 2 if, and only if, its units digit is divisible by 2; that is, if and only if, the units digit is even.

Theorem 4-4: Divisibility Test for 5

A whole number is divisible by 5 if, and only if, its units digit is divisible by 5; that is, if, and only if, the units digit is 0 or 5.

Theorem 4-5: Divisibility Test for 10

A whole number is divisible by 10 if, and only if, its units digit is divisible by 10; that is, if, and only if, the units digit is 0.

Divisibility Tests for 4 and 8

When we consider divisibility tests for 4 and 8, we see that $4 \nmid 10$ and $8 \nmid 10$, so it is not a matter of checking the units digit for divisibility by 4 or 8. However, 4 (which is 2^2) divides 10^2, and 8 (which is 2^3) divides 10^3.

We first develop a divisibility rule for 4. Consider any four-digit whole number n such that $n = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$. The *subgoal* is to *write the given number as a sum of two numbers*, one of which is as great as possible and divisible by 4. We know that $4 \mid 10^2$ because $10^2 = 4 \cdot 25$ and, consequently, $4 \mid 10^3$. Because $4 \mid 10^2$, then $4 \mid b \cdot 10^2$; and because $4 \mid 10^3$, then $4 \mid a \cdot 10^3$. Finally, $4 \mid a \cdot 10^3$ and $4 \mid b \cdot 10^2$ imply $4 \mid (a \cdot 10^3 + b \cdot 10^2)$. Now the divisibility of $a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$ by 4 depends on the divisibility of $(c \cdot 10 + d)$ by 4. The number represented by the last two digits in the given number n is $c \cdot 10 + d$.

The preceding derivation of the divisibility test for 4 can be somewhat shortened by observing that if $abcd$ is a base ten numeral (a, b, c, d are the digits), then $abcd = ab00 + cd = ab \cdot 100 + cd$. Since $4 \mid 100$, by Theorem 4-1 this implies that $4 \mid ab \cdot 100$. Thus, if $4 \mid cd$, then by Theorem 4-2(a), $4 \mid ab \cdot 100 + cd$; that is, $4 \mid abcd$. If, however, $4 \nmid cd$, then Theorem 4-2(b) implies that $4 \nmid ab \cdot 100 + cd$; that is, $4 \nmid abcd$. We summarize this in the following theorem.

Theorem 4-6: Divisibility Test for 4

A whole number is divisible by 4 if, and only if, the last two digits of the number represent a number divisible by 4.

To investigate divisibility by 8, we note that the least positive power of 10 divisible by 8 is 10^3 since $10^3 = 8 \cdot 125$. Consequently by Theorem 4-1, all whole number powers of 10 greater than 10^3 also are divisible by 8. Hence, the following is a divisibility test for 8.

Theorem 4-7: Divisibility Test for 8

A whole number is divisible by 8 if, and only if, the last three digits of the whole number represent a number divisible by 8.

Example 4

a. Determine whether 97,128 is divisible by 2, 4, and 8.
b. Determine whether 83,026 is divisible by 2, 4, and 8.

Solution

a. $2 \mid 97{,}128$ because $2 \mid 8$
$4 \mid 97{,}128$ because $4 \mid 28$
$8 \mid 97{,}128$ because $8 \mid 128$

b. $2 \mid 83{,}026$ because $2 \mid 6$
$4 \nmid 83{,}026$ because $4 \nmid 26$
$8 \nmid 83{,}026$ because $8 \nmid 026$

Example 5

Use Theorems 4-1 and 4-2 to show why the divisibility test for 8 works in Example 4(a).

Solution We write 97,128 as 97,000 + 128. Because $8|1000$, then $8|97 \cdot 1000$ or $8|97{,}000$ (Theorem 4-1). Next check if $8|128$. It does, so $8|(97{,}000 + 128)$ or $8|97{,}128$ (Theorem 4-2).

In Example 4(a), it would have been sufficient to check that the given whole number a is divisible by 8 because if $8|a$, then $a = 8k$ for some whole number k. Hence $a = 2(4k)$ and $a = 4(2k)$, which implies that $2|a$ and $4|a$. This relationship is shown in Figure 4.

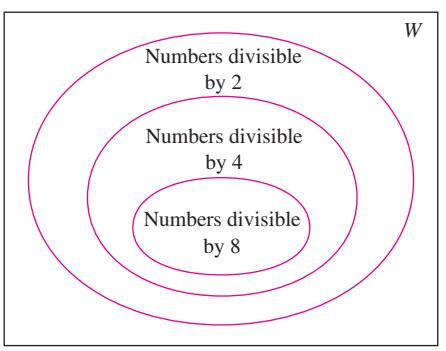


Figure 4

Notice that if $8 \nmid a$, we cannot conclude that $4 \nmid a$ or $2 \nmid a$. (Why?)

Divisibility Tests for 3 and 9

Consider a divisibility test for 3. No power of 10 is divisible by 3, but the numbers 9, and 99, and 999, and other similar numbers are divisible by 3. For example, to determine whether 5721 is divisible by 3, we rewrite the number using 999, 99, and 9, as follows:

$$\begin{aligned} 5721 &= 5 \cdot 10^3 + 7 \cdot 10^2 + 2 \cdot 10 + 1 \\ &= 5(999 + 1) + 7(99 + 1) + 2(9 + 1) + 1 \\ &= 5 \cdot 999 + 5 \cdot 1 + 7 \cdot 99 + 7 \cdot 1 + 2 \cdot 9 + 2 \cdot 1 + 1 \\ &= (5 \cdot 999 + 7 \cdot 99 + 2 \cdot 9) + (5 + 7 + 2 + 1) \end{aligned}$$

The sum in the first set of parentheses in the last line is divisible by 3, so the divisibility of 5721 by 3 depends on the sum in the second set of parentheses. In this case, $5 + 7 + 2 + 1 = 15$ and $3|15$, so by Theorem 4-2(a), $3|5721$. Hence, to test 5721 for divisibility by 3, we test $5 + 7 + 2 + 1$ for divisibility by 3. Notice that $5 + 7 + 2 + 1$ is the sum of the digits of 5721. This suggests the following test for divisibility by 3.

Theorem 4-8: Divisibility Test for 3

A whole number is divisible by 3 if, and only if, the sum of its digits is divisible by 3.

Proof

A proof similar to the above argument for 5721 works for any whole number. We illustrate the proof for any four-digit whole number $n = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$. Even though

$a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$ is not necessarily divisible by 3, the number $a \cdot 999 + b \cdot 99 + c \cdot 9$ *is* divisible by 3. We have:

$$\begin{aligned} a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d &= a \cdot 1000 + b \cdot 100 + c \cdot 10 + d \\ &= a \cdot (999 + 1) + b \cdot (99 + 1) + c \cdot (9 + 1) + d \\ &= (a \cdot 999 + b \cdot 99 + c \cdot 9) + (a \cdot 1 + b \cdot 1 + c \cdot 1 + d) \\ &= (a \cdot 999 + b \cdot 99 + c \cdot 9) + (a + b + c + d) \end{aligned}$$

Because $3 \mid 9, 3 \mid 99$, and $3 \mid 999$, it follows that $3 \mid (a \cdot 999 + b \cdot 99 + c \cdot 9)$. If $3 \mid (a + b + c + d)$, then $3 \mid [(a \cdot 999 + b \cdot 99 + c \cdot 9) + (a + b + c + d)]$; that is, $3 \mid n$. If, on the other hand, $3 \nmid (a + b + c + d)$, it follows from Theorem 4-2(b) that $3 \nmid n$. ■

Since $9 \mid 9, 9 \mid 99, 9 \mid 999$, and so on, a test similar to that for divisibility by 3 applies to divisibility by 9. (Why?)

Theorem 4-9: Divisibility Test for 9

A whole number is divisible by 9 if, and only if, the sum of the digits of the whole number is divisible by 9.

Example 6

Use divisibility tests to determine whether each of the following numbers is divisible by 3 and divisible by 9.

a. 1002 **b.** 14,238

Solution

a. Because $1 + 0 + 0 + 2 = 3$ and $3 \mid 3$, it follows that $3 \mid 1002$. Because $9 \nmid 3$, it follows that $9 \nmid 1002$.

b. Because $1 + 4 + 2 + 3 + 8 = 18$ and $3 \mid 18$, it follows that $3 \mid 14{,}238$. Because $9 \mid 18$, it follows that $9 \mid 14{,}238$.

Example 7

The store manager has an invoice for 72 calculators. The first and last digits on the invoice are illegible. The manager can read

$$\$\blacksquare 67.9\blacksquare.$$

What are the missing digits, and what is the cost of each calculator?

Solution

Let the missing digits be x and y so that the base ten notation for the number is $x67.9y$ dollars, or $x679y$ cents. Because there were 72 calculators sold, the number on the invoice must be divisible by 72. Because the number is divisible by 72 and $72 = 8 \cdot 9$, it must be divisible by 8 and 9, which are factors of 72. For the number on the invoice to be divisible by 8, the three-digit number $79y$ must be divisible by 8. Because $79y$ must be divisible by 8, it is an even number. Therefore, $79y$ must be either 790, 792, 794, 796, or 798. Only the number 792 is divisible by 8, so the last digit, y, on the invoice must be 2.

Because the number on the invoice must be divisible by 9, we know that 9 must divide $x + 6 + 7 + 9 + 2$, or $(x + 24)$. Since 3 is the only single digit that will make $(x + 24)$ divisible by 9, then x must be 3. Therefore, the number on the invoice must be \$367.92. The calculators must cost $\$367.92/72 = \$5.11$ each.

Divisibility Tests for 11 and 6

The divisibility test for 7 is usually harder to use than actually performing the division, so we omit the test.

To develop a test for divisibility by 11, we search for numbers close to powers of 10 that are divisible by 11. Notice that $11|11, 11|99, 11|1001, 11|9999, \ldots$. As for divisibility by 3 and 9, we illustrate our argument on a four-digit number n and split it into two parts, one divisible by 11 and the other to be tested.

$$\begin{aligned} n &= a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d \\ &= a(1001 - 1) + b(99 + 1) + c(11 - 1) + d \\ &= (a \cdot 1001 + b \cdot 99 + c \cdot 11) + (b + d - a - c) \end{aligned}$$

Because 11 divides the number in the first parentheses, the test depends on whether $b + d - a - c = b + d - (a + c)$ is divisible by 11 or not. Thus we have the following theorem.

Theorem 4-10: Divisibility Test for 11

A whole number is divisible by 11 if, and only if, the sum of the digits in the places that are even powers of 10 minus the sum of the digits in the places that are odd powers of 10 is divisible by 11.
(If the sums are different, then subtract the lesser from the greater.)

For example, to test whether 8,471,986 is divisible by 11, we check whether 11 divides the difference $(6 + 9 + 7 + 8) - (8 + 1 + 4) = 17$. Because $11 \nmid 17$, it follows from the divisibility test for 11 that $11 \nmid 8{,}471{,}986$. A number like 2772 is divisible by 11 because $(2 + 7) - (7 + 2) = 9 - 9 = 0$ and 0 is divisible by 11.

The divisibility test for 6 is related to the divisibility tests for 2 and 3. In Section 4-2, we show that if $2|n$ and $3|n$, then $(2 \cdot 3)|n$, and in general: if a and b have no factors in common and if $a|n$ and $b|n$, then we can conclude that $ab|n$. Consequently, the following divisibility test is true.

Theorem 4-11: Divisibility Test for 6

A whole number is divisible by 6 if, and only if, the whole number is divisible by both 2 and 3.

Example 8

The number 57,729,364,583 has too many digits for most calculator displays. Determine whether it is divisible by each of the following:

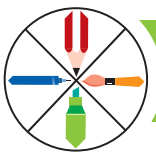
a. 2 **b.** 3 **c.** 5 **d.** 6 **e.** 8 **f.** 9 **g.** 10 **h.** 11

Solution

a. No; the units digit 3 is not divisible by 2.
b. No; the sum of the digits is 59, which is not divisible by 3.
c. No; the units digit is neither 0 nor 5.
d. No; the number is not divisible by 2 (see Theorem 4-11).
e. No; the number formed by the last three digits is 583, which is not divisible by 8.
f. No; the sum of the digits is 59, which is not divisible by 9.
g. No; the units digit is not 0.
h. Yes; $(3 + 5 + 6 + 9 + 7 + 5) - (8 + 4 + 3 + 2 + 7) = 35 - 24 = 11$ and 11 is divisible by 11.



Divisibility tests for other numbers are explored in Assessments 4-1A and 4-1B. The student page shown on page 184 illustrates divisibility rules.



School Book Page

Factors, Multiples, and Divisibility

Lesson 5-1

Common Core

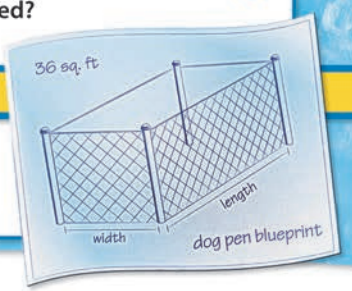
This lesson reinforces concepts and skills required for Topic 9.

Factors, Multiples, and Divisibility

How are factors and divisibility related?

How many different ways can Raul build a dog pen that has an area of 36 square feet and has sides measured in whole feet?

Choose an Operation Divide to find the possible measurements.



Another Example How can you use divisibility rules?

Is 126 divisible by 2, 3, 4, 6, 9, or 10? You can use divisibility rules to find out.

Divisibility Rules

A whole number is divisible by	Examples:	Is 126 divisible by the number?
2 if the ones digit is an even number.	10; 6; 108	Yes, the ones digit is even.
3 if the sum of the digits is divisible by 3.	3; 627; 891	Yes, 1 + 2 + 6 = 9. 9 is divisible by 3.
4 if the last two digits of the number are divisible by 4.	64; 5,888	No, 26 is not divisible by 4.
5 if the ones digit is 0 or 5.	380; 9,005	No, the ones digit is 6.
6 if the number is divisible by both 2 and 3.	240; 8,982	Yes, 126 is divisible by both 2 and 3.
9 if the sum of the digits is divisible by 9.	189; 1,035	Yes, 1 + 2 + 6 = 9. 9 is divisible by 9.
10 if the ones digit is 0.	170; 1,380	No, the ones digit is not 0.

What is a multiple of a number?

Is 126 a multiple of 7? A multiple of a given number is a product of that number and a whole number greater than 0.

Since 126 ÷ 7 = 18, 126 is the product of 7 and 18. So, 126 is a multiple of 7. Other multiples of 7 are 7, 14, 21, 28, 35, 42, 49, and so on.

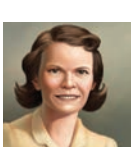
Explain It

1. Is 53,802 a multiple of 9? Explain your answer.

120

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Historical Note



A twentieth century mathematician who did research in number theory was American Julia Robinson (1919–1985). Robinson was the first woman mathematician to be elected to the National Academy of Sciences and the first woman president of the American Mathematical Society.

Problem Solving Maple Syrup

In a maple syrup bottling warehouse in New Hampshire, bottles are packed with 12 bottles in a case or 6 bottles in a case. The manager checks the inventory and notices that there are 12,338 bottles in the warehouse. He immediately knows that this is wrong. How does he know?

Understanding the Problem The problem is to determine how the manager knows 12,338 cannot be the correct inventory. To solve the problem, we assume that each case contains exactly 12 or 6 bottles.

Devising a Plan Because the cases are for 6 or 12, the total number of bottles has to be divisible by 6 or 12. Since 6 divides 12, we only need to check for divisibility by 6.

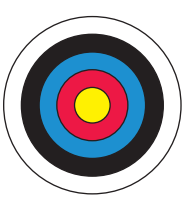
Carrying Out the Plan We can see that 6 divides 12,000 and only need to determine if 6 divides 338. Furthermore, knowing that 6 divides 300 would leave us to check if 6 divides 38, and since it does not, we conclude that the manager is correct and 12,338 is the incorrect inventory.

Looking Back Another way to look at the problem is to create an equation to model the total number of bottles. If we let n be the number of cases with 6 bottles and m be the number of cases with 12 bottles, then we have a total of $6n + 12m$ bottles. The manager claimed the total was incorrect, so $6n + 12m \neq 12{,}338$. We need to show that these are not equal. We can factor $6n + 12m = 6(n + 2m)$. If the total 12,338 is correct, it would have to be divisible by the factor 6. The divisibility test for 6 states that a number is divisible by 6 if, and only if, the number is divisible by both 2 and 3. Because 12,338 ends in 8, it is divisible by 2. However, 3 does not divide the sum of the digits $1 + 2 + 3 + 3 + 8 = 17$. So $6 \nmid 12338$, and therefore, the manager's claim was correct.

Assessment 4-1A

1. If you multiply an even number by 2 and add 1, is your answer even or odd?
2. Classify each of the following statements as true or false. If false, tell why.
 - **a.** 5 is a multiple of 24.
 - **b.** 10 is a divisor of 30.
 - **c.** $8 \mid 324$
 - **d.** 24 is divisible by 0.
3. Without using a calculator, test each of the following numbers for divisibility by 2, 3, 4, 5, 6, 8, 9, 10, and 11.
 - **a.** 4,201,012
 - **b.** 1573
 - **c.** 15,810
4. Using divisibility tests, solve each of the following problems.
 - **a.** There are 1379 children signed up in a baseball league. If exactly 9 players are to be placed on each team, will any team be short of players?
 - **b.** An orchard has 1354 apple trees to be planted. Can the apple trees be planted in rows of 11?
 - **c.** If there are 1216 pencils in a box and 8 classrooms, can each classroom receive the same number of pencils?
5. Find all possible single digits that can be placed in the square so that the number 1,427,4□2 is divisible by the given number.
 - **a.** 2
 - **b.** 3
 - **c.** 4
 - **d.** 9
6. Fill each of the following blanks with the greatest digit that makes the statement true.
 - **a.** $3 \mid 74__$
 - **b.** $9 \mid 83__45$
 - **c.** $11 \mid 6__55$
 - **d.** $5 \mid 136__$
 - **e.** $6 \mid 2__44$
 - **f.** $8 \mid 38__4$
7. Justify each of the following statements.
 - **a.** $7 \mid 280$
 - **b.** $19 \mid (3800 + 19)$
 - **c.** $15 \mid 2^4 \cdot 3^5 \cdot 5$
 - **d.** $19 \nmid (3800 + 37)$

8. Without using a calculator, classify each of the following statements as true or false. Justify your answers.
 a. $13 \mid 390{,}026$
 b. $13 \nmid 260{,}033$
 c. $17 \mid 34{,}015$
 d. $17 \mid 34{,}051$
 e. $19 \mid 19{,}031$
 f. $31 \mid 93^{11}$.
9. Classify each of the following statements as true or false, for all whole numbers a, b, and c, with $b \neq 0$.
 a. $7 \mid 21$ implies $7 \mid 21^2$.
 b. If $b \mid a$, then $(b + c) \mid (a + c)$.
 c. If $b \mid a$, then $b^2 \mid a^3$.
 d. If $b \mid a$, then $b \mid (a + b)$.
10. Find a whole number solution that makes each of the following true.
 a. Divisible by 2 and 3 but not by 5.
 b. Divisible by 2 and 4 but not by 8.
 c. Divisible by 5 and 10 but not by 2.
11. A group of people ordered pencils. The bill was \$2.09. If the original price of each pencil was 12¢ and the price has risen, how much does each cost?
12. Classify each of the following statements as true or false.
 a. The product of three consecutive whole numbers is divisible by 3.
 b. The product of two consecutive whole numbers is divisible by 4.
13. An archery target consists of five concentric circles as shown below. The value of the arrow landing in each region starting from the inner circle is 9, 7, 5, 3, 1 points. In how many ways could five scoring arrows earn 29 points?



14. Classify each of the following statements as true or false.
 a. If every digit of a number is divisible by 3, then the number is divisible by 3.
 b. If a number is divisible by 3, then every digit of the number is divisible by 3.
 c. A number is divisible by 3 if, and only if, every digit of the number is a multiple of 3.
15. To find the remainder when a number is divided by 3 you could use long division. Alternatively, you could find the sum of the digits and then find the remainder when that sum is divided by 3.
 a. Find the remainder when 7,242,815 is divided by 3 using long division and then using the sum of the digits.
 b. Explain why the sum of the digits approach is valid.
 c. Will the sum of the digits approach work to find the remainder when a number is divided by 9? Justify your answer.
16. If $45 \mid n$, what other whole numbers divide n? Why?
17. Devise a test for divisibility by 16.
18. A test for checking arithmetic calculations is called *casting out nines*. Consider the sum $193 + 24 + 786 = 1003$. The remainders when 193, 24, and 786 are divided by 9 are 4, 6, and 3, respectively. The sum of the remainders, 13, has a remainder of 4 when divided by 9, as does 1003. Checking the remainders in this manner provides a quasi-check for the computation. Find the following sums and use *casting out nines* to check the sums:
 a. $12{,}343 + 4546 + 56$
 b. $987 + 456 + 8765$
 c. Try the check on the subtraction $1003 - 46$.
 d. Try the check on the multiplication $345 \cdot 56$.
 e. Would it make sense to try the check on division? Why or why not?
19. A palindrome is a number that reads the same forward as backward.
 a. Are all five digit palindromes divisible by 11? Why or why not?
 b. Are all six-digit palindromes divisible by 11? Why or why not?
20. Prove the test for divisibility by 9 for any-five-digit whole number.

Assessment 4-1B

1. If you multiply an odd number by 2 and add, 1 is your answer even or odd?
2. Classify each of the following statements as true or false. If false, tell why.
 a. 5 is a divisor of 20.
 b. 10 is a multiple of 20.
 c. $5 \mid 20$
 d. 30 is divisor of 6.
3. Without using a calculator, test each of the following numbers for divisibility by 2, 3, 4, 5, 6, 8, 9, 10, and 11.
 a. 7163
 b. 34,200
 c. 199,990
4. Using divisibility tests, solve each of the following problems.
 a. Nine friends win the lottery. The payoff is \$1,111,500. Can the money be divided equally?
 b. There are 548 students at a baseball game. Can they be divided equally into four sections?
 c. The average amount of student loan debt is \$26,600. Can this amount be paid back equally over 12 years?
5. Find all possible single digits that can be placed in the square so that the number $1{,}182{,}1\square 6$ is divisible by the given number.
 a. 2
 b. 3
 c. 4
 d. 9

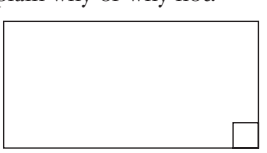
6. Without using a calculator, classify each of the following statements as true or false. Justify your answers.
 a. $5 \mid (2 \cdot 3 \cdot 5 \cdot 7)$
 b. $5 \mid (2 \cdot 3 \cdot 5 \cdot 7) + 1$
 c. $6 \mid (2^3 \cdot 3^2 \cdot 17^4)$
 d. $7 \nmid 4222$
7. Justify each of the following statements.
 a. $26 \mid (13^4 \cdot 100)$
 b. $13 \nmid (2^4 \cdot 5^3 \cdot 26 + 1)$
 c. $2^4 \nmid (2 \cdot 4 \cdot 6 \cdot 8 \cdot 17^{10} + 1)$
 d. $2^4 \mid (10^4 + 6^4)$
8. Without using a calculator, classify each of the following statements as true or false. Justify your answers.
 a. $12 \mid 24{,}013$
 b. $12 \mid 24{,}036$
 c. $19 \mid 38{,}037$
 d. $19 \nmid 3{,}800{,}018$
 e. $23 \nmid 23{,}023$
 f. $23 \mid 46^{10}$
9. Classify each of the following statements as true or false, assuming that a, b, and c are whole numbers.
 a. $3 \nmid 50$ and $3 \mid 30$ imply $3 \nmid (50 + 30)$.
 b. $3 \nmid 50$ and $3 \mid 30$ imply $3 \nmid (50 \cdot 30)$.
 c. $3 \mid (a + b)$ and $3 \nmid c$ imply $3 \nmid (a + b + c)$.
 d. $3 \mid b$ implies $3 \mid b^2$.
10. Answer each of the following questions without actually performing the division. Explain how you did it in each case.
 a. Is 24,013 divisible by 12?
 b. Is 24,036 divisible by 12?
 c. Is 17,034 divisible by 17?
 d. Is $2 \cdot 3 \cdot 5 \cdot 7$ divisible by 3?
 e. Is $(2 \cdot 3 \cdot 5 \cdot 7) + 1$ divisible by 6?
11. The bookstore marked some notepads down from $2.00 but still kept the price over $1.00. It sold all of them. The total amount of money from the sale of the pads was $31.45. How many notepads were sold? What was the price of each notepad?
12. Classify each of the following statements as true or false.
 a. The sum of every three consecutive numbers is divisible by 3.
 b. The sum of every four consecutive numbers is divisible by 4.
13. In a football game, a touchdown with an extra point is worth 7 points and a field goal is worth 3 points. Suppose that in a game the only scoring done by teams are touchdowns with extra points and field goals.
 a. Which of the scores 1 to 25 are impossible for a team to score?
 b. List all possible ways for a team to score 40 points.
 c. A team scored 57 points with 6 touchdowns and 6 extra points. How many field goals did the team score?
14. Classify each of the following statements as true or false.
 a. If a number is divisible by 6, then it is divisible by 2 and 3.
 b. If a number is divisible by 2 and 3, then it is divisible by 6.
 c. If a number is divisible by 2 and 4, then it is divisible by 8.
 d. If a number is divisible by 8, then it is divisible by 2 and 4.
 e. A number is divisible by 8 if, and only if, it is divisible by 2 and by 4.
15. Complete the following table where n is the given whole number.

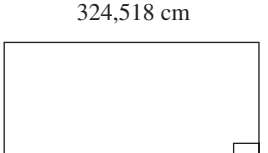
n	Remainder when n is divided by 9	Sum of the digits of n	Remainder when the sum of the digits of n is divided by 9
a. 31			
b. 143			
c. 345			
d. 2987			
e. 7652			

 f. Make a conjecture about the remainders when a whole number and the sum of its digits are divided by 9.
16. If $28 \mid n$, what other whole numbers divide n? Why?
17. Devise a test for divisibility by 25.
18. Use the casting out nines approach outlined in exercise 18 of Assessment 4-1 A to show that the following computations are wrong.
 a. $99 + 28 = 227$
 b. $11{,}199 - 21 = 11{,}168$
 c. $99 \cdot 26 = 2575$
19. A palindrome is a number that reads the same forward as backward.
 a. Check the following four-digit palindromes for divisibility by 11.
 i. 4554
 ii. 9339
 iii. 2002
 iv. 2222
 b. Are all four-digit palindromes divisible by 11?
20. Prove the test for divisibility by 9 for any four-digit number.

Mathematical Connections 4-1

Reasoning

1. Prove the following statements, assuming n is a natural number.
 a. If n^2 is odd, then n is odd.
 b. If n^2 is even, then n is even.
2. A customer wants to mail a package. The postal clerk determines the cost of the package to be $18.95, but only 5¢ and 10¢ stamps are available. Can the available stamps be used for the exact amount of postage for the package? Why or why not?
3. a. Jim uses his calculator to see if a number n having eight or fewer digits is divisible by a number d. He finds that $n \div d$ has a display of 32 digits. Does $d \mid n$? Why or why not?
 b. If $n \div d$ gives a display of 16.8, does $d \mid n$? Why or why not?
4. Is the area (in cm^2) of each of the following rectangles divisible by 4? The area is the length times the width. Explain why or why not.
 a.


 b.


5. Answer each of the following and justify your answers.
 a. If a number is not divisible by 5, can it be divisible by 10?
 b. If a number is not divisible by 10, can it be divisible by 5?
6. A number in which each digit except 0 appears exactly 3 times is divisible by 3. For example, 777,555,222 and 414,143,313 are divisible by 3. Explain why this statement is true.
7. Enter any three-digit number on the calculator; for example, enter 243. Repeat it: 243,243. Divide by 7. Divide by 11. Divide by 13. What is the answer? Try it again with any other three-digit number. Will this always work? Why?
8. Arrange the digits 0 through 9 so that the number formed by the first (left) digit is divisible by 1, the number formed by the first 2 digits is divisible by 2, the number formed by the first 3 digits is divisible by 3, and so forth. What is the 10-digit number? If you were only allowed to use the digits 1 through, what would the 9-digit number be?
9. Three tiles marked X, Y, and Z need three different nonzero digits painted on them. The whole numbers XYZ, as well as ZXY and YZX, are all divisible by 6 and 9. What are the values of X, Y, and Z?

Open-Ended

10. A breakfast-food company had a contest in which the numbers were placed in breakfast-food boxes. A prize of $1000 was awarded to anyone who could collect numbers whose sum was 100. The company had thousands of cards made with the following numbers on them:

3	12	15	18	27	33
45	51	66	75	84	90

 a. If the company did not make any more cards, is there a winning combination?
 b. If the company is going to add one more number to the list and it wants to make sure the contest has at most 1000 winners, suggest a strategy for it to use.
11. How would you use concrete materials to explain the following to young children?
 a. The concept of even and odd numbers.
 b. A number being divisible by 3 or not being divisible by 3.
 c. If $9 \mid a$, then $3 \mid a$.

Cooperative Learning

12. In your group, discuss if the sum of two consecutive numbers is divisible by 2, the sum of three consecutive numbers is divisible by 3, the sum of four consecutive numbers is divisible by 4, etc. Make a general statement about the sum of n consecutive numbers and its divisibility by n.
13. In your groups use base-ten blocks to illustrate the divisibility rules for 3, 4, 5, 8, 9, and 10. Use a three-digit number as your example.

Connecting Mathematics to the Classroom

14. Amy claims 10 is both even and odd because it is divisible by the even number 2 and the odd number 5. How would you respond?
15. Jane claimed that a number is divisible by 4 if each of the last two digits is divisible by 4. Is this claim correct? If not, how would you suggest that Jane change it?
16. Your seventh-grade class has just completed a unit on divisibility rules. One student asks why divisibility by 6 cannot be tested by dividing the sum of the digits by 6. How do you respond?
17. A student claims that a number is divisible by 24 if, and only if, it is divisible by 6 and by 4, and, in general, a number is divisible by $a \cdot b$ if, and only if, it is divisible by a and by b. What is your response?
18. Brian claims zero is neither even nor odd because in order to be even, zero would have to be divided into two equal piles and in order to be odd, zero would not be able to be divided into two equal piles. What is your response to Brian to convince him that zero is even?

National Assessments

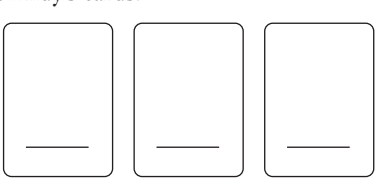
National Assessment of Educational Progress (NAEP) Questions

Andy has three cards, A, B, and C. Each card has one number on it.

One card has the number 4 on it.
One card has the number 6 on it.
One card has the number 10 on it.

The number on card B is a factor of 8, and the number on card C is a factor of 12.

Write the correct number on each of the cards below so that they are the same as Andy's cards.



NAEP, Grade 4, 2013

Sam did the following problems.

$$2 + 1 = 3$$
$$6 + 1 = 7$$

Sam concluded that when he adds 1 to any whole number, his answer will always be odd.

Is Sam correct? _____
Explain your answer.

NAEP, Grade 4, 2009

4-2 Prime and Composite Numbers

4-2 Objectives

Students will be able to understand and explain

- Prime and composite numbers.
- The number of divisors of any whole number.
- The Fundamental Theorem of Arithmetic.
- The factorization of whole numbers.

CCSS **The grade 4 *Common Core Standards* states that students should be able to:**

Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite. (p. 29)

One method used in elementary schools to determine the positive factors of a nonzero whole number is to use squares of paper or cubes to represent the number as a rectangle. Such a rectangle resembles a candy bar formed with small squares. The dimensions of the rectangle are divisors or factors of the number. For example, Figure 5 shows rectangles to represent 12.

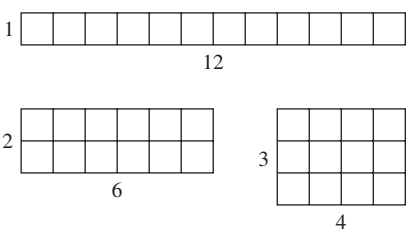


Figure 5

As seen in Figure 5, the number 12 has six positive divisors: 1, 2, 3, 4, 6, and 12. If we use rectangles to find the divisors of 7, then we would find only a 1×7 rectangle, as shown in Figure 6. Thus, 7 has exactly two divisors: 1 and 7.



Figure 6

Table 1 illustrates the number of positive divisors of a nonzero whole number. Below each number listed across the top, we identify the numbers less than or equal to 37 that have that number of whole number divisors. For example, 12 is in the 6 column because it has six divisors, and 7 is in the 2 column because it has only two divisors.

NOW TRY THIS 3

a. Will there be other entries in the 1 column? Why?
b. What are the next three numbers in the 3 column?
c. What is the next number in the 5 column?
d. Find an entry for the 7 column.
e. What kinds of numbers have an odd number of factors? Why?
f. What other patterns do you see in Table 1?

The numbers in the 2 column in Table 1 are of particular importance: they have exactly two positive divisors, namely, 1 and themselves. Any positive integer with exactly two distinct, positive divisors is a *prime number*, or a **prime**. Any whole number greater than 1 that has a positive factor

Table 1

Number of Positive Divisors								
1	2	3	4	5	6	7	8	9
1	2	4	6	16	12		24	36
	3	9	8		18		30	
	5	25	10		20			
	7		14		28			
	11		15		32			
	13		21					
	17		22					
	19		26					
	23		27					
	29		33					
	31		34					
	37		35					

other than 1 and itself is a *composite number*, or a **composite**. For example, 4, 6, and 16 are composites because they have positive factors other than 1 and themselves. The number 1 has only one positive factor, so it is neither prime nor composite. From the 2 column in Table 1, we see that the first 12 primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37. The number 2 is sometimes called the "oddest" prime because it is the only even prime.

Example 9

Show that the following numbers are composite.

a. 1564 **b.** 2781
c. 1001 **d.** $3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1$

Solution

a. Since $2 \mid 4$, 1564 is divisible by 2 and is composite.
b. Since $3 \mid (2 + 7 + 8 + 1)$, 2781 is divisible by 3 and is composite.
c. Since $11 \mid [(1 + 0) - (0 + 1)]$, 1001 is divisible by 11 and is composite.
d. Because a product of odd numbers is odd (why?), $3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ is odd. If we add 1 to an odd number, the sum is even. Any even natural number has a factor of 2 besides 1 and itself, and is therefore composite.

Prime Factorization

Composite numbers can be expressed as products of two or more whole numbers greater than 1. For example, $18 = 2 \cdot 9$, $18 = 3 \cdot 6$, or $18 = 2 \cdot 3 \cdot 3$. Each expression of 18 as a product of factors is a **factorization**.

A factorization containing only prime numbers is a **prime factorization**. To find a prime factorization of a given composite number, we first rewrite the number as a product of two whole numbers greater than 1. We continue the process, factoring these numbers until all factors are primes. For example, consider 260:

$$260 = 26 \cdot 10 = (2 \cdot 13)(2 \cdot 5) = 2 \cdot 2 \cdot 5 \cdot 13 = 2^2 \cdot 5 \cdot 13$$

The procedure for finding a prime factorization of a number can be organized using a **factor tree**, as Figure 7(a) demonstrates. The last branches of the tree display the prime factors of 260.

A second way to factor 260 is shown in Figure 7(b). The two trees produce the same prime factorization.

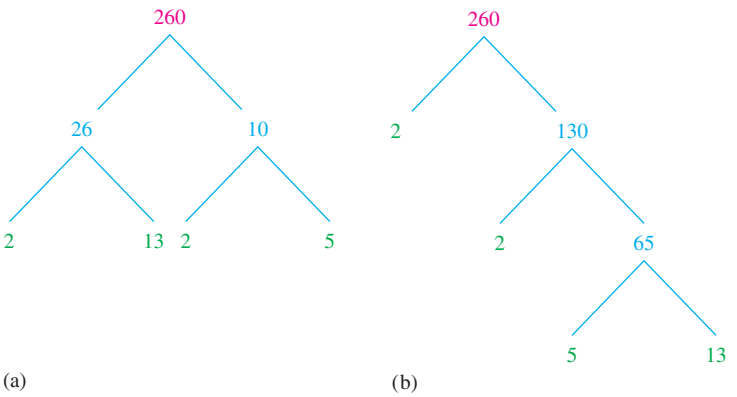


Figure 7

The *Fundamental Theorem of Arithmetic*, or the *Unique Factorization Theorem*, states that in general, if order is disregarded, the prime factorization of a number is unique.

Theorem 4-12: Fundamental Theorem of Arithmetic

Each composite number can be written as a product of primes in one, and only one, way except for the order of the prime factors in the product.

The Fundamental Theorem of Arithmetic assures that once we find a prime factorization of a number, a different prime factorization of the same number cannot be found. For example, consider 260. We start with the least prime, 2, and see whether it divides 260. If not, we try the next greater prime and check for divisibility by this prime. Once we find a prime that divides the number in question, we must find the quotient when the number is divided by the prime. This step in the prime factorization of 260 is shown in Figure 8(a). Next, if the quotient is a composite number, we check whether the prime divides the quotient. If so, we repeat the process; if not, we try the next greater prime, 3, and check to see if it divides the quotient. We see that 130 divided by 2 yields 65, as shown in Figure 8(b). We continue the procedure, using greater primes, until a quotient of 1 is reached. The original number is the product of all the prime divisors used. The complete procedure for 260 is shown in Figure 8(c). An alternative form of this procedure is shown in Figure 8(d).

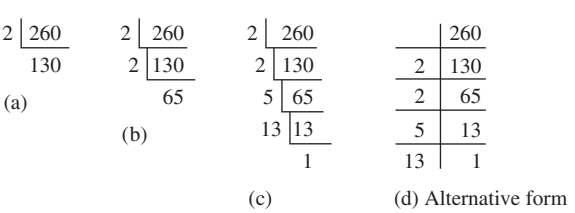
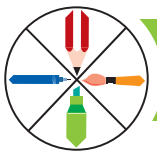


Figure 8

The primes in the prime factorization of a number are typically listed in increasing order from left to right, and if a prime appears in a product more than once, exponential notation is used. Thus, the factorization of 260 is written as $2^2 \cdot 5 \cdot 13$. Prime factorization is demonstrated in the student page shown on page 192. Notice that the factor tree is developed in two different ways leading to the same result.



School Book Page

Prime Factorization

Lesson 5-2

Common Core

This lesson reinforces concepts and skills required for Topic 9.

Prime Factorization

How can you write the prime factorization of a number?

Whole numbers greater than 1 are either prime or composite numbers.

A prime number has exactly two factors, 1 and itself. The numbers 2, 3, and 5 are prime numbers.

Model	Dimension	Factors
	1 × 2	1, 2
	1 × 3	1, 3
	1 × 4	1, 4
	2 × 2	2
	1 × 5	1, 5

A composite number has more than two factors and can be written as the product of its prime factors. This is called its prime factorization.

4 is a composite number. The factors of 4 are 1, 2, and 4. 2 is its only prime factor. The prime factorization of 4 is 2×2, or 2^2.

To find the prime factorization of 60, write its factors, beginning with the smallest prime factor.

$60 = 2 \times 30$ ← 2 is a factor of 60.

$= 2 \times 2 \times 15$ ← 2 is a factor again.

$= 2 \times 2 \times 3 \times 5$ ← 3 and 5 are factors.

$= 2^2 \times 3 \times 5$ ← Use exponents.

Another Example How can you use a factor tree to find the prime factorization of a number?

One Way

To find the prime factorization of 72, begin with the smallest prime factor. Write factors until all the factors are prime numbers.

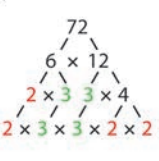


$72 = 2 \times 2 \times 2 \times 3 \times 3$

$72 = 2^3 \times 3^2$

Another Way

To find the prime factorization of 72, begin with any two factors of 72. Write factors until all the factors are prime numbers.



Arrange prime factors in order.

$72 = 2 \times 2 \times 2 \times 3 \times 3$

$72 = 2^3 \times 3^2$

There is only one prime factorization for any number.

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NOW TRY THIS 4

Colored rods are used in the elementary-school classroom to teach many concepts. The rods vary in length from 1 cm to 10 cm. Various lengths have different colors; for example, the 5 rod is yellow. The rods and their colors are shown in Figure 9(a). A row with all the same color rods is called a *one-color train*.

a. What rods can be used to form a one-color train for 18?
b. What one-color trains are possible for 24?
c. How many one-color trains of two or more rods are possible for each prime number?
d. In Figure 9(b), a number represented by an all-red train, an all-green train, and an all-purple train is shown. How many factors does this number have? What are the factors?

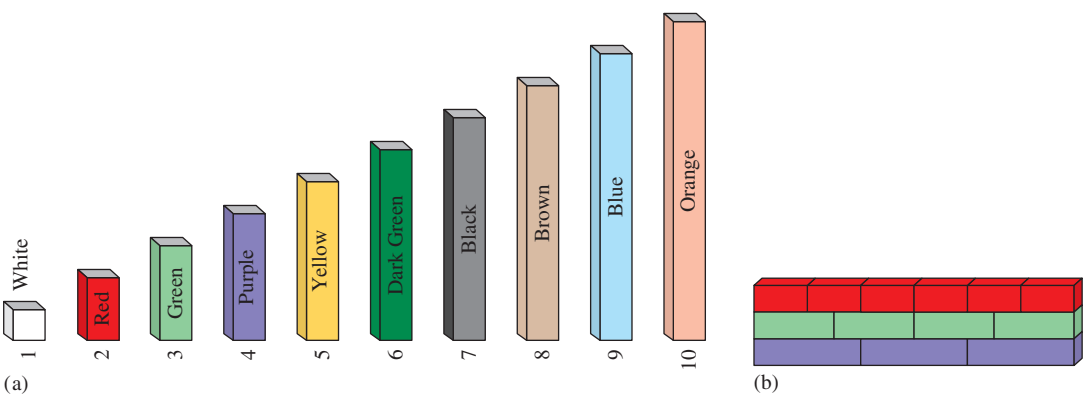


Figure 9

Number of Whole Number Divisors



Figure 10

How many whole number divisors does 24 have? To aid in answering the question, we group divisors as in Figure 10. The whole number divisors of 24 occur in pairs, where the product of the divisors in each pair is 24. If 3 is a divisor of 24, then $24/3 = 8$, is also a divisor of 24. In general, if a whole number k is a divisor of 24, then $24/k$ is also a divisor of 24.

Another way to think of the number of whole number divisors of 24 is to consider the prime factorization $24 = 2^3 \cdot 3$. The whole number divisors of 2^3 are $2^0, 2^1, 2^2$, and 2^3. The whole number divisors of 3 are 3^0 and 3^1. We know that 2^3 has $(3 + 1) = 4$ divisors and 3^1 has $(1 + 1) = 2$ divisors. Because each divisor of 24 is the product of a divisor of 2^3 and a divisor of 3^1, we use the Fundamental Counting Principle to conclude that 24 has $4 \cdot 2 = 8$ positive divisors. This is summarized in Table 2.

Table 2

Divisors of 2^3	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$
Divisors of 3^1	$3^0 = 1$	$3^1 = 3$		
Divisors of $3^1 \cdot 2^3$ (Divisors of 24)	$3^0 \cdot 2^0 = 1$ $3^1 \cdot 2^0 = 3$	$3^0 \cdot 2^1 = 2$ $3^1 \cdot 2^1 = 6$	$3^0 \cdot 2^2 = 4$ $3^1 \cdot 2^2 = 12$	$3^0 \cdot 2^3 = 8$ $3^1 \cdot 2^3 = 24$

This discussion can be generalized as follows: If p is any prime and n is any whole number, then the whole number divisors of p^n are $p^0, p^1, p^2, p^3, \ldots, p^n$. Therefore, there are $(n + 1)$ whole number divisors of p^n. Using the Fundamental Counting Principle, we can find the number of divisors of any number whose prime factorization is known.

Theorem 4-13

If p and q are different primes, and n, m are whole numbers, then $p^n q^m$ has $(n + 1)(m + 1)$ whole number divisors. In general, if $p_1, p_2, \ldots, p_k$ are primes and $n_1, n_2, \ldots, n_k$ are whole numbers, then $p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_k^{n_k}$ has $(n_1 + 1)(n_2 + 1) \cdot \ldots \cdot (n_k + 1)$ whole number divisors.

Example 10

Find the number of whole number divisors of each given value.

a. 1,000,000 **b.** 210^{10}

Solution

a. We first find the prime factorization of 1,000,000.

$$\begin{aligned} 1{,}000{,}000 = 10^6 = (2 \cdot 5)^6 &= (2 \cdot 5)(2 \cdot 5)(2 \cdot 5)(2 \cdot 5)(2 \cdot 5)(2 \cdot 5) \\ &= (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5) \\ &= 2^6 \cdot 5^6 \end{aligned}$$

Because 2^6 has $6 + 1$ divisors and 5^6 has $6 + 1$ divisors, then by Theorem 4-13, $2^6 \cdot 5^6$ has $(6 + 1)(6 + 1) = 49$ divisors.

b. We first find the prime factorization of 210^{10}:

$$\begin{aligned} 210 &= 21 \cdot 10 = 3 \cdot 7 \cdot 2 \cdot 5 = 2 \cdot 3 \cdot 5 \cdot 7, \\ 210^{10} &= (2 \cdot 3 \cdot 5 \cdot 7)^{10} = 2^{10} \cdot 3^{10} \cdot 5^{10} \cdot 7^{10} \end{aligned}$$

By Theorem 4-13, the number of divisors of 210^{10} is $(10 + 1)(10 + 1)(10 + 1)(10 + 1) = 11^4 = 14{,}641$.

Determining Whether a Number Is Prime

Suppose we want to check whether 97 is prime and we find that 2, 3, 5, and 7 do not divide 97. Could a greater prime divide 97? If p is a prime greater than 7, then $p \geq 11$. If $p \mid 97$, then $97/p$ also divides 97. However, because $p \geq 11$ then $97/p$ must be less than 9 and hence cannot divide 97. (Why?) We see that there is no need to check for divisibility by numbers other than 2, 3, 5, and 7. These ideas are generalized in the following theorems.

Theorem 4-14

If d is a divisor of n, where $n \neq 0$, then $\frac{n}{d}$ is also a divisor of n.

Suppose that p is the *least* divisor of a composite number n and $p \neq 1$. Such a divisor must be prime. (Why?) Then $n = pk$, for some whole number k, $k \neq 1$. Since $k \mid n$ and p was the least divisor of n, $k \geq p$. Therefore, $n = pk \geq pp = p^2$. Since $n \geq p^2$, we have $p^2 \leq n$. This idea is summarized in the following theorem.

Theorem 4-15

If n is composite, then n has a prime factor p such that $p^2 \leq n$.

Theorem 4-15 can be used to help determine whether a given number is prime or composite. For example, consider the number 109. If 109 is composite, it must have a prime divisor p such that $p^2 \leq 109$. The primes whose squares do not exceed 109 are 2, 3, 5, and 7. Mentally, we can see that $2 \nmid 109$, $3 \nmid 109$, $5 \nmid 109$, and $7 \nmid 109$. Hence, 109 is prime. The argument used leads to the following theorem.

Theorem 4-16

If n is a whole number greater than 1 and not divisible by any prime p such that $p^2 \leq n$, then n is prime.

Example 11

a. Is 397 composite or prime?
b. Is 91 composite or prime?

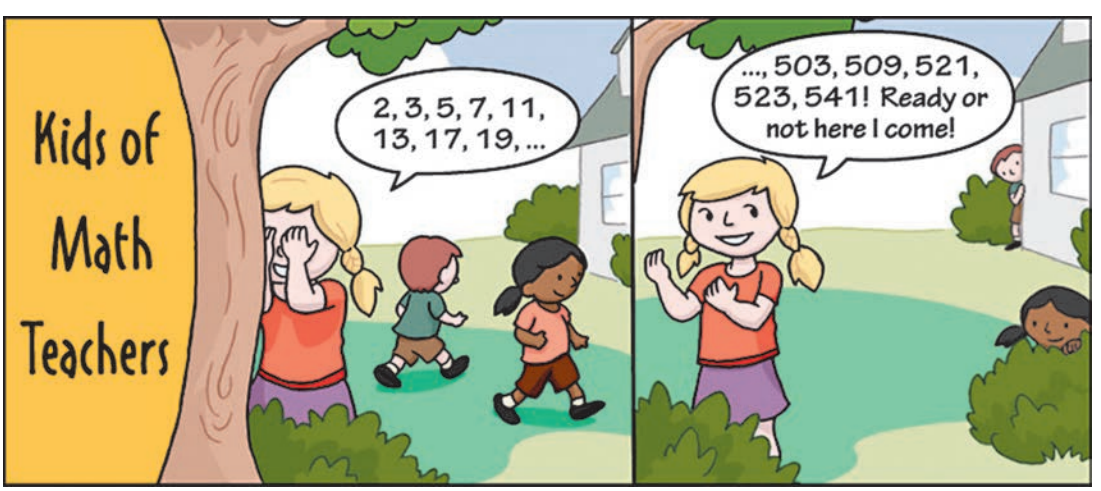
Solution

a. The possible primes p such that $p^2 \leq 397$ are 2, 3, 5, 7, 11, 13, 17, and 19. Because $2 \nmid 397$, $3 \nmid 397$, $5 \nmid 397$, $7 \nmid 397$, $11 \nmid 397$, $13 \nmid 397$, $17 \nmid 397$, and $19 \nmid 397$, the number 397 is prime.
b. The possible primes p such that $p^2 \leq 91$ are 2, 3, 5, and 7. Because 91 is divisible by 7, it is composite.

NOW TRY THIS 5

The cartoon below concerns divisibility and prime numbers. Answer the following questions based on the cartoon.

a. Find all the factors of the numbers shown. What can you conclude?
b. How many numbers in all did she say?

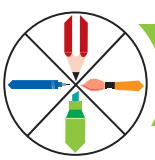


One way to find all the primes less than a given number is to use the *Sieve of Eratosthenes*, named after the Greek mathematician Eratosthenes (ca. 276–194 BCE). If all the whole numbers greater than 1 are considered (or placed in the sieve), the numbers that are not prime are methodically crossed out (or drop through the holes of the sieve). The remaining numbers are prime. The partial student page shown on page 196 illustrates this process.

Historical Note



Eratosthenes (276–194 BCE) spent most of his life in Alexandria as a chief librarian. In mathematics Eratosthenes is best known for his "sieve"—a systematic procedure for isolating the prime numbers—and for a simple method for calculating the circumference of the Earth. ●



School Book Page — Problem Solving

Problem Solving MATHEMATICAL PRACTICES

36. Use the following steps devised by the Greek mathematician Eratosthenes to create a list of prime numbers from 1 to 100. How many prime numbers are there between 1 and 100?

- Write all the numbers from 1 to 100.
- Draw a triangle around number 1; it is not prime nor composite.
- Circle 2 and cross out all other multiples of 2.
- Circle 3 and cross out all other multiples of 3.
- Circle 5, the next number that is not crossed out. Cross out other multiples of 5.
- Continue in the same way. When you have finished, the circled numbers are prime.



Lesson 11-2 261

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The Sieve of Eratosthenes is another way to demonstrate Theorem 4-16. Notice the observations from the sieve in Table 3 as we crossed out numbers.

Table 3

Prime	Observation
2	Once 2 is circled, the first number not crossed out that 2 divides is $4 = 2^2$.
3	Once 3 is circled, the first number not crossed out that 3 divides is $9 = 3^2$.
5	Once 5 is circled, the first number not crossed out that 5 divides is $25 = 5^2$.
7	Once 7 is circled, the first number not crossed out that 7 divides is $49 = 7^2$.

Problem Solving Perfect Squares

A perfect square is a whole number that is a square of a whole number. For example, 81 is a perfect square because $81 = 9^2$. Show that a nonzero whole number is a perfect square if, and only if, it has an odd number of divisors.

Understanding the Problem We need to show that all perfect squares—such as 36—have an odd number of divisors, and if a number is not a perfect square—such as 24—then it has an even number of divisors. Indeed, $36 = 6^2 = 2^2 \cdot 3^2$ and it has $(2 + 1)(2 + 1) = 9$ divisors, which is odd. In contrast, $24 = 4 \cdot 6 = 2^3 \cdot 3^1$ has $(3 + 1)(1 + 1) = 8$ divisors, an even number. We need to prove the statement in the problem for all nonzero whole numbers.

Devising a Plan We write the prime factorization of a perfect square and use Theorem 4-13 to find an expression for the number of divisors. Similarly, if a number is not a perfect square, we characterize its prime factorization and again use Theorem 4-13.

Carrying Out the Plan Every perfect square can be written as m^2, where m is a whole number. If the prime factorization of m is $p_1^{n_1} p_2^{n_2} \cdot \ldots \cdot p_k^{n_k}$, then $m^2 = p_1^{2n_1} p_2^{2n_2} \cdot \ldots \cdot p_k^{2n_k}$. Theorem 4-13 tells us that the number of divisors of m^2 is $(2n_1 + 1)(2n_2 + 1) \cdot \ldots \cdot (2n_k + 1)$. Each of these factors is odd. Since a product of odd numbers is odd, the above product is odd.

If a number is not a perfect square, then the exponents to which the primes in its prime factorization are raised cannot be all even. If they were all even, the number would be a perfect square. Thus, at least one of the exponents is odd. Suppose that exponent is n_j; then by Theorem 4-13, the number of divisors would be a product in which one of the factors in that product is $n_j + 1$. Since n_j is odd, $n_j + 1$ is even. When an even whole number is multiplied by any whole number, the result is even and hence the product is even.

Looking Back We could approach the problem differently. Theorem 4-14 states that if $d|n$, where $n \neq 0$, then n/d is also a divisor of n. Consequently, if for all divisors d of n, $d \neq n/d$, then each divisor can be paired with a different divisor of n. Therefore n must have an even number of divisors. If for some divisor d, $d = n/d$, then $n = d^2$, and all the divisors of n except d are paired with a different divisor. Hence the number of divisors is odd. It follows that n has an even number of divisors if, and only if, n is a perfect square.

More About Primes

There are infinitely many whole numbers, infinitely many odd whole numbers, and infinitely many even whole numbers. Are there infinitely many primes? The answer to this question is not obvious. Euclid was the first to prove that there are infinitely many primes.

Mathematicians have long looked for a formula that produces only primes, but no one has found one. One result was the expression $n^2 - n + 41$, where n is a whole number. Substituting $0, 1, 2, 3, \ldots, 40$ for n in the expression always results in a prime number. However, substituting 41 for n gives $41^2 - 41 + 41 = 41^2$, which is a composite number. In 1998, Roland Clarkson, a 19-year-old student at California State University, showed that $2^{3,021,377} - 1$ is prime. The number has 909,526 digits. The full decimal expansion of the number would fill several hundred pages. Since then, more large primes have been discovered, including $2^{57,885,161} - 1$ (17,425,170 digits) in 2013. The latest primes are examples of *Mersenne primes*. A Mersenne prime, named after the French monk Marin Mersenne (1588–1648), is a prime of the form $2^n - 1$, where n is prime.

Another type of interesting prime is a *Sophie Germain prime*, which is an odd prime p for which $2p + 1$ is also a prime. Notice that $p = 3$ is a Sophie Germain prime, since $2 \cdot 3 + 1$, or 7, is also a prime. Check that 5, 11, and 23 are also such primes. The primes were named after the French mathematician Sophie Germain. The greatest Sophie Germain prime discovered as of 2012 is $18{,}543{,}637{,}900{,}515 \cdot 2^{666,667} - 1$ with 200,701 digits.

Historical Note



Sophie Germain (1776–1831) was born in Paris and grew up during the French Revolution. She wanted to study at the prestigious École Polytechnique, but women students were not allowed. Consequently, she studied from lecture notes and from Gauss's monograph on number theory. Germain's work was highly regarded by Gauss, who recommended her for an honorary degree from the University of Göttingen, but she died before the degree could be awarded. ●

Problem Solving How Many Bears?

A large toy store carries one kind of stuffed bear. On Monday the store sold a certain number of the stuffed bears for \$1843, and on Tuesday, without changing the price, the store sold a certain number of the stuffed bears for \$1957. How many toy bears were sold each day if the price of each bear is a whole number and greater than \$1?

Understanding the Problem One day a store sold a number of stuffed bears for \$1843 and on the next day a number of them for a total of \$1957. We don't know the number of stuffed bears sold on each day nor do we know the cost.

However, we know the following:

Day 1: number of bears sold times the cost per bear is \$1843.

Day 2: number of bears sold times the cost per bear is \$1957.

The total for two days is (number of bears sold on day 1 plus number of bears sold on day 2) times the cost per bear is \$1843 + \$1957.

We need to find the number of bears sold each day.

Devising a Plan If x bears were sold the first day and y bears the second day, and if the price of each bear was c dollars, we would have $cx = 1843$ and $cy = 1957$. Thus, 1843 and 1957 should have a common factor—the price c. We could factor each number and find the possible factors. If the problem is to have a unique solution, the two numbers should have only one common factor other than 1. Any common factor of 1957 and 1843 will also be a factor of $1957 - 1843 = 114$ and the factors of 114 are easier to find.

Carrying Out the Plan We have $114 = 2 \cdot 57 = 2 \cdot 3 \cdot 19$. Thus, if 1957 and 1843 have a common prime factor, it must be 2, 3, or 19. Neither 2 nor 3 divides 1957 or 1843. Hence the only possible common factor is 19. We divide each number by 19 and find

$$1843 = 19 \cdot 97$$
$$1957 = 19 \cdot 103.$$

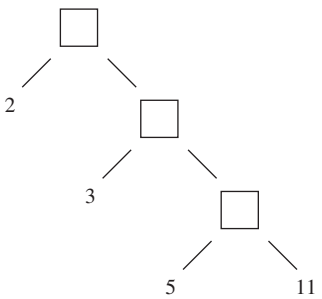
Consequently, the price of each bear is \$19. The first day 97 bears were sold, and the next day 103 bears were sold.

Looking Back The problem has a unique solution because the only common factor (greater than 1) of the two numbers is 19. We could create similar problems by having the price of the item be a prime number and the number of items sold each day also be prime numbers. For example, the total sale on the first day could have been $23 \cdot 101$, or \$2323, and on the second day $23 \cdot 107$, or \$2461 (23, 101, and 107 are prime numbers).

To find a common factor of 1957 and 1843, we found all the common factors of $1957 - 1843 = 114 = 2 \cdot 3 \cdot 19$ and checked which of the factors of the difference was a common factor of the original numbers. We used Theorem 4-2(c): If $d|a$ and $d|b$, then $d|(a - b)$. This theorem assures that every common factor of a and b is also a factor of $a - b$ if $a \geq b$. Thus the set of all common factors of a and b is a subset of all the factors of $a - b$.

Assessment 4-2A

1. a. Fill in the missing numbers in the following factor tree.



b. How could you find the top number without finding the other two numbers?

2. Use a factor tree to find the prime factorization for each of the following numbers.

a. 504 **b.** 2475 **c.** 11,250

3. Find the prime factorization of the following numbers.

a. $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$

b. $10^2 \cdot 13 \cdot 49^{10}$

c. 251

d. 100^{10}

4. What is the greatest prime that must be considered to test whether 769 is prime?

5. Determine which of the following numbers are primes.

a. 103 **b.** 119

c. 31 **d.** 303

e. 463 **f.** 97

g. $2 \cdot 3 \cdot 5 \cdot 7 + 1$

h. $2 \cdot 3 \cdot 5 \cdot 7 + 11$

6. Find the least natural number that is divisible by three different primes.

7. Find the least three-digit whole number that has exactly five positive factors.

8. a. Use the Fundamental Theorem of Arithmetic to justify that if $2 \mid n$ and $3 \mid n$, then $6 \mid n$.

b. Is it always true that if $a \mid n$ and $b \mid n$, then $ab \mid n^2$? Either prove the statement or give a counterexample.

9. Find the prime factorization of each of the following numbers.

a. $36^{10} \cdot 49^{20} \cdot 6^{15}$

b. $100^{60} \cdot 300^{40}$

c. $2 \cdot 3^4 \cdot 5^{110} \cdot 7 + 4 \cdot 3^4 \cdot 5^{110}$

d. $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1$

10. Explain why $2^3 \cdot 3^2 \cdot 25^3$ is not a prime factorization and find the prime factorization of the number.

11. Is it possible to find nonzero whole numbers x, y, and z such that $3^x \cdot 5^y = 8^z$? Justify your answer.

12. If $32n = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$, explain why $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11^6$ is a factor of n.

13. Is $7^4 \cdot 11^3$ a factor of $7^5 \cdot 11^3$? Explain why or why not.

14. a. When the U.S. flag had 48 stars, the stars formed a 6 by 8 rectangular array. In what other rectangular arrays could they have been arranged?

b. How many rectangular arrays of stars could there be if there were only 47 states?

15. Miss Cotton wants to put her classroom in a rectangular array. For each of the following number of desks, find all possible numbers of rows if each row has the same number of desks.

a. 30 **b.** 28 **c.** 23 **d.** 120

16. Briah and Emma are on the same swim team for the summer. The coach placed each team member on a different training schedule. As a result, Briah swims laps every other day and Emma swims laps every three days. The schedule begins on the first day of summer, with both girls swimming laps. How many times will Briah and Emma swim laps on the same day during the first ten weeks of summer?

17. A woman with a basket of eggs finds that if she removes the eggs from the basket 3 or 5 at a time, there is always 1 egg left. However, if she removes the eggs 7 at a time, there are no eggs left. If the basket holds up to 100 eggs, how many eggs does she have?

18. A band of 17 pirates stole a sack of golden coins. When they tried to divide them into equal portions there were 3 leftover. In the brawl over who would get the extra 3 coins a pirate was killed. They tried to divide the coins into equal portions again, but this time 10 coins were left over. Again there was a brawl and another pirate was killed. Now when the coins were divided none were left over. What is the least number of coins the pirates stole?

19. A license plate is in two parts, a two-digit number and a three-digit number. The two-digit number is prime and the sum of the two digits is a two-digit prime. The tens digit is greater than the units digit. In the three digit part, the digits are all odd and different. The sum of the three digits is palindromic. The sum of the first and third digit is one-half the sum of the first and second. Determine the license plate number.

20. The prime numbers 11 and 13 are called *twin primes* because they differ by 2. (The existence of infinitely many twin primes has not been proved.) Find all the twin primes less than 200.

21. A whole number is a *perfect* number if the sum of its factors (other than the number itself) is equal to the number. For example, 6 is a perfect number because its factors sum to 6; that is, $1 + 2 + 3 = 6$.

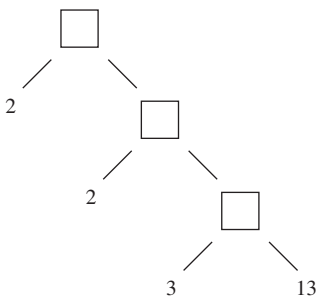
a. Determine which of the following numbers are perfect.

i. 12 **ii.** 28 **iii.** 35

b. Find at least one more number that is perfect.

Assessment 4-2B

1. a. Fill in the missing numbers in the following factor tree.



b. How could you find the top number without finding the other two numbers?

2. Use a factor tree to find the prime factorization for each of the following numbers.
a. 544 **b.** 3010 **c.** 9550

3. Find the prime factorization of the following numbers.
a. 1001
b. 1001^2
c. 999^{10}
d. $111^{10} - 111^9$

4. What is the greatest prime that must be considered to test whether 671 is prime?

5. Determine which of the following numbers are primes.
a. 101 **b.** 113
c. 153 **d.** 189
e. 541 **f.** 601
g. $2 \cdot 3 \cdot 5 \cdot 7 + 5$ **h.** $2 \cdot 3 \cdot 5 \cdot 7 - 5$

6. Find the least natural number that is divisible by four different primes.

7. Find the greatest four-digit whole number that has exactly three positive factors.

8. Use the Fundamental Theorem of Arithmetic to prove that if $4 \mid n$ and $9 \mid n$, then $36 \mid n$.

9. Find the prime factorization of each of the following numbers.
a. $16^4 \cdot 81^4 \cdot 6^6$
b. $8^4 \cdot 32^5$
c. $2^2 \cdot 3^5 \cdot 7^{55} + 2^4 \cdot 3^4 \cdot 7^{55}$
d. $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 + 1$

10. Explain why $2^2 \cdot 5^3 \cdot 9^2$ is not a prime factorization and find the prime factorization of the number.

11. Is it possible to find nonzero whole numbers x, y, and z such that $2^x \cdot 3^y = 5^z$? Justify your answer.

12. If $2n = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$, explain why $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ is a factor of n.

13. Is $3^2 \cdot 2^4$ a factor of $3^3 \cdot 2^2$? Explain why or why not.

14. a. A rectangular solar panel is made up of small squares. A particular solar panel is 4 squares by 6 squares. In what other rectangular arrays could the squares have been arranged?
b. If there were 43 squares, how many rectangular arrays can there be?

15. Mr. Todd wants to build a new rectangular patio. For each of the following numbers of square tiles, find all possible lengths and widths.
a. 15 **b.** 24 **c.** 17 **d.** 200

16. Barbara is getting in shape, and every other day she goes for a run and every fifth day she goes to dance class. How many times in a 30-day month does she run and dance in the same day?

17. Jose has fewer than 100 baseball cards. If he puts them in piles of 2, 3, 4, 5, or 6 he has one card left over each time. How many baseball cards does Jose have?

18. Isabelle, Noelle, and Juliette acquired a pile of chocolate coins. They go to bed, leaving their dad to guard the coins. An hour later Juliette gets up, gives two chocolate coins from the pile to her dad as hush money, divides the remaining pile of coins evenly into three parts, takes one part, and goes back to bed. A little later, Noelle gets up, gives two chocolate coins to her dad as hush money, divides the remaining pile evenly into three parts, takes one part, and goes back to bed. A little later, Isabelle gets up and does the very same thing as her siblings. In the morning, when they count the remaining chocolate coins, there are 100 of them left. How many were in the pile originally?

19. Anya had a collection of red, green, and blue beads. She noticed that the number of beads in each color was a prime number and that the numbers were all different. She also observed that if she multiplied the number of red beads by the total number of red and green beads she obtained a number exactly 120 greater than the number of blue beads. How many beads of each color did she have?

20. A prime such as 7331 is a *superprime* because any integers obtained by deleting digits from the right of 7331 are prime, namely, 733, 73, and 7.
a. For a prime to be a superprime, what digit(s) cannot appear in the number?
b. Of the digits that can appear in a superprime, what digit(s) cannot be the leftmost digit of a superprime?
c. Find all two-digit superprimes.
d. Find a three-digit superprime other than 733.

21. A whole number is an *abundant* number if the sum of its factors (other than the number itself) is greater than the number itself. A *deficient* number is a number with factors (other than the number itself) whose sum is less than the number itself.
a. Determine which of the following numbers are abundant or deficient.
i. 12 **ii.** 28 **iii.** 35
b. Find at least one more number that is deficient and one that is abundant.

Mathematical Connections 4-2

Reasoning

1. Explain why the product of any three consecutive whole numbers is divisible by 6.
2. Explain why the product of any four consecutive whole numbers is divisible by 24.
3. In the Sieve of Eratosthenes for numbers less than 100, explain why, after crossing out all the multiples of 2, 3, 5, and 7, the remaining numbers are prime.
4. Let $M = 2 \cdot 3 \cdot 5 \cdot 7 + 11 \cdot 13 \cdot 17 \cdot 19$. Without multiplying, show that none of the primes less than or equal to 19 divides M.
5. The numbers 2, 3, 5, 7, 11, 13, and 17 are not factors of 353. Explain why it is possible to conclude that 353 is prime without checking for more prime factors.
6. Explain why, when a number is composite, its least whole number divisor, other than 1, must be prime.
7. Euclid proved that given any finite list of primes, there exists a prime not in the list. Read the following argument and answer the questions that follow.

 Let $2, 3, 5, 7, \ldots, p$ be a list of all primes less than or equal to a certain prime p. We will show that there exists a prime not on the list. Consider the product

 $$2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p$$

 Every prime in the list divides that product. However, if we add 1 to the product, that is, form the number $n = (2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p) + 1$, then none of the primes in the list will divide n. Whether n is prime or composite, some prime q must divide n. Because no prime in the list divides n, q is not one of the primes in the list. Consequently $q > p$. We have shown that there exists a prime greater than p.
 - **a.** Explain why no prime in the list will divide n.
 - **b.** Explain why some prime must divide n.
 - **c.** Does the argument show that there are infinitely many primes? Why or why not?
8. A class of 23 students was using square tiles to build rectangular shapes. Each student had more than 1 tile and each had a different number of tiles. Each student was able to build only one shape of rectangle. All tiles had to be used to build a rectangle and the rectangle could not have holes. For example, a 2 by 6 rectangle uses 12 tiles and is considered the same as a 6 by 2 rectangle but is different from a 3 by 4 rectangle. The class did the activity using the least number of tiles. How many tiles did the class use? Explore the various rectangles that could be made.
9. Explain why each of the following numbers is composite.
 - **a.** $3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
 - **b.** $(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8) + 2$
 - **c.** $(3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) + 5$
 - **d.** $10! + 7$ (*Note*: $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.)
10. After a game of marbles with his three friends, Jacob said, "If only I had 1 more marble, I would have 4 times as many as Felipe, 5 times as many as Bella, and 7 times as many as Jessie." What is the least number of marbles Jacob could have had?
11. When students asked their teacher how old her children were, she said, "I have three children. The product of their ages is 72 and the sum of their ages is the number of this room." The children then asked for the door to be opened to verify the room number. Then Carly told the teacher she needed more information to solve the problem. The teacher said, "My oldest child is good at math." Then Carly announced the correct ages of her teacher's children. What are the ages of the teacher's children?

Open-Ended

12. **a.** In which of the following intervals are there more primes? Why?
 - **i.** 0–99
 - **ii.** 100–199

 b. What is the longest string of consecutive composite numbers in the intervals?
 c. How many twin primes are there in each interval?
 d. What patterns, if any, do you see for any of the preceding questions? Predict what might happen in other intervals.

Cooperative Learning

13. There are 100 people waiting in line for a baseball spring training game. The manager gives each person a free hat and then asks them to participate in a contest where the people with their hats on at the end of the game win free admission. The manager asks the first person in line and then every person after that to put on their hat and then asks the second person in line and then every other person after that to take their hat off, every third person to change their status (if they had their hat on, they take it off and if they had their hat off they put it on), every fourth person to change their status, and so on. All 100 people played the game. Which people won free admission? Work in your group to come up with a strategy to solve this problem and write out the solution.
14. In the game *Catch the Divisors*, player *A* circles a number on the grid in blue and player *B* circles all the divisors of the number in red. Player *A*'s score is the number they circled and player *B*'s score is the sum of the divisors they circled. Once a number is circled it cannot be used again. The game is done when all numbers are circled and the winner is the player with the highest total score. Play the game in pairs and discuss winning strategies.

1	2	2	2	2	2	3
3	3	3	3	4	4	4
4	5	5	5	5	6	6
6	7	7	8	8	9	9
10	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	30
32	33	34	35	36	38	39
40	42	44	45	46	48	49
50	51	52	54	55	56	60

Connecting Mathematics to the Classroom

15. In order to test for divisibility by 12, one student checked to determine divisibility by 3 and 4; another checked for divisibility by 2 and 6. Are both students using a correct approach to divisibility by 12? Why or why not?

16. In the book *A Remainder of One* by Elinor J. Pinczes (1995), ant soldier Joe is part of the 25th squadron. If the squadron divides into 2, 3, 4, and 5 queues, how many soldiers are in each queue? Make all possible arrays with 25 ants and find the one without a remainder. Discuss how modeling this story can help students find a solution.
17. Mary says that her factor tree for 72 begins with 3 and 24 so her prime factors will be different from Larry's because he is going to start with 8 and 9. What do you tell Mary?
18. Bob says that to check whether a number is prime he just uses the divisibility rules he knows for 2, 3, 4, 5, 6, 8, and 10. He says if the number is not divisible by these numbers, then it is prime. How do you respond?
19. Joe says that every odd number greater than 3 can be written as the sum of two primes. To convince the class, he wrote $7 = 2 + 5, 5 = 2 + 3$, and $9 = 7 + 2$. How do you respond?
20. A sixth-grade student argues that there are infinitely many primes because "there is no end to numbers." How do you respond?
21. A student claims 1 is prime because it only has 1 and itself as factors. How do you respond?

Review Problems

22. Classify the following statements as true or false.
 a. 13 is a factor of 130.
 b. 1001 is a multiple of 11.
 c. $7 \mid 1001$ and $7 \nmid 12$ imply $7 \nmid (1001 - 12)$.
23. Check each of the following numbers for divisibility by 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11.
 a. 438,162
 b. 2,345,678,910
24. Prove that if a number is divisible by 15, then it is divisible by 3.

National Assessments

National Assessment of Educational Progress (NAEP) Questions

A. If c and d are different prime numbers less than 10 and the sum $c + d$ is a composite number greater than 10, what is one possible pair of values for c and d?

$c =$ _____ $d =$ _____

B. If j and k are different prime numbers less than 10 and the sum $j + k$ is a prime number less than 10, what is one possible pair of values for j and k?

$j =$ _____ $k =$ _____

C. If s and t are different prime numbers greater than 10, explain why the sum $s + t$ cannot be a prime number.

$s =$ _____ $t =$ _____

NAEP, Grade 8, 2013

Which of the following true statements proves that 119 is not a prime number?
A. $17 \cdot 7 = 119$
B. $119 \cdot 1 = 119$
C. 119 is greater than 100.
D. 119 is an odd number.
E. 119 is not divisible by 3.

NAEP, Grade 8, 2011

4-3 Greatest Common Divisor and Least Common Multiple

4-3 Objectives

Students will be able to understand and explain

- The greatest common divisor and least common multiple of two or more whole numbers by using a variety of methods.
- Applications of GCD and LCM to word problems and their solutions.
- The Euclidean Algorithm in relation to the greatest common divisor.

CCSS **In the grade 6 *Common Core Standards*, students**

find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12." Furthermore they are asked to "use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express $36 + 8$ as $4(9 + 2)$. (p. 42)

Consider the following problem:

There are two fourth-grade classrooms at Haven Elementary School. Class 4-A has 24 students and class 4-B has 30 students. During recess, the two classes will play many games with each other, but they have to pick teams based on the following rules:

1. Teams must consist of students from the same class.
2. All teams must be of equal size.
3. Every student must be on exactly one team.
4. Teams should have as many students as possible.

How many teams can be made? How many students will be on each team?

The students could be in teams of 2, but this does not satisfy the condition of having the largest team possible. The number of students must divide both 24 and 30. (Why?). Numbers that divide both 24 and 30 are 1, 2, 3, and 6. The greatest of these numbers is 6, so there should be 6 students on each team. Class 4-A would have 4 teams and class 4-B would have 5 teams. In this problem, we have found the greatest number that divides both 24 and 30, that is, the *greatest common divisor* (GCD) of 24 and 30. We denote this as $\text{GCD}(24, 30) = 6$. Another name for the greatest common divisor is the *greatest common factor* (GCF).

Definition

The **greatest common divisor (GCD)** or the **greatest common factor (GCF)** of two whole numbers a and b, not both 0, is the greatest whole number that divides both a and b.

We provide several methods for finding GCDs.

Colored Rods Method

We can build a model of two or more nonzero whole numbers with colored rods to determine their GCD. For example, find the GCD of 6 and 8 using the 6 rod and the 8 rod, as in Figure 11.

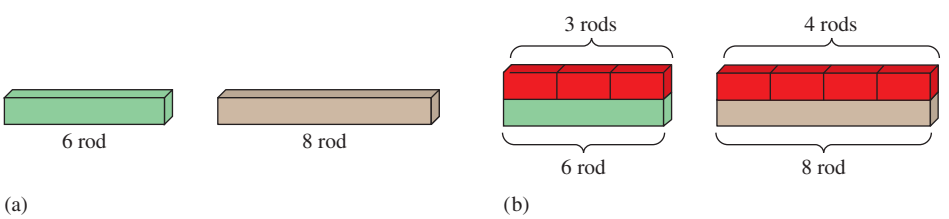


Figure 11

To find the GCD of 6 and 8, we must find the longest rod such that we can use multiples of that rod to build both the 6 rod and the 8 rod. The 2 rods can be used to build both the 6 and 8 rods, as shown in Figure 11(b); the 3 rods can be used to build the 6 rod but not the 8 rod; the 4 rods can be used to build the 8 rod but not the 6 rod; the 5 rods can be used to build neither; and the 6 rods cannot be used to build the 8 rod. Therefore, the GCD of 6 and 8 is 2. We denote this as $\text{GCD}(6, 8) = 2$.

The Intersection of Sets Method

In the *intersection of sets* method, we list all members of the set of whole number divisors of the two numbers, then find the set of all *common divisors*, and, finally, pick the *greatest* element in that set. For example, to find the GCD of 20 and 32, we write the sets of divisors of 20 and 32, denoted by D_{20} and D_{32}, respectively.

$$\begin{array}{rl} D_{20} = & \{1, 2, 4, 5, 10, 20\} \\ & \updownarrow\ \updownarrow\ \updownarrow \\ D_{32} = & \{1, 2, 4, 8, 16, 32\} \end{array}$$

The set of all common whole number divisors of 20 and 32 is

$$D_{20} \cap D_{32} = \{1, 2, 4\}.$$

Because the greatest number in the set of common whole number divisors is 4, $\text{GCD}(20, 32) = 4$.

NOW TRY THIS 6

The Venn diagram in Figure 12 shows the factors of 24 and 40.

a. What is the meaning of each of the shaded regions shown?
b. Which factor is the GCD of 24 and 40?
c. Draw a similar Venn diagram to find the GCD of 36 and 44.

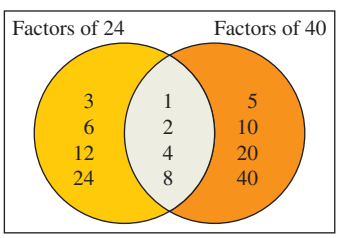


Figure 12

The Prime Factorization Method

The intersection of sets method is rather time consuming and tedious if the numbers have many divisors. Another, more efficient, method is the *prime factorization method*. To find GCD(180, 168), we first consider the prime factorization of the numbers:

$$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = (2^2 \cdot 3)3 \cdot 5$$

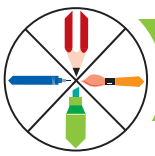
and

$$168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = (2^2 \cdot 3)2 \cdot 7$$

We see that 180 and 168 have two factors of 2 and one of 3 in common. These common primes divide both 180 and 168. In fact, the only numbers other than 1 that divide both 180 and 168 must have no more than two 2s and one 3 and no other prime factors in their prime factorizations. The possible common divisors are $1, 2, 2^2, 3, 2 \cdot 3$, and $2^2 \cdot 3$. Hence, the greatest common divisor of 180 and 168 is $2^2 \cdot 3$ or 12. The procedure for finding the GCD of two or more numbers by using the prime factorization method is summarized as follows:

> *To find the GCD of two or more nonzero whole numbers, first find the prime factorizations of the given numbers and then identify each common prime factor of the given numbers. The GCD is the product of the common prime factors, each raised to the lowest power of that prime that occurs in any of the prime factorizations.*

If we apply the prime factorization technique to finding GCD(4, 9), we see that 4 and 9 have no common prime factors. But that does not mean there is no GCD. We still have 1 as a common divisor, so GCD(4, 9) = 1. Numbers, such as 4 and 9, whose GCD is 1 are said to be **relatively prime**. Both the intersection of sets method and the prime factorization method are found on the student page shown on page 205.



School Book Page

Greatest Common Factor

4-4 Greatest Common Factor

Check Skills You'll Need

1. **Vocabulary Review** Write a sentence about math using the words *factor* and *product*.

Find the prime factorization.

2. 45
3. 21
4. 99

GO for Help Lesson 4-3

What You'll Learn

To find the GCF of two or more numbers

New Vocabulary common factor, greatest common factor (GCF)

Why Learn This?



A stamp club president distributes equally one set of 18 stamps and another set of 30 stamps to members present at a meeting. No stamps are left over. You can use factors to find the greatest possible number of club members at the meeting.

To find the greatest possible number of club members, you can find the factors that 18 and 30 share. A factor that two or more numbers share is a **common factor.**

The **greatest common factor (GCF)** of two or more numbers is the greatest factor shared by all the numbers. You can find the GCF of two numbers by listing their factors.

EXAMPLE 1 Using Lists of Factors

Find the greatest common factor of 18 and 30.

List the factors of 18 and the factors of 30. Then circle the common factors.

Factors of 18: (1), (2), (3), (6), 9, 18
Factors of 30: (1), (2), (3), 5, (6), 10, 15, 30

← The common factors are 1, 2, 3, and 6.

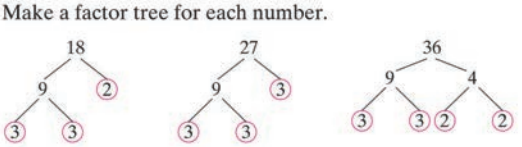
The greatest common factor (GCF) is 6.

EXAMPLE 3 Using Factor Trees



A volunteer divides 18 adults, 27 girls, and 36 boys into groups to clean up the park. He divides the adults, girls, and boys equally among the groups. What is the largest possible number of groups he can make?

Make a factor tree for each number.



$18 = 3 \times 3 \times 2$ ← Write the prime factorization for each number.
$27 = 3 \times 3 \times 3$
$36 = 3 \times 3 \times 2 \times 2$

↑ Identify common factors.

The GCF of 18, 27, and 36 is 9. The largest number of groups is 9.

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EXAMPLE 12

Find each of the following:

a. GCD(108, 72) **b.** GCD(0, 13)
c. GCD(x, y) if $x = 2^3 \cdot 7^2 \cdot 11 \cdot 13$ and $y = 2 \cdot 7^3 \cdot 13 \cdot 17$
d. GCD(x, y, z) if $z = 2^2 \cdot 7$, using x and y from part (c)
e. GCD(x, y), where $x = 5^4 \cdot 13^{10}$ and $y = 3^{10} \cdot 11^{20}$

Solution

a. Since $108 = 2^2 \cdot 3^3$ and $72 = 2^3 \cdot 3^2$, it follows that GCD$(108, 72) = 2^2 \cdot 3^2 = 36$.
b. Because $13 \mid 0$ and $13 \mid 13$, it follows that GCD$(0, 13) = 13$.
c. GCD$(x, y) = 2 \cdot 7^2 \cdot 13 = 1274$.
d. Because $x = 2^3 \cdot 7^2 \cdot 11 \cdot 13$, $y = 2 \cdot 7^3 \cdot 13 \cdot 17$, and $z = 2^2 \cdot 7$, the GCD$(x, y, z) = 2 \cdot 7 = 14$. Notice that GCD$(x, y, z)$ can also be obtained by finding the GCD of z and 1274, the answer from part (c).
e. Because x and y have no common prime factors, GCD$(x, y) = 1$.

Calculator Method



Calculators with a [Simp] key can be used to find the GCD of two numbers. For example, to find GCD(120, 180), use the following sequence of buttons to start: First, press [1] [2] [0] [/] [1] [8] [0] [Simp] [=] to obtain the display [N/D → n/d 60/90]. By pressing the [$x \circ y$] button, we see [2] on the display as a common divisor of 120 and 180. By pressing the [$x \circ y$] button again and pressing [Simp] [=] [$x \circ y$], we see 2 again as a factor. The process is repeated to reveal 3 and 5 as other common factors. The GCD of 120 and 180 is the product of the common prime factors $2 \cdot 2 \cdot 3 \cdot 5$, or 60.

Some calculators have a built-in GCD feature, probably found in the *MATH* menu. With this feature, select GCD and enter the numbers separated by a comma and enclosed within parentheses; for example, GCD(120, 180). When the [=] is pressed, the GCD of 60 will be displayed.

Euclidean Algorithm Method

For large numbers, a more efficient method than factorization for finding the GCD is available. For example, suppose we want to find GCD(676, 221). If we could find two smaller numbers whose GCD is the same as GCD(676, 221), the task would be easier. From Theorem 4-2(c), every divisor of 676 and 221 is also a divisor of $676 - 221$ and 221. Conversely, every divisor of $676 - 221$ and 221 is also a divisor of 676 and 221. Thus, the set of all the common divisors of 676 and 221 is the same as the set of all common divisors of $676 - 221$ and 221. Consequently, GCD$(676, 221) =$ GCD$(676 - 221, 221)$. This process can be continued to subtract three 221s from 676 so that GCD$(676, 221) =$ GCD$(676 - 3 \cdot 221, 221) =$ GCD$(13, 221)$. Because 13 is a prime, we can conclude now that the GCD is either 1 or 13. Since $13 \mid 221$, GCD$(13,221) = 13$. However, it is often advantageous to continue the process until the remainder 0 is obtained: GCD$(13, 221) =$ GCD$(221, 13) =$ GCD$(221 - 17 \cdot 13, 13) =$ GCD$(0, 13) = 13$. Consequently GCD$(676, 221) = 13$. To determine how many 221s can be subtracted from 676, and how many 13s from 221, we could have divided as follows:

$$\begin{array}{rr} & 3 \text{ R } 13 \quad\quad 17 \text{ R } 0 \\ 221\overline{)676} & \quad 13\overline{)221} \end{array}$$

When 0 is reached as a remainder, the divisions are complete. Based on this illustration, the following generalization is given in Theorem 4-17.

Theorem 4-17

If a and b are any whole numbers with $b \neq 0$ and $a \geq b$, then GCD$(a, b) =$ GCD(r, b), where r is the remainder when a is divided by b.

Because $\text{GCD}(x, y) = \text{GCD}(y, x)$ for all whole numbers x and y, not both 0, Theorem 4-17 can be written

$$\text{GCD}(a, b) = \text{GCD}(b, r).$$

We have seen that $\text{GCD}(676, 221) = \text{GCD}(676 - 221, 221)$

$$= \text{GCD}(676 - 3 \cdot 221, 221).$$

This is true in general, as stated in Theorem 4-18:

Theorem 4-18

If a and b are any nonzero whole numbers and $a \geq b$, then:

a. $\text{GCD}(a, b) = \text{GCD}(a - b, b)$

b. $\text{GCD}(a, b) = \text{GCD}(a - kb, b)$ for any whole number k for which $a - kb$ is a whole number.

Finding the GCD of two numbers by repeatedly using Theorem 4-17 until the remainder 0 is reached is referred to as the **Euclidean algorithm**. This method is found in Book VII of Euclid's *Elements* (300 BCE). Figure 13 is a flowchart for using the Euclidean algorithm.

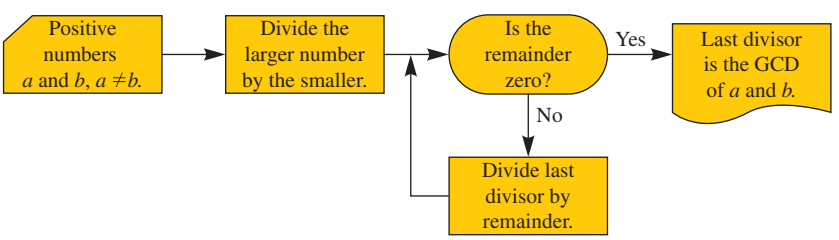


Figure 13

EXAMPLE 13

Use the Euclidean algorithm to find $\text{GCD}(10764, 2300)$.

Solution

$$\begin{array}{r} 4 \\ 2300\overline{)10764} \\ -9200 \\ \hline 1564 \end{array}$$

Thus, $\text{GCD}(10764, 2300) = \text{GCD}(2300, 1564)$.

$$\begin{array}{r} 1 \\ 1564\overline{)2300} \\ -1564 \\ \hline 736 \end{array}$$

Thus, $\text{GCD}(2300, 1564) = \text{GCD}(1564, 736)$.

$$\begin{array}{r} 2 \\ 736\overline{)1564} \\ -1472 \\ \hline 92 \end{array}$$

Thus, $\text{GCD}(1564, 736) = \text{GCD}(736, 92)$.

$$\begin{array}{r} 8 \\ 92\overline{)736} \\ -736 \\ \hline 0 \end{array}$$

Thus, $\text{GCD}(736, 92) = \text{GCD}(92, 0)$.

Because $\text{GCD}(92, 0) = 92$, it follows that $\text{GCD}(10764, 2300) = 92$.

The procedure for finding the GCD by using the Euclidean algorithm can be stopped at any step at which the GCD is obvious. Sometimes shortcuts can be used to find the GCD of two or more numbers, as seen in the following example.

EXAMPLE 14

Find each of the following:

a. GCD(134791, 6341, 6339)
b. The GCD of any two consecutive whole numbers

Solution

a. Any common divisor of three numbers is also a common divisor of any two of them (why?). Consequently, the GCD of three numbers cannot be greater than the GCD of any two of the numbers. The numbers 6341 and 6339 are close to each other, and therefore it is easy to find their GCD:

$$\begin{aligned} \text{GCD}(6341, 6339) &= \text{GCD}(6341 - 6339, 6339) \\ &= \text{GCD}(2, 6339) \\ &= 1 \end{aligned}$$

Because GCD(134791, 6341, 6339) cannot be greater than 1, it follows that it must equal 1.

b. GCD(4, 5) = 1, GCD(5, 6) = 1, GCD(6, 7) = 1, and GCD(99, 100) = 1. It seems that the GCD of any two consecutive whole numbers is 1. To justify this conjecture, we show that for all whole numbers n, GCD$(n, n + 1) = 1$. Using Theorem 4-18 we have:

$$\begin{aligned} \text{GCD}(n, n + 1) = \text{GCD}(n + 1, n) &= \text{GCD}(n + 1 - n, n) \\ &= \text{GCD}(1, n) \\ &= 1 \end{aligned}$$

Least Common Multiple

Consider the following problem:

It is Morgan's birthday and we need to buy cups and plates for the party. Plastic cups are sold in packages of 8 and paper plates are sold in packages of 12. What is the least number of each we need to buy to have an equal number of plates and cups? The number of cups that we need to buy are the multiples of 8, that is, 8, 16, 24, 32, 40, ... Likewise the number of plates we need to buy are multiples of 12, that is, 12, 24, 36, 48, ... In this problem we are interested in the least number we need to buy, which is 24. So, we need to buy 3 packages of cups and 2 packages of plates. We found the **least common multiple (LCM)** of 8 and 12, that is LCM(8, 12) = 24.

Definition

The **least common multiple (LCM)** of two nonzero whole numbers a and b is the least nonzero whole number that is simultaneously a multiple of a and a multiple of b.

As with GCDs, there are several methods for finding least common multiples.

Number Line Method

A number line can be used to find the LCM of two numbers. For example, to find LCM(3, 4), we show the multiples of 3 and 4 on the number line using intervals of 3 and 4, as shown in Figure 14.

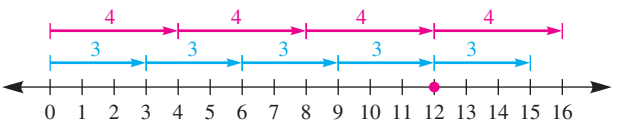


Figure 14

Beginning at 0, the arrows do not coincide until the point 12 on the number line. If the line were continued, the arrows would coincide again at 24, 36, 48, and so on. There are an infinite number of common multiples of 3 and 4, but the least common multiple is 12. This number line approach is instructive and promotes understanding but is not practical for large numbers.

Colored Rods Method

Colored rods are used to determine the LCM of two numbers. For example, consider the 3 rod and the 4 rod in Figure 15(a). Build trains of 3 rods and 4 rods until they are the same length, as shown in Figure 15(b). The LCM is the common length of the train.

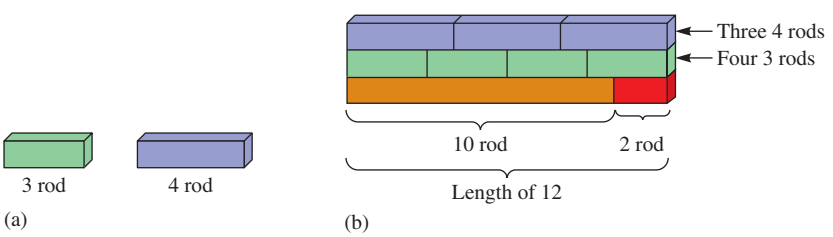


Figure 15

The Intersection of Sets Method

In the *intersection of sets* method, we first find the set of all positive *multiples* of both the first and second numbers, then find the set of all *common multiples* of both numbers, and finally pick the *least* element in that set. For example, to find the LCM of 8 and 12, we denote the sets of nonzero whole number multiples of 8 and 12 by M_8 and M_{12}, respectively.

$$M_8 = \{8, 16, 24, 32, 40, 48, 56, 64, 72, \ldots\}$$

$$M_{12} = \{12, 24, 36, 48, 60, 72, 84, 96, 108, \ldots\}$$

The set of common multiples is

$$M_8 \cap M_{12} = \{24, 48, 72, \ldots\}.$$

Because the least number in $M_8 \cap M_{12}$ is 24, the LCM of 8 and 12 is 24, or LCM(8, 12) = 24.

NOW TRY THIS 7

Draw a Venn diagram showing M_8 and M_{12} and show how to find LCM(8, 12) using the diagram.

The Prime Factorization Method

The intersection of sets method for finding the LCM is often lengthy, especially when it is used to find the LCM of three or more nonzero whole numbers. Another, more efficient, method for finding the LCM of several numbers is the *prime factorization method*. For example, to find LCM(40, 12), we first find the prime factorizations of 40 and 12: $40 = 2^3 \cdot 5$ and $12 = 2^2 \cdot 3$.

If $m = \text{LCM}(40, 12)$, then m is a multiple of 40 and must contain both 2^3 and 5 as factors. Also, m is a multiple of 12 and must contain 2^2 and 3 as factors. Since 2^3 is a multiple of 2^2, then $m = 2^3 \cdot 5 \cdot 3 = 120$. In general, we have the following:

> *To find the LCM of two nonzero whole numbers, we first find the prime factorization of each number. Then we take each of the primes that are factors of either of the given numbers. The LCM is the product of these primes, each raised to the greatest power of the prime that occurs in either of the prime factorizations.*

EXAMPLE 15

Find the LCM of 2520 and 10,530.

Solution

$$2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$$
$$10{,}530 = 2 \cdot 3^4 \cdot 5 \cdot 13$$
$$\text{LCM}(2520, 10{,}530) = 2^3 \cdot 3^4 \cdot 5 \cdot 7 \cdot 13 = 294{,}840$$

The prime factorization method can also be used to find the LCM of more than two numbers. For example, to find LCM(12, 108, 120), we proceed as follows:

$$12 = 2^2 \cdot 3$$
$$108 = 2^2 \cdot 3^3$$
$$120 = 2^3 \cdot 3 \cdot 5$$

Then, $\text{LCM}(12, 108, 120) = 2^3 \cdot 3^3 \cdot 5 = 1080$.

The GCD-LCM Product Method

To see the connection between the GCD and LCM, we consider the GCD and LCM of 24 and 30. The prime factorizations of these numbers are

$$24 = 2^3 \cdot 3$$
$$30 = 2 \cdot 3 \cdot 5.$$

Figure 16 shows a diagram with the prime factors.

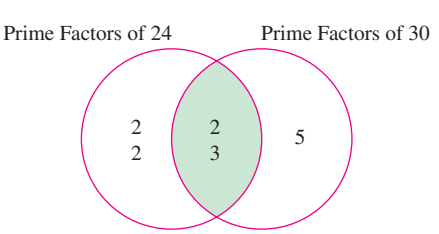


Figure 16

$\text{GCD}(24, 30) = 2 \cdot 3$ and is the product of the factors in the shaded region, and $\text{LCM}(24, 30) = 2^3 \cdot 3 \cdot 5$ is the product of the prime factors in the combined regions. Also:

$$\text{GCD}(24, 30) \cdot \text{LCM}(24, 30) = (2 \cdot 3)(2^3 \cdot 3 \cdot 5) = (2^3 \cdot 3)(2 \cdot 3 \cdot 5) = 24 \cdot 30$$

This shows that the product of the GCD and LCM of 24 and 30 is equal to $24 \cdot 30$. In general, the connection between the GCD and LCM of any pair of nonzero whole numbers is given by Theorem 4-19.

Theorem 4-19

For any two nonzero whole numbers a and b,

$$\text{GCD}(a, b) \cdot \text{LCM}(a, b) = ab.$$

Theorem 4-19 can be justified in several ways. A specific example suggesting how the theorem might be proved is given below:

$$\text{Let} \quad a = 5^{13} \cdot 7^{20} \cdot 11^4 \quad \text{and} \quad b = 5^{10} \cdot 7^{25} \cdot 11^6 \cdot 13.$$
$$\text{Then, LCM}(a, b) = 5^{13} \cdot 7^{25} \cdot 11^6 \cdot 13 \quad \text{and} \quad \text{GCD}(a, b) = 5^{10} \cdot 7^{20} \cdot 11^4.$$

We have $\text{LCM}(a, b) \cdot \text{GCD}(a, b) = 5^{13+10} \cdot 7^{25+20} \cdot 11^{6+4} \cdot 13$ and $ab = 5^{13+10} \cdot 7^{20+25} \cdot 11^{6+4} \cdot 13$.

For the preceding values of a and b, Theorem 4-19 is true. However, in general we reason as follows: in the product $\text{LCM}(a, b) \cdot \text{GCD}(a, b)$, we have all the powers of the primes appearing in a or in b, because for the LCM we take the greater of the powers of the common primes and for the GCD the lesser. Also, ab contains all the powers of the primes appearing in a or b. Hence, Theorem 4-19 is true in general.

The Euclidean Algorithm Method

Theorem 4-19 is useful for finding the LCM of two numbers a and b when their prime factorizations are not easy to find. $\text{GCD}(a, b)$ can be found by the Euclidean algorithm, the product ab can be found by multiplication, and $\text{LCM}(a, b)$ can be found by division.

EXAMPLE 16

a. Find $\text{LCM}(731, 952)$.
b. If $b \mid a$, find $\text{LCM}(a, b)$ in terms of a or b.

Solution

a. By the Euclidean algorithm, $\text{GCD}(731, 952) = 17$. By Theorem 4-19,

$$17 \cdot \text{LCM}(731, 952) = 731 \cdot 952.$$

Consequently,

$$\text{LCM}(731, 952) = \frac{731 \cdot 952}{17} = 40{,}936.$$

b. Because $b \mid a$, a is a multiple of b; thus $\text{LCM}(a, b) = a$.

The Division-by-Primes Method

Another procedure for finding the LCM of several nonzero whole numbers involves *division by primes*. For example, to find $\text{LCM}(12, 75, 120)$, start with the least prime that divides at least one of the given numbers and divide as follows:

$$\begin{array}{r|l} 2 & 12, 75, 120 \\ \hline & 6, 75, \ \ 60 \end{array}$$

Because 2 does not divide 75, we simply bring down the 75. To obtain the LCM using this procedure, we continue the division process until the row of answers consists of relatively prime numbers.

$$\begin{array}{r|l} 2 & 12, 75, 120 \\ \hline 2 & 6, 75, 60 \\ \hline 2 & 3, 75, 30 \\ \hline 3 & 3, 75, 15 \\ \hline 5 & 1, 25, 5 \\ \hline & 1, 5, 1 \end{array}$$

Thus, $\text{LCM}(12, 75, 120) = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 1 \cdot 5 \cdot 1 = 2^3 \cdot 3 \cdot 5^2 = 600$.

Assessment 4-3A

1. Use colored rods to find the GCD and the LCM of 6 and 8.

2. Find the GCD and the LCM for each of the following groups of numbers using the intersection-of-sets method.
- **a.** 18 and 12
- **b.** 20 and 36
- **c.** 8, 24, and 64
- **d.** 7 and 9

3. Find the GCD and the LCM for each of the following groups of numbers using the prime factorization method.
- **a.** 132 and 504
- **b.** 65 and 1690
- **c.** 96, 900, and 630
- **d.** 108 and 360

4. Find the GCD for each of the following groups of numbers using the Euclidean algorithm.
- **a.** 220 and 2924
- **b.** 14,595 and 10,856

5. Find the LCM for each of the following groups of numbers using any method.
- **a.** 24 and 36
- **b.** 72, 90, and 96
- **c.** 90, 105, and 315
- **d.** 16^{100} and 25^{100}

6. The product of two numbers is 1734 and their GCD is 17. Find their LCM.

7. The GCD of two numbers is 19 and their LCM is 228. One of the numbers is 57. Find the other number.

8. Assume a and b are natural numbers and answer the following questions.
- **a.** If $\text{GCD}(a, b) = 1$, find $\text{LCM}(a, b)$.
- **b.** Find $\text{GCD}(a, a)$ and $\text{LCM}(a, a)$.
- **c.** Find $\text{GCD}(a^2, a)$ and $\text{LCM}(a^2, a)$.
- **d.** If $a|b$, find $\text{GCD}(a, b)$ and $\text{LCM}(a, b)$.

9. Classify each of the following statements as true or false where a and b are whole numbers.
- **a.** If $\text{GCD}(a, b) = 1$, then a and b cannot both be even.
- **b.** If $\text{GCD}(a, b) = 2$, then both a and b are even.
- **c.** If a and b are even, then $\text{GCD}(a, b) = 2$.

10. To find GCD(24, 20, 12), it is possible to find GCD(24, 20), which is 4, and then find GCD(4, 12), which is 4. Use this approach and the Euclidean algorithm to compute the following.
- **a.** GCD(120, 75, 105)
- **b.** GCD(34578, 4618, 4619)

11. Show that 97,219,988,751 and 4 are relatively prime.

12. Draw Venn diagrams to show the prime factors for each of the following sets of three numbers.
- **a.** 10, 15, 60
- **b.** 8, 16, 24

13. Find three pairs (a, b) such that $\text{LCM}(a, b) = 48$.

14. Find all whole numbers x such that $\text{GCD}(49, x) = 1$ and $1 \leq x \leq 49$.

15. In Quinn's dormitory room, there are three snooze-alarm clocks, each of which is set at a different time. Clock A goes off every 15 min, clock B goes off every 40 min, and clock C goes off every 60 min. If all three clocks go off at 6:00 A.M., answer the following questions.
- **a.** How long will it be before the clocks go off simultaneously again after 6:00 A.M.?
- **b.** Would the answer to (a) be different if clock B went off every 15 min and clock A went off every 40 min?

16. A company is running a promotion on Facebook. Every twelfth and thirteenth "like" earns a badge and every twentieth "like" wins free movie tickets. Which "like" will be first to get two badges and a free movie ticket?

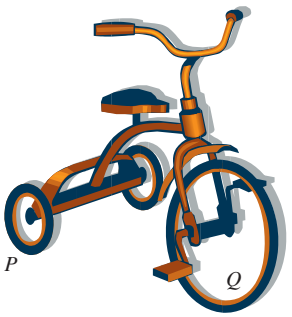
17. If there were to be 9 boys and 6 girls at a party and the host wanted each to be given exactly the same number of candies that could be bought in packages containing 12 candies, what is the fewest number of packages that could be bought?

18. Three motorcyclists ride around a circular course starting at the same place and the same time. The first passes the starting point every 12 min, the second every 18 min, and the third every 16 min. After how many minutes will all three pass the starting point again at the same time? Explain your reasoning.

19. Anna has made 42 chocolate chip cookies and 54 sugar cookies. She wants to put them in bags for her friends with the same number of cookies in each bag, the same kind of cookies in each bag, and no cookies left over. What is the greatest number of cookies she can put in each bag?

20. The front wheel of a tricycle has a circumference of 100 in. and the back wheels have a circumference of 18 in. If points P and Q are both touching the sidewalk when Jose starts to ride, how many ft will the tricycle

travel before P and Q first touch the sidewalk at the same time again?

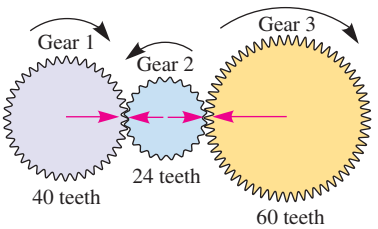


21. Jazz has 72 yd of red ribbon and 42 yd of blue ribbon. What are the largest possible pieces of ribbon Jazz can cut so that each piece is the same length and all the ribbon is used?

Assessment 4-3B

1. Use colored rods to find the GCD and the LCM of 4 and 10.
2. Find the GCD and the LCM for each of the following groups of numbers using the intersection-of-sets method.
 a. 12 and 30 **b.** 18 and 58
 c. 6, 18, and 24 **d.** 36 and 13
3. Find the GCD and the LCM for each of the following groups of numbers using the prime factorization method.
 a. 11 and 19 **b.** 140 and 320
 c. 800, 75, and 450 **d.** 103 and 320
4. Find the GCD for each of the following groups of numbers using the Euclidean algorithm.
 a. 14,560 and 8250
 b. 8424 and 2520
5. Find the LCM for each of the following groups of numbers using any method.
 a. 25 and 36
 b. 82, 90, and 50
 c. 80, 105, and 315
 d. 8^{100} and 50^{100}
6. The product of two numbers is 5880 and their GCD is 14. Find their LCM.
7. The GCD of two numbers is 11 and their LCM is 330. One of the numbers is 66. Find the other number.
8. Assume a and b are natural numbers and answer the following questions.
 a. If a and b are two different primes, find $\text{GCD}(a, b)$ and $\text{LCM}(a, b)$.
 b. What is the relationship between a and b if $\text{GCD}(a, b) = a$?
 c. What is the relationship between a and b if $\text{LCM}(a, b) = a$?
9. Classify each of the following statements as true or false where a and b are distinct natural numbers.
 a. $\text{LCM}(a, b) \mid \text{GCD}(a, b)$
 b. $\text{LCM}(a, b) \mid ab$
 c. $\text{GCD}(a, b) \leq a$
 d. $\text{LCM}(a, b) \geq a$
10. To find GCD(24, 20, 12), it is possible to find GCD(24, 20), which is 4, and then find GCD(4,12), which is 4. Use this approach and the Euclidean algorithm to find the following.
 a. GCD(180, 240, 306)
 b. GCD(5284, 1250, 1280)
11. Show that 181,345,913 and 11 are relatively prime.
12. Draw Venn diagrams to show the prime factors for each of the following sets of three numbers.
 a. 12, 14, 70 **b.** 6, 8, 18
13. Find three pairs (a, b) such that $\text{LCM}(a, b) = 60$.
14. Find all whole numbers x such that $\text{GCD}(25, x) = 1$ and $1 \leq x \leq 25$.
15. A light display has three colored lights, each of which flashes on for an instant at regular intervals. The yellow one flashes 15 times per minute, the blue one flashes 6 times per minute, and the red one flashes on 12 times per minute. You see all three lights flash on simultaneously. How much longer must you wait before the three lights flash on simultaneously again?
16. A movie rental store gives a free popcorn to every 15th customer and a free movie rental to every 40th customer. Which customer was the first to win both prizes?
17. Brian purchases 245 white bouncy balls, 238 yellow bouncy balls, and 84 orange bouncy balls. He asked his son to divide them up into bags, so that each bag of balls had at least one of each color in them, and that each bag had the same number of total balls, and that there were no balls left over. What is the greatest number of balls he can put into each bag?
18. A college runs two different bus routes starting at 7:00 A.M. Route A takes 75 min, while route B takes 90 min. If they take off from the same location, when will both busses be there at that same location again?
19. By selling cookies at 24¢ each, Jose made enough money to buy several cans of pop costing 45¢ per can. If he had no money left over after buying the pop, what is the least number of cookies he could have sold?

20. Determine how many complete revolutions each gear in the following diagram must make before the red arrows are lined up again.



21. Kobe has collected 300 football cards and 264 baseball cards. He wants to divide them into piles so that each pile has only one type of card, there is the same number of cards in each pile, and each pile has the greatest possible number of cards. How many cards will be in each pile?

Mathematical Connections 4-3

Reasoning

1. Can two nonzero whole numbers have a greatest common multiple? Explain your answer.
2. Is it true that $\text{GCD}(a, b, c) \cdot \text{LCM}(a, b, c) = abc$? Explain your answer.
3. Suppose that $\text{GCD}(a, b, c) = 1$. Is it necessarily true that $\text{GCD}(a, b) = \text{GCD}(b, c) = 1$? Explain your reasoning.
4. Suppose that $\text{GCD}(a, b) = \text{GCD}(b, c) = 2$. Does that always imply that $\text{GCD}(a, b, c) = 2$? Justify your answer.
5. Is it true that every common divisor of two nonzero whole numbers a and b is a divisor of the $\text{GCD}(a, b)$? Explain your answer.
6. How can you tell from the prime factorization of two numbers if their LCM equals the product of the numbers? Explain your reasoning.
7. Can the LCM of two nonzero whole numbers ever be greater than the product of the two numbers? Explain your reasoning.

Open-Ended

8. Find three pairs of numbers for which the LCM of the numbers in a pair is less than the product of the two numbers.
9. Describe infinitely pairs of numbers whose GCD is equal to the following numbers.
a. 2 **b.** 6 **c.** 91
10. A large gear is used to turn a smaller gear. If the larger gear makes 72 revolutions per minute and the smaller gear makes 1500 revolutions per minute, how many teeth does each gear have? Give three different possibilities. What is the least number of teeth possible?

Cooperative Learning

11. **a.** In your group, discuss whether the Euclidean algorithm for finding the GCD of two numbers should be introduced in middle school (To all students? To some?). Why or why not?
b. If you decide that it should be introduced in middle school, discuss how it should be introduced. Report your group's decision to the class.

Connecting Mathematics to the Classroom

12. Describe to a sixth-grade student the difference between a divisor and multiple.
13. Eleanor claims that the $\text{GCD}(0, a) = 0$. Is she correct? What does she understand about GCD? What does she not understand?
14. Aiko says to find the LCM you can just multiply the two numbers. As a teacher, how do you respond?
15. A student wants to know how many whole numbers between 1 and 10,000 inclusive are either multiples of 3 or multiples of 5. She wonders if it is correct to find the number of those whole numbers that are multiples of 3 and add the number of those that are multiples of 5. How do you respond?

Review Problems

16. Find the greatest digit that makes the following statements true.
a. $3 \mid 83_51$ **b.** $11 \mid 8_691$ **c.** $23 \mid 103_6$
17. Find the prime factorization of the following numbers.
a. 17,496 **b.** 32,715 **c.** $2^4 \cdot 8^2 \cdot 27^3$
18. Is 2223 prime? Justify your answer.
19. Find a number that has exactly five prime factors.
20. Find the least positive number that is divisible by 2, 4, 6, 8, and 10.
21. What is the greatest prime that must be used to determine if 3359 is prime?

National Assessments

National Assessment of Educational Progress (NAEP) Question

The least common multiple of 8, 12, and a third number is 120. Which of the following could be the third number?
A. 15
B. 16
C. 24
D. 32
E. 48

NAEP, Grade 8, 1990

Hint for Solving the Preliminary Problem

Try to find in how many days two cycles peak before finding when all three cycles peak again.

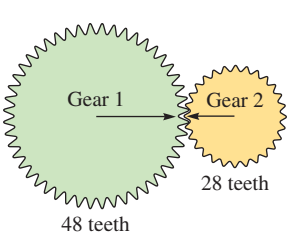
Chapter 4 Summary

KEY CONCEPTS	DEFINITIONS, DESCRIPTIONS, AND THEOREMS
Section 4-1	
Even whole number (p. 175)	A whole number that has remainder 0 when divided by 2.
Odd whole number (p. 175)	A whole number that has remainder 1 when divided by 2.
Divides (p. 175)	For any whole numbers a and b, where $b \neq 0$, b divides a (written $b \mid a$), if, and only if, there is a unique whole number q such that $a = bq$. If a and b are any whole numbers and $b \neq 0$, then b does *not* divide a, written $b \nmid a$ if, and only if, the remainder when a is divided by b is not 0.
Terms for divisibility (p. 177)	If $b \mid a$, then • b *divides* a. • b is a *factor* of a. • b is a *divisor* of a. • a is *divisible by* b. • a is a *multiple* of b.
Divisibility theorems (p. 177)	*Theorem:* For any whole numbers a and d, if $d \mid a$ and n is any whole number, then $d \mid na$. *Theorem:* For any whole numbers a, b, and d, • If $d \mid a$ and $d \mid b$, then $d \mid (a + b)$. • If $d \mid a$ and $d \nmid b$, then $d \nmid (a + b)$. • If $d \mid a$, $d \mid b$, and $a > b$, then $d \mid (a - b)$. • If $d \mid a$, $d \nmid b$, and $a > b$, then $d \nmid (a - b)$. • If $d \nmid a$, $d \mid b$, and $a > b$, then $d \nmid (a - b)$.
Divisibility rules (p. 179)	*Divisibility Tests* *for* 2: A whole number is divisible by 2 if, and only if, its units digits is divisible by 2; that is if, and only if, the units digit is even. *for* 5: A whole number is divisible by 5 if, and only if, its units digits is divisible by 5; that is if, and only if, the units digit is 0 or 5. *for* 10: A whole number is divisible by 10 if, and only if, its units digits is divisible by 10; that is if, and only if, the units digit is 0. *for* 4: A whole number is divisible by 4 if, and only if, the last two digits of the number represent a number divisible by 4. *for* 8 A whole number is divisible by 8 if, and only if, the last three digits of the number represent a number divisible by 8. *for* 3: A whole number is divisible by 3 if, and only if, the sum of its digits is divisible by 3. *for* 9 A whole number is divisible by 9 if, and only if, the sum of its digits is divisible by 9. *for* 11: A whole number is divisible by 11 if, and only if, the sum of the digits in the places that are even powers of 10 minus the sum of the digits in the places that are odd powers of 10 is divisible by 11. (If the sums are different, then subtract the smaller from the greater.) *for* 6: A whole number is divisible by 6 if, and only if, the number is divisible by 2 and 3.

Section 4-2	
Prime number (p. 190)	Any whole number with exactly two divisors.
Composite number (p. 190)	Any whole number greater than 1 that is not prime.
Prime factorization (p. 190)	A *factorization* (a product of factors) containing only prime numbers. A prime factorization of any composite number can be found by using a *factor tree*.
Fundamental Theorem of Arithmetic (p. 191)	Each composite number can be written as a product of primes in one, and only one way, except for the order of the prime factors in the product.
Finding prime numbers (p. 191)	*Theorem*: If d is a divisor of n, where $n \neq 0$, then $\frac{n}{d}$ is also a divisor of n. *Theorem*: If n is composite, then n has a prime factor p such that $p^2 \leq n$. *Theorem*: If n is a whole number greater than 1 and not divisible by any prime p such that $p^2 \leq n$, then n is prime. *Sieve of Eratosthenes*: A method of finding all the primes less than a given number.
Section 4-3	
The greatest common divisor (GCD) (p. 203)	For two whole numbers a and b, not both 0, the greatest whole number that divides both a and b.
Methods for finding the GCD (p. 203)	• *Colored rods* • *Intersection of sets* • *Prime factorization* • *Calculator* • *Euclidean Algorithm* *Theorem*: If a and b are any whole numbers with $b \neq 0$, and $a \geq b$, then $\text{GCD}(a, b) = \text{GCD}(r, b)$, where r is the remainder when a is divided by b. *Theorem*: If a and b are any nonzero whole numbers and $a \geq b$, then • $\text{GCD}(a, b) = \text{GCD}(a - b, b)$. • $\text{GCD}(a, b) = \text{GCD}(a - kb, b)$ for any whole number k for which $a - kb$ is a whole number.
Relatively prime (p. 204)	Two whole numbers are relatively prime if they share no common positive divisors except 1.
The least common multiple (LCM) (p. 208)	For two nonzero whole numbers a and b, the least natural number that is simultaneously a multiple of both a and b.
Methods for finding the LCM (p. 210)	• *Number line* • *Colored rods* • *Intersection of sets* • *Prime factorization* • *The GCD-LCM product* *Theorem*: For any two nonzero whole numbers a and b, $\text{GCD}(a, b) \cdot \text{LCM}(a, b) = ab$. • *Division-by-primes*

Chapter 4 Review

1. Classify each of the following statements as true or false.
 a. $12 \mid 4$
 b. $0 \mid 8$
 c. $8 \mid 0$
 d. If a number is divisible by 4 and by 6, then it is divisible by 24.
 e. If a number is not divisible by 12, then it is not divisible by 3.
2. Test each of the following numbers for divisibility by 2, 3, 4, 5, 6, 8, 9, 10, and 11.
 a. 125,160
 b. 12,193
3. Find the greatest digit to make each of the following statements true.
 a. $6 \mid 87_4$
 b. $24 \mid 4_856$
 c. $29 \mid 87__4$
4. A student claims that the sum of five consecutive whole numbers is always divisible by 5.
 a. Check the student's claim for a few cases.
 b. Prove or disprove the student's claim.
5. How can you tell if a number is divisible by 24? Check 4152 for divisibility by 24.
6. Construct a number that has exactly five positive divisors. Explain your construction.
7. Find all the positive divisors of 144.
8. Prove the test for divisibility by 9 using a three-digit number n such that $n = a \cdot 10^2 + b \cdot 10 + c$.
9. If 1009 is prime, prove that 1026 is not divisible by 17.
10. Determine whether each of the following numbers is prime or composite.
 a. 147
 b. 373
11. Explain why each of the following numbers is composite.
 a. $7 \cdot 11 \cdot 13 \cdot 17 + 17$
 b. $10! + k$ where $k = 2, 3, 4, 5, 6, 7, 8, 9,$ or 10 (*Note*: $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.)
12. The triplets 3, 5, 7 consist of consecutive odd whole numbers that are all prime. Give a convincing argument that this is the only triplet of consecutive odd integers that are all prime. (*Hint*: use the division algorithm.)
13. Find the prime factorization of each of the following numbers.
 a. 111
 b. 144
 c. 188
 d. 520
14. Find the prime factorization of each of the following numbers.
 a. 10^{10}
 b. 89^4
 c. $8^3 \cdot 6^4 \cdot 13^2$
 d. $2^3 \cdot 3^2 + 2^4 \cdot 3^3 \cdot 7$
 e. $2^4 \cdot 3 \cdot 5^7 + 2^4 \cdot 5^6$
15. Find the least nonzero whole number that is divisible by every nonzero whole number less than or equal to 10.
16. Find the GCD for each of the following pairs of numbers.
 a. 24 and 52
 b. 5767 and 4453
17. Find the LCM for each of the following groups of numbers.
 a. $2^3 \cdot 5^2 \cdot 7^3$, $2 \cdot 5^3 \cdot 7^2 \cdot 13$ and $2^4 \cdot 5 \cdot 7^4 \cdot 29$
 b. 277 and 278
18. Is the LCM of two numbers always greater than the GCD of the numbers? Justify your answer.
19. Explain how to find the LCM of three numbers with the help of the Euclidean algorithm.
20. If the GCD of two positive whole numbers is 1, what can you say about the LCM of the two numbers? Explain your reasoning.
21. Candy bars priced at 50¢ each were not selling, so the price was reduced. Then they all sold in one day for a total of \$31.93. What was the reduced price of each candy bar?
22. June, an owner of a coffee stand, marked down the price of a latte between 7:00 A.M. and 8:00 A.M. from \$2.00 a cup. If she grossed \$98.69 from the latte sale and we know that she never sells a latte for less than a dollar, how many lattes did she sell between 7:00 A.M. and 8:00 A.M.? Explain your reasoning. (*Note*: $71 \mid 9869$.)
23. Two bells ring at 8:00 A.M. For the remainder of the day, one bell rings every half hour and the other bell rings every 45 min. What time will it be when the bells ring together again?
24. Midas has 120 gold coins and 144 silver coins. He wants to place his gold coins and his silver coins in stacks so that there are the same number of coins in each stack. What is the greatest number of coins that he can place in each stack?
25. Jane and Ramon are running laps on a track. If they start at the same time and place and go in the same direction, with Jane running a lap in 5 min and Ramon running a lap in 3 min, how long will it take for them to be at the starting place at the same time if they continue to run at these speeds?
26. There are 60 oranges and 24 apples in Ivan's house. These fruits have to be kept in separate plastic bags containing the same kind and number in each. Find the greatest number of possible fruits in each plastic bag.
27. Determine how many complete revolutions Gear 2 in the following diagram must make before the arrows are lined up again.



Chapter 5

Integers

5-1 Addition and Subtraction of Integers

5-2 Multiplication and Division of Integers



Preliminary Problem

In a research study, seventh-grade students were paired according to ability, based on previously taken tests. An experimental teaching method was used, and tests were given at the end of the experiment to determine how the paired students had fared. Pairs of numbers were used to represent the scores of paired students with the first number in each pair being the score of the student having the experimental treatment and the second number being the control student without the experimental treatment. Use sums of the differences of the pairs to explain whether you would call the experiment a success and why. The scores are (20, 17), (15, 18), (21, 19), (17, 22), (18, 18), (19, 15), (16, 14), (20, 18), (21, 20), and (18, 25).

If needed, see Hint on page 252.

The grade 6 *Common Core Standards* say that students should

understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation. (p. 43)

Mount Everest is 29,028 ft above sea level, and the Dead Sea is 1293 ft below sea level. We may symbolize these elevations as 29,028 and $^{-}1293$.

In mathematics, the need for negative integers arises because subtractions cannot always be performed in the set of whole numbers. To compute $4 - 6$ using the definition of subtraction for whole numbers, we must find a whole number n such that $6 + n = 4$. There is no such whole number n. To perform the computation, we must invent a new number, a *negative integer*. If we attempt to calculate $4 - 6$ on a number line, we must draw intervals to the left of 0. In Figure 1, $4 - 6$ is pictured as an arrow that starts at 0 and ends 2 units to the left of 0. The new number that corresponds to the point 2 units to the left of 0 is *negative two*, symbolized by $^{-}2$.

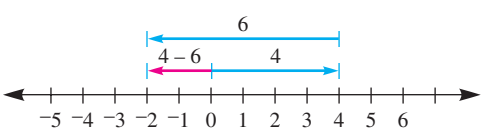


Figure 1

Other numbers to the left of 0 are created similarly. The new set of numbers $\{^{-}1, ^{-}2, ^{-}3, ^{-}4, \ldots\}$ is the set of **negative integers**. The set $\{1, 2, 3, 4, \ldots\}$ is the set of **positive integers**. The integer 0 is neither positive nor negative.

Definition

The union of the set of negative integers, the set of positive integers, and $\{0\}$ is the set of **integers**, denoted by I.

$$I = \{\ldots, ^{-}4, ^{-}3, ^{-}2, ^{-}1, 0, 1, 2, 3, 4, \ldots\}$$

Historical Note



The Hindu mathematician Brahmagupta (ca. 598–665 CE) provided the first systematic treatment of negative numbers and of zero. Only about 1000 years later did the Italian mathematician Gerolamo Cardano (1501–1576) consider negative solutions of equations. Still uncomfortable with the concept of negative numbers, he called them "fictitious" numbers. ●

5-1 Addition and Subtraction of Integers

5-1 Objectives

Students will be able to understand and explain

- The meaning of integers and their representation on a number line.
- Models for addition and subtraction of integers.
- Properties of addition and subtraction of integers.

Representations of Integers

It is unfortunate that the symbol "−" is used to indicate both a subtraction and a negative sign. To reduce confusion between the uses of this symbol in this text, a raised "−" sign is used for negative numbers, as in $^{-}2$, and for the opposite of a number (described later), in contrast to the lower sign for subtraction. To emphasize that an integer is positive, sometimes a raised plus sign is used, as in $^{+}3$. In this text, we use the plus sign for addition only and write $^{+}3$ simply as 3.

The negative integers are **opposites** of the positive integers. For example, the opposite of 5 is $^{-}5$. Similarly, the positive integers are the opposites of the negative integers. As depicted in Figure 1, opposites are on "opposite" sides of 0 and the same number of units from 0. Because the opposite of 4 is denoted $^{-}4$, the opposite of $^{-}4$ can be denoted $^{-}(^{-}4)$, which equals 4. The opposite of 0 is 0. In the set of integers I, every element has an opposite that is also in I.

CCSS In grade 6 *Common Core Standards* as in Example 1(b), students should "recognize that the opposite of the opposite of a number is the number itself." (p. 43) Using addition of integers, we shall soon see that when the opposite of an integer is added to the integer the sum is 0. In fact, ^{-}a can be defined as the solution of $x + a = 0$.

Example 1

For each of the following equations, find the opposite of x.

a. $x = 3$ **b.** $x = {}^{-}5$ **c.** $x = 0$

Solution

a. $^{-}x = {}^{-}3$

b. $^{-}x = {}^{-}(^{-}5) = 5$

c. $^{-}x = {}^{-}0 = 0$

The value of ^{-}x in Example 1(b) is 5. Note that ^{-}x is the opposite of x and might *not represent a negative number*. In other words, x is a variable that can be replaced by some number either positive, zero, or negative. *Note: ^{-}x is read "the opposite of x" not "minus x" or "negative x."*

We next investigate informal ways to introduce operations on integers, beginning with the addition of integers.

Historical Note



The dash has not always been used for both the subtraction operation and the negative sign. Other notations were developed but never adopted universally. One such notation was used by Abu al-Khwarizmi (ca. 825), who indicated a negative number by placing a small circle over it. For example, $^{-}4$ was recorded as $\mathring{4}$. The Hindus denoted a negative number by enclosing it in a circle; for example, $^{-}4$ was recorded as ④. The symbols + and − first appeared in print in European mathematics in the late fifteenth century, at which time the symbols referred not to addition or subtraction nor positive or negative numbers, but to surpluses and deficits in business problems.

E-Manipulative Activity
Use the *Token Addition* activity for additional practice using the chip or charged-field models.

Integer Addition

Hands-on materials can be an aid when working with integers. Several models are presented to illustrate integer addition.

Chip Model for Addition

In the chip model, positive integers are represented by black chips and negative integers by red chips. One red chip neutralizes one black chip. Hence, the integer $^{-}1$ can be represented by 1 red chip, or 2 red and 1 black, or 3 red and 2 black, and so on. Similarly, every integer can be represented in many ways using chips, as shown in Figure 2.

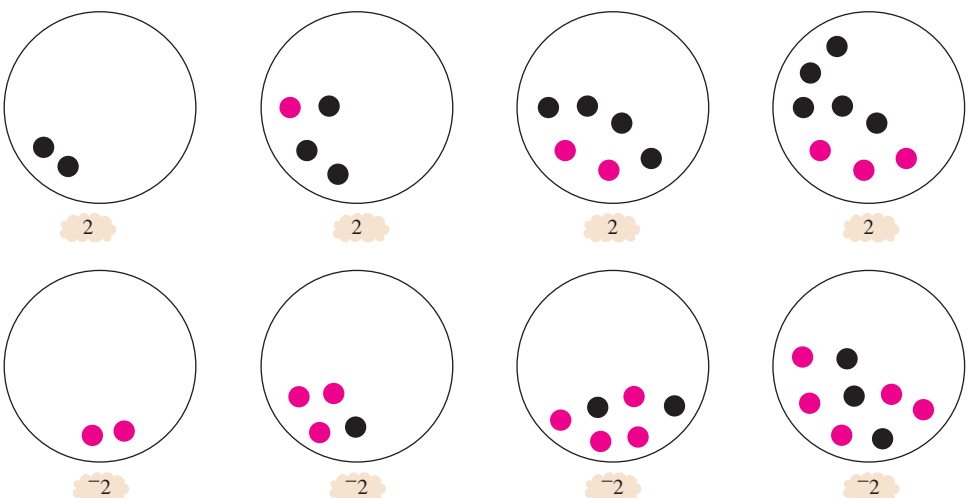


Figure 2



An example of the use of colored chips for modeling integer addition is seen on the student page shown on page 222.

On the student page, note the sentence, "*To add integers with different signs, use zero pairs.*" Observe that one chip of each of two different colors pair to equal 0. Also, note the sentence, "*Removing a zero pair does not change the sum.*" Additionally, as seen later in the chapter, adding zero pairs to aid in computation with integers does not change the computation.

Figure 3 shows a chip model for the addition $^{-}4 + 3$. Place 4 red chips together with 3 black chips. Because 3 red chips neutralize 3 black ones, Figure 3 represents the equivalent of 1 red chip or $^{-}1$.

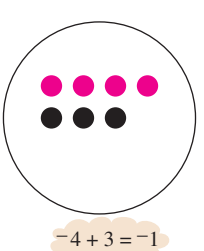


Figure 3

Charged-Field Model for Addition

A model similar to the chip model uses positive and negative charges. A field has 0 charge if it has the same number of positive (+) and negative (−) charges. As in the chip model, a given integer

School Book Page

Modeling Integer Addition and Subtraction

1-2a Activity Lab

Modeling Integer Addition and Subtraction

You can use models to add and subtract integers. Use chips of two different colors. Let one color represent positive integers and the other color represent negative integers.

ACTIVITY

1. Find 5 + 2.

Show 5 "+" chips. Then add 2 "+" chips.

There are 7 "+" chips. So 5 + 2 = 7.



2. Find −5 + (−2).

Show 5 "−" chips. Then add 2 "−" chips.

There are 7 "−" chips. So −5 + (−2) = −7.



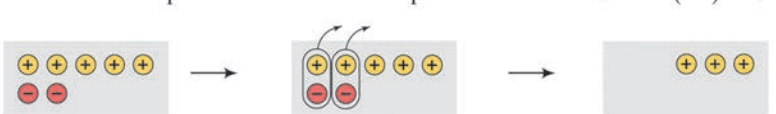
To add integers with different signs, use zero pairs. These chips ⊕ ⊖ are a *zero pair* because ⊕ ⊖ = 0. Removing a zero pair does not change the sum.

3. Find 5 + (−2).

Show 5 "+" chips. Then add 2 "−" chips.

Pair the "+" and "−" chips. Remove the pairs.

There are 3 "+" chips left. So 5 + (−2) = 3.

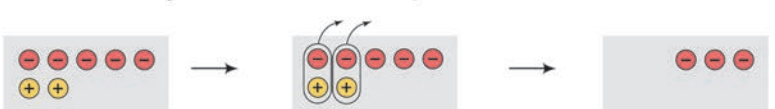


4. Find −5 + 2.

Show 5 "−" chips. Then add 2 "+" chips.

Pair the "+" and "−" chips. Remove the pairs.

There are 3 "−" chips left. So −5 + 2 = −3.



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can be represented in many ways using the charged-field model. Figure 4 uses the model for $3 + {}^{-}5$. Because 3 positive charges "neutralize" 3 negative charges, the net result is 2 negative ones. Hence, $3 + {}^{-}5 = {}^{-}2$.

E-Manipulative Activity
Use the *Number Line* activity to illustrate the number-line model.

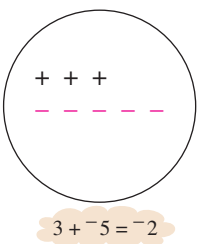


Figure 4

Number-Line Model for Addition

Another model for addition of integers involves a number line, and it can be introduced with the idea of a hiker walking the number line, as seen in Figure 5(a) illustrating $3 + 2 = 5$ and in Figure 5(b) illustrating ${}^{-}3 + {}^{-}2 = {}^{-}5$. Without the hiker, ${}^{-}3 + {}^{-}5 = {}^{-}8$ is pictured in Figure 6(a) and $3 + {}^{-}5 = {}^{-}2$ is in Figure 6(b).

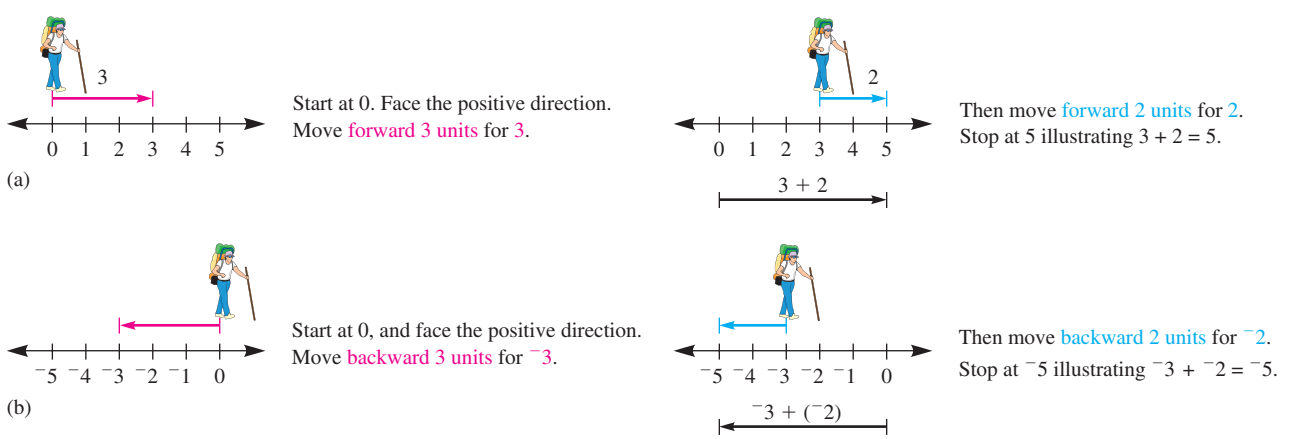


Figure 5

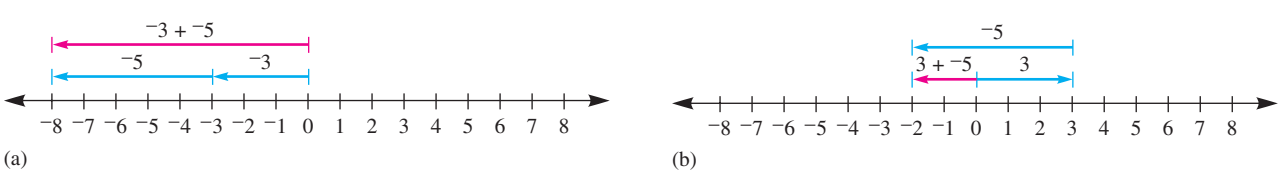


Figure 6

NOW TRY THIS 1

Explain whether the sum of two negative integers is always negative.

Example 2 involves a thermometer with a scale in the form of a vertical number line.

Example 2

The temperature was $^{-}4$°C. In an hour, it rose 10°C. What is the new temperature?

Solution Figure 7 shows that the new temperature is 6°C and that $^{-}4 + 10 = 6$.

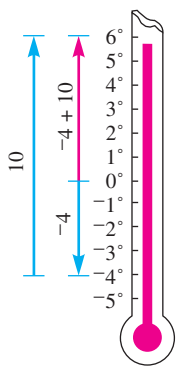


Figure 7

Pattern Model for Addition

Addition of whole numbers was discussed in Chapter 3. Addition of integers can also be motivated using patterns of addition of whole numbers. Notice in the left column of Table 1, the first four facts are known from whole number addition. Also notice that as the 4 stays fixed and the numbers added to 4 decrease by 1, the sum decreases by 1. Following this pattern, we get $4 + {}^{-}1 = 3$, and we can continue the pattern as shown. Similar inductive reasoning can be used to complete the computations in the right column of Table 1, where $^{-}2$ stays fixed and the other numbers decrease by 1 each time.

Table 1

$4 + 3 = 7$	$^{-}2 + 4 = 2$
$4 + 2 = 6$	$^{-}2 + 3 = 1$
$4 + 1 = 5$	$^{-}2 + 2 = 0$
$4 + 0 = 4$	$^{-}2 + 1 = {}^{-}1$
$4 + {}^{-}1 = 3$	$^{-}2 + 0 = {}^{-}2$
$4 + {}^{-}2 = 2$	$^{-}2 + {}^{-}1 = {}^{-}3$
$4 + {}^{-}3 = 1$	$^{-}2 + {}^{-}2 = {}^{-}4$
$4 + {}^{-}4 = 0$	$^{-}2 + {}^{-}3 = {}^{-}5$
$4 + {}^{-}5 = {}^{-}1$	$^{-}2 + {}^{-}4 = {}^{-}6$
$4 + {}^{-}6 = {}^{-}2$	$^{-}2 + {}^{-}5 = {}^{-}7$

However, we know that the inductive reasoning shown in Table 1 does not constitute a proof that the sums are correct.

Absolute Value

Because 4 and $^{-}4$ are opposites, they are on opposite sides of 0 on the number line and are the same distance (4 units) from 0, as shown in Figure 8.

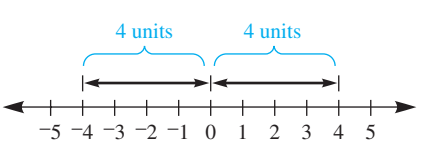


Figure 8

Distance is always a positive number or zero. The distance between the point corresponding to an integer and 0 is the **absolute value** of the integer. Thus, the absolute value of both 4 and $^{-}4$ is 4, written $|4| = 4$ and $|{}^{-}4| = 4$, respectively. Notice that if $x \geq 0$, then $|x| = x$, and if $x < 0$, then ^{-}x is positive. Therefore, we have the following definition.

Definition of Absolute Value

For any integer x,

$|x| = x$, if $x \geq 0$.

$|x| = {}^{-}x$, if $x < 0$.

Example 3

Evaluate each of the following expressions.

a. $|20|$ **b.** $|{}^{-}5|$ **c.** $|0|$

d. ${}^{-}|{}^{-}3|$ **e.** $|2 + {}^{-}5|$

Solution

a. $|20| = 20$ **b.** $|{}^{-}5| = 5$ **c.** $|0| = 0$

d. ${}^{-}|{}^{-}3| = {}^{-}3$ **e.** $|2 + {}^{-}5| = |{}^{-}3| = 3$

CCSS **In grade 6 *Common Core Standards*, we find that students should**

understand the absolute value of a rational number [integer] as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. (p. 43)

Example 4

If Omar's bank balance is ${}^{-}\$75$, what could be said about his indebtedness to the bank?

Solution

Because the bank balance is ${}^{-}\$75$, we could interpret his indebtedness as $|{}^{-}75| = 75$. We might say that he is in debt \$75.

In Example 4, indebtedness is reported as a positive number \$75 while Omar's balance is ${}^{-}\$75$. This different use of positives and negatives at the same time in discussions of real-world applications has to be carefully considered as it may cause confusion.

NOW TRY THIS 2

Write each of the following in simplest form without the absolute value notation in the final answer. Show your work.

a. $|x| + x$ if $x \leq 0$

b. ${}^{-}|x| + x$ if $x \leq 0$

c. ${}^{-}|x| + x$ if $x \geq 0$

Integer Addition

Addition of two integers can be defined in cases using what we have learned with the hands-on models in this section. Because the set of integers includes the set of whole numbers, any definition of addition of integers should be valid for the addition of whole numbers.

Examples taken from the models are seen in Table 2. With a knowledge of addition of whole numbers, the definition of opposites for integers, and the model examples, generalizations for addition of integers are also given in Table 2.

Table 2

Examples from Models	Generalizations
$^-2 + 0 = {}^-2 = 0 + {}^-2$	$m + 0 = m = 0 + m$ for all integers m
$4 + 3 = 7$	If m and n are nonnegative integers, they are added as whole numbers.
$^-3 + {}^-5 = {}^-(3 + 5)$	If $m \geq 0$ and $n \geq 0$, then $^-m + {}^-n = {}^-(m + n)$.
$4 + {}^-3 = 4 - 3$	If $n > 0$ and $m \geq n$, then $m + {}^-n = m - n$.
$3 + {}^-5 = {}^-(5 - 3)$	If $n > m$ and $m \geq 0$, then $m + {}^-n = {}^-(n - m)$.

The generalizations for addition of integers in Table 2 do not conflict with the addition of whole numbers and are used in the following definition.

Definition of Addition of Integers

Let m and n be any integers as described.

a. If m is any integer, then $0 + m = m = m + 0$.
b. If $m \geq 0$ and $n \geq 0$, then $m + n$ is defined using addition of whole numbers.
c. If $m \geq 0$ and $n \geq 0$, then $^-m + {}^-n = {}^-(m + n)$.
d. If $n > 0$ and $m \geq n$, then $m + {}^-n = {}^-n + m = m - n$.
e. If $n > m$ and $m \geq 0$, then $m + {}^-n = {}^-n + m = {}^-(n - m)$.

CCSS **The *Common Core Standards* for grade 7 indicate that students should**

understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. (p. 48)

While not a definition, the statement provides an understanding of how integers may be added. The ideas here lead to a definition of integer addition using absolute values as follows:

To add two integers with the same sign, add the absolute values of the integers. The sum has the same sign as the integers being added.

To add two integers with different signs, subtract the lesser absolute value from the greater one. The sum has the same sign as the integer with the greater absolute value.

Thus, an alternative definition of integer addition using absolute values is given below.

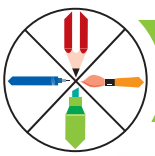
Definition of Integer Addition Using Absolute Values

Let m and n be any integers as described.

If m is any integer, then $m + 0 = m = 0 + m$.
If m and n are non-negative integers, then $m + n$ is the sum of whole numbers m and n.
If m and n are negative integers, then $m + n = {}^-(|m| + |n|)$.
If m is a positive integer and n is a negative integer, then $m + n$ is defined as follows:

If $|m| \geq |n|$, then $m + n = n + m = |m| - |n|$.
If $|m| < |n|$, then $m + n = n + m = {}^-(|n| - |m|)$.

The use of absolute values to compute integer addition is seen on the partial student page.



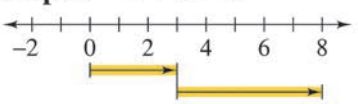
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Adding Integers

KEY CONCEPTS **Adding Integers**

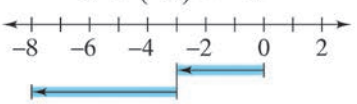
Start at zero. Move to the first integer. Find the absolute value of the second integer and move that distance. If the second integer is positive, move in the positive direction (right). If negative, move in the negative direction (left).

Examples $3 + 5 = 8$



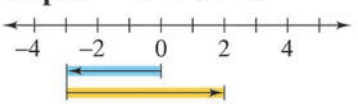
Move to 3. $|5| = 5$. Move 5 to the right.

$-3 + (-5) = -8$



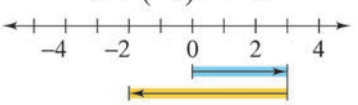
Move to −3. $|-5| = 5$. Move 5 to the left.

Examples $-3 + 5 = 2$



Move to −3. $|5| = 5$. Move 5 to the right.

$3 + (-5) = -2$



Move to 3. $|-5| = 5$. Move 5 to the left.

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Properties of Integer Addition

Integer addition has all the properties of whole number addition. These properties can all be proved using the definition of addition of integers.

Theorem 5-1: Properties of Integer Addition

For all integers a, b, and c,

a. **Closure property of addition of integers** $a + b$ is a unique integer.
b. **Commutative property of addition of integers** $a + b = b + a$.
c. **Associative property of addition of integers** $(a + b) + c = a + (b + c)$.
d. **Identity property of addition of integers** 0 is the unique integer, the **additive identity,** such that, for all integers a, $0 + a = a = a + 0$.

We have seen that every integer has an opposite. This opposite is the **additive inverse** of the integer. The fact that each integer has a unique (one and only one) additive inverse is stated in Theorem 5-2.

Theorem 5-2: Additive Inverse Property of Integers

For every integer a, there exists a unique integer ^-a, the **additive inverse** of a, such that $a + {}^-a = 0 = {}^-a + a$.

By definition, the additive inverse, ^{-}a, is the solution of the equation $x + a = 0$. The fact that the additive inverse is unique is equivalent to saying that the preceding equation has only one solution. In fact, for any integers a and b, the equation $x + a = b$ has a unique solution, $b + {}^{-}a$.

The uniqueness of additive inverses can be used to justify other theorems; for example: The opposite, or the additive inverse, of ^{-}a can be written $^{-}(^{-}a)$. However, because $a + {}^{-}a = 0$, the additive inverse of ^{-}a is also a. Because the additive inverse of ^{-}a must be unique, we have $^{-}(^{-}a) = a$. ■

Other theorems of addition of integers can be investigated by considering previously discussed notions. For example, if we have two representations of the same integer, we may use the number line or other models to show that if we add the same integer to each representation, we have equal results. For example, we know that if $3 + {}^{-}4 = {}^{-}2 + 1$, then $(3 + {}^{-}4) + 3 = ({}^{-}2 + 1) + 3$. This is generalized in Theorem 5-3(b) as the Addition Property of Equality for Integers. Also, we saw that $^{-}2 + {}^{-}4 = {}^{-}(2 + 4)$. This relationship is true in general and is stated in Theorem 5-3(c).

Theorem 5-3

Let a, b, and c be any integers.

a. $^{-}(^{-}a) = a$

b. Addition Property of Equality for Integers: If $a = b$, then $a + c = b + c$.

c. $^{-}a + {}^{-}b = {}^{-}(a + b)$

We prove Theorem 5-3(c) as follows:

By definition $^{-}(a + b)$ is the additive inverse of $(a + b)$; that is, $(a + b) + {}^{-}(a + b) = 0$. If we could show that $^{-}a + {}^{-}b$ is also the additive inverse of $a + b$, the uniqueness of the additive inverse implies that $^{-}(a + b)$ and $^{-}a + {}^{-}b$ are equal. To show that $^{-}a + {}^{-}b$ is the additive inverse of $a + b$, we need only to show that their sum is 0. Using the associative and commutative properties of integer addition and the definition of the additive inverse, we have the following:

$$\begin{aligned}(a + b) + ({}^{-}a + {}^{-}b) &= (a + b) + ({}^{-}b + {}^{-}a)\\ &= [(a + b) + {}^{-}b] + {}^{-}a\\ &= [a + (b + {}^{-}b)] + {}^{-}a\\ &= (a + 0) + {}^{-}a\\ &= a + {}^{-}a\\ &= 0\end{aligned}$$

Since $(a + b) + ({}^{-}a + {}^{-}b) = 0$ and $(a + b) + {}^{-}(a + b) = 0$, by the uniqueness of the additive inverse we have $^{-}(a + b) = {}^{-}a + {}^{-}b$. ■

Example 5

Find the additive inverse of each of the following expressions.

a. $^{-}(3 + x)$ **b.** $a + {}^{-}4$ **c.** $^{-}3 + {}^{-}x$

Solution

a. $3 + x$

b. $^{-}(a + {}^{-}4)$, which can be written as $^{-}a + {}^{-}(^{-}4)$, or $^{-}a + 4$

c. $^{-}(^{-}3 + {}^{-}x)$, which can be written $^{-}(^{-}3) + {}^{-}(^{-}x)$, or $3 + x$

Integer Subtraction

As with integer addition, we explore several models for integer subtraction.

E-Manipulative Activity
Use the *Token Subtraction* activity for additional practice using the chip and charged-field models.

Chip Model for Subtraction

To find $3 - {}^{-}2$, we want to subtract ${}^{-}2$ (take away 2 red chips) from 3 black chips. As seen in Figure 9(a), if we just have 3 black chips, we cannot take away 2 red ones. Therefore, we need to represent 3 so that at least 2 red chips are present using zero pairs. To insure the problem does not change because we need 2 red chips, we can add 2 black chips and 2 red chips forming two zero pairs. In Figure 9(b), we now see 3 represented using 5 black chips and 2 red chips. Now when the 2 red chips are "taken away," in Figure 9(c), 5 black chips are left and hence, $3 - {}^{-}2 = 5$.

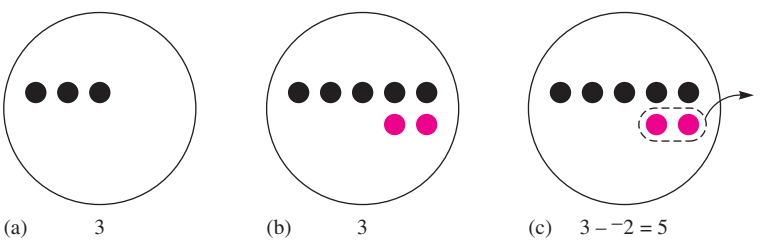


Figure 9

Charged-Field Model for Subtraction

Integer subtraction can be modeled with a charged field. For example, consider ${}^{-}3 - {}^{-}5$. To subtract ${}^{-}5$ from ${}^{-}3$, we represent ${}^{-}3$ in Figure 10(a) so at least 5 negative charges are present as in Figure 10(b). To subtract ${}^{-}5$, remove the 5 negative charges as in Figure 10(c), leaving 2 positive charges, as in Figure 10(d). Hence, ${}^{-}3 - {}^{-}5 = 2$.

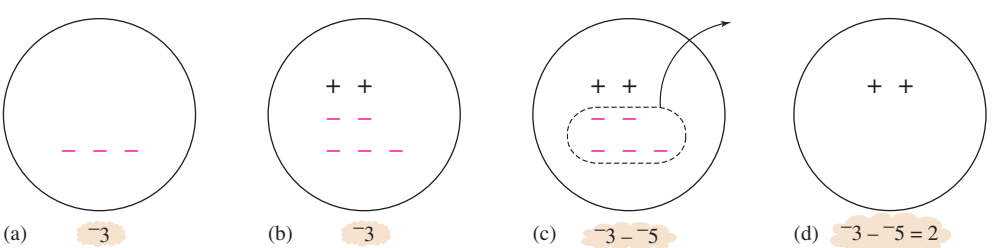


Figure 10

Number-Line Model for Subtraction

The number-line model used for integer addition can also be used to model integer subtraction. While integer addition is modeled by maintaining the same direction and moving forward or backward depending on whether a positive or negative integer is added, subtraction is modeled by turning around. This is seen in Figure 11 with $3 - {}^{-}2$.

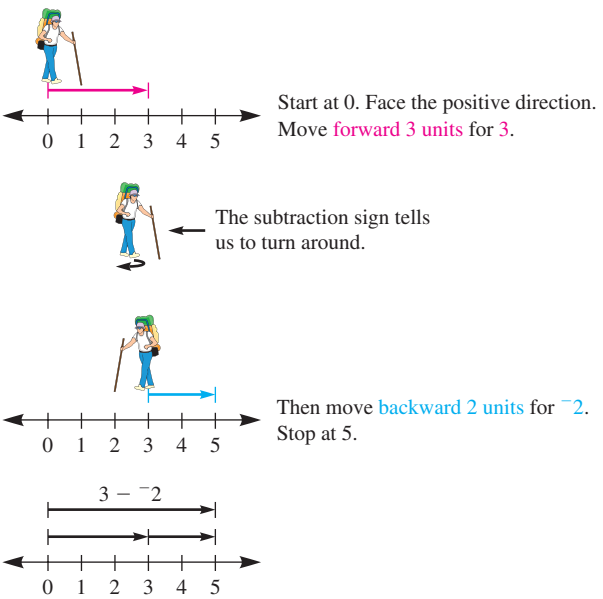


Figure 11

NOW TRY THIS 3

Model the subtraction $4 - {}^{-}3$ on a number line.

Example 6

a. Suppose a mail carrier brings us three letters, one with a check for \$25, and the other two with bills for \$15 and \$20, respectively. What is the result to our balance?

b. Suppose the very next day, we find that the bill for \$20 was mis-delivered, and we give it back to the mail carrier. What is our new balance?

Solution

a. We record the transaction as $25 + {}^{-}15 + {}^{-}20 = {}^{-}10$. Thus, we have \$10 less than we had initially.

b. From part (a), we have the $^{-}10$ less than the original balance to start but a bill for \$20 was taken away so the transaction results in $^{-}10 - {}^{-}20 = 10$. Thus, we have \$10 more than the original balance.

Pattern Model for Subtraction

By using inductive reasoning, we find the difference of two integers by considering the patterns that result from subtractions we know. Both the following pattern on the left and the pattern on the right of Table 3 start with $3 - 2 = 1$.

Table 3

$3 - 2 = 1$	$3 - 2 = 1$
$3 - 3 = 0$	$3 - 1 = 2$
$3 - 4 = ?$	$3 - 0 = 3$
$3 - 5 = ?$	$3 - {}^{-}1 = ?$
$3 - 6 = ?$	$3 - {}^{-}2 = ?$

In the pattern on the left, the difference decreases by 1. If we continue the pattern, we have $3 - 4 = {}^{-}1$, $3 - 5 = {}^{-}2$, and $3 - 6 = {}^{-}3$. In the pattern on the right, the difference increases by 1. If we continue the pattern, we have $3 - {}^{-}1 = 4$ and $3 - {}^{-}2 = 5$.

Defining Integer Subtraction

Subtraction of integers, like subtraction of whole numbers, can be defined in terms of addition. Using the missing-addend approach, $5 - 3$ can be computed by finding a whole number n as follows:

$$5 - 3 = n \quad \text{if, and only if,} \quad 5 = 3 + n$$

Because $3 + 2 = 5$, $n = 2$.

Similarly, we compute $3 - 5$ as follows:

$$3 - 5 = n \quad \text{if, and only if,} \quad 3 = 5 + n$$

Because $5 + {}^{-}2 = 3$, $n = {}^{-}2$. In general, for integers a and b, we have the following definition of subtraction.

Definition of Integer Subtraction

For all integers a and b, $a - b$ is the unique integer n such that $a = b + n$.

Addition is the inverse of subtraction; it "undoes" subtraction; that is, $(a - b) + b = a$. Also, subtraction "undoes" addition; that is, $(a + b) - b = a$.

Example 7

Use the definition of subtraction to compute the following subtractions:

a. $3 - 10$ **b.** ${}^{-}2 - 10$

Solution

a. Let $3 - 10 = n$. Then $n + 10 = 3$, so $n = {}^{-}7$. Therefore, $3 - 10 = {}^{-}7$.

b. Let ${}^{-}2 - 10 = n$. Then $n + 10 = {}^{-}2$, so $n = {}^{-}12$. Therefore, ${}^{-}2 - 10 = {}^{-}12$.

Subtraction Using Adding the Opposite Approach



In grade 7 *Common Core Standards*, we find that students should "understand subtraction of rational numbers [integers] as adding the additive inverse, $p - q = p + {}^{-}q$." (p. 48)



This approach is shown with a number line on the partial student page shown on page 232.

From previous work, we know that $3 - 5 = {}^{-}2$ and $3 + {}^{-}5 = {}^{-}2$. Hence, $3 - 5 = 3 + {}^{-}5$. In general, the following is true.

Theorem 5-4

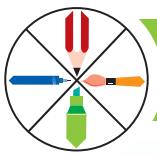
For all integers a and b, $a - b = a + {}^{-}b$.

The preceding theorem can be justified using the fact that the equation $b + x = a$ has a unique solution for x. From the definition of subtraction, the solution of the equation is $a - b$. To show that $a - b = a + {}^{-}b$, we show that $a + {}^{-}b$ is also a solution to $b + x = a$ by checking whether $b + (a + {}^{-}b) = a$:

$$\begin{aligned} b + (a + {}^{-}b) &= b + ({}^{-}b + a) \\ &= (b + {}^{-}b) + a \\ &= 0 + a \\ &= a \end{aligned}$$

Consequently, $a - b = a + {}^{-}b$. ■

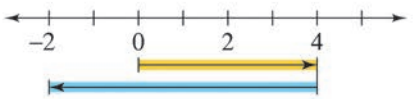
Theorem 5-4 is frequently used as an alternative definition of subtraction.



School Book Page — Subtracting Integers

EXAMPLES Subtracting Integers

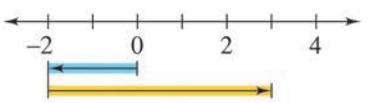
3 Find $4 - 6$.



Start at 0. Move to 4. Then add the opposite of 6, which is −6.

$4 - 6 = 4 + (-6) = -2$

4 Find $-2 - (-5)$.



Start at 0. Move to −2. Then add the opposite of −5, which is 5.

$-2 - (-5) = -2 + 5 = 3$

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NOW TRY THIS 4

a. Is the set of integers closed under subtraction? Why?
b. Do the commutative, associative, or identity properties hold for subtraction of integers? Why?

Example 8

Using the fact that $a - b = a + {}^{-}b$, compute each of the following.

a. $2 - 8$ **b.** $2 - {}^{-}8$ **c.** ${}^{-}12 - {}^{-}5$ **d.** ${}^{-}12 - 5$

Solution

a. $2 - 8 = 2 + {}^{-}8 = {}^{-}6$
b. $2 - {}^{-}8 = 2 + {}^{-}({}^{-}8) = 2 + 8 = 10$
c. ${}^{-}12 - {}^{-}5 = {}^{-}12 + {}^{-}({}^{-}5) = {}^{-}12 + 5 = {}^{-}7$
d. ${}^{-}12 - 5 = {}^{-}12 + {}^{-}5 = {}^{-}17$

Example 9

Rewrite each expression without parentheses.

a. ${}^{-}(b - c)$
b. $a - (b + c)$

Solution

a. ${}^{-}(b - c) = {}^{-}(b + {}^{-}c) = {}^{-}b + {}^{-}({}^{-}c) = {}^{-}b + c$
b. $a - (b + c) = a + {}^{-}(b + c) = a + ({}^{-}b + {}^{-}c) = (a + {}^{-}b) + {}^{-}c = a + {}^{-}b + {}^{-}c$
$= a - b - c$

Example 10

Simplify each of the following expressions.

a. $2 - (5 - x)$ **b.** $5 - (x - 3)$ **c.** $^{-}(x - y) - y$

Solution

a. $\begin{aligned} 2 - (5 - x) &= 2 + {}^{-}(5 + {}^{-}x) \\ &= 2 + {}^{-}5 + {}^{-}({}^{-}x) \\ &= 2 + {}^{-}5 + x \\ &= {}^{-}3 + x \text{ or } x + {}^{-}3 \text{ or } x - 3 \end{aligned}$

b. $\begin{aligned} 5 - (x - 3) &= 5 + {}^{-}(x + {}^{-}3) \\ &= 5 + {}^{-}x + {}^{-}({}^{-}3) \\ &= 5 + {}^{-}x + 3 \\ &= 8 + {}^{-}x \\ &= 8 - x \end{aligned}$

c. $\begin{aligned} {}^{-}(x - y) - y &= {}^{-}(x + {}^{-}y) + {}^{-}y \\ &= [{}^{-}x + {}^{-}({}^{-}y)] + {}^{-}y \\ &= ({}^{-}x + y) + {}^{-}y \\ &= {}^{-}x + (y + {}^{-}y) \\ &= {}^{-}x + 0 \\ &= {}^{-}x \end{aligned}$

Subtraction of Integers on a Calculator



Many calculators have a change of sign key, either [CHS], [+/−], or use [(−)], a key that allows computation with integers. For example, to compute $8 - ({}^{-}3)$, we press [8] [−] [3] [+/−] [=]. Investigate what happens if you press [8] [−] [−] [3] [=].

Order of Operations

Subtraction in the set of integers is neither commutative nor associative, as illustrated in these counterexamples:

$$5 - 3 \neq 3 - 5 \quad \text{because} \quad 2 \neq {}^{-}2$$

$$(3 - 15) - 8 \neq 3 - (15 - 8) \quad \text{because} \quad {}^{-}20 \neq {}^{-}4$$

An expression such as $3 - 15 - 8$ is ambiguous unless we know in which order to perform the subtractions. Mathematicians agree that $3 - 15 - 8$ means $(3 - 15) - 8$; that is, the subtractions in $3 - 15 - 8$ are performed in order from left to right. Similarly, $3 - 4 + 5$ means $(3 - 4) + 5$ and not $3 - (4 + 5)$. Thus, $(a - b) - c$ may be written without parentheses as $a - b - c$. Order of operations for integers will be revisited after multiplication and division are discussed.

Example 11

Compute each of the following.

a. $2 - 5 - 5$ **b.** $3 - 7 + 3$ **c.** $3 - (7 - 3)$

Solution

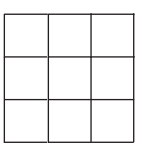
a. $2 - 5 - 5 = {}^{-}3 - 5 = {}^{-}8$

b. $3 - 7 + 3 = {}^{-}4 + 3 = {}^{-}1$

c. $3 - (7 - 3) = 3 - 4 = {}^{-}1$

Assessment 5-1A

1. Find the additive inverse of each of the following integers. Write the answer in the simplest possible form.
 a. 2 b. $^-6$
 c. m d. 0
 e. ^-m f. $a + b$
2. Simplify each of the following expressions.
 a. $^-(^-2)$ b. $^-(^-m)$ c. $^-0$
3. Evaluate each of the following expressions.
 a. $|^-5|$ b. $|10|$
 c. $^-|^-5|$ d. $^-|5|$
4. Demonstrate each of the following additions using the charged-field or chip model.
 a. $5 + ^-3$ b. $^-2 + 3$
 c. $^-3 + 2$ d. $^-3 + ^-2$
5. Demonstrate each of the additions in exercise 4 using a number-line model.
6. Use the absolute value definition of addition to find the following sums.
 a. $7 + ^-13$ b. $^-7 + ^-13$
7. Compute each of the following subtractions using $a - b = a + ^-b$.
 a. $3 - ^-2$ b. $^-3 - 2$ c. $^-3 - ^-2$
8. Answer each part of exercise 7 using subtraction with the missing-addend approach.
9. Write an addition equation that corresponds to each of the following sentences and then answer the question.
 a. A certain stock dropped 17 points and the following day gained 20 points. What was the net change in the stock's worth?
 b. The temperature was $^-10$°C and then it rose by 8°C. What is the new temperature?
 c. The plane was at 35,000 ft and dropped 1000 ft. What is the new altitude of the plane?
10. On January 1, Jane's bank balance was $300. During the month, she wrote checks for $45, $55, $165, $35, and $100 and made deposits of $75, $25, and $400.
 a. If a check is represented by a negative integer and a deposit by a positive integer, express Jane's transactions as a sum of positive and negative integers.
 b. What was the balance in Jane's account at the end of the month?
11. Use a number-line model to compute the following subtractions.
 a. $^-4 - ^-1$ b. $^-4 - ^-3$
12. Use patterns to show the following statements.
 a. $^-4 - ^-1 = ^-3$ b. $^-2 - 1 = ^-3$
13. Do exercise 12 using the charged-field model.
14. Compute each of the following expressions.
 a. $^-2 + (3 - 10)$ b. $(8 - ^-5) - 10$
 c. $^-2 - 7 + 10$
15. In each of the following problems, write both a subtraction and an addition expression that correspond to the question and then answer the question.
 a. The temperature is 55°F and is supposed to drop 60°F by midnight. What is the expected midnight temperature?
 b. Moses has overdraft privileges at his bank. If he had $200 in his checking account and he wrote a $220 check, what would be his balance if the bank allowed the overdraft?
16. Simplify each of the following expressions as much as possible. Show all work.
 a. $3 - (2 - 4x)$ b. $x - (^-x - y)$
17. For which integers a, b, and c, does $a - b + c = a - (b - c)$? Justify your answer.
18. Let W stand for the set of whole numbers, I the set of integers, I^+ the set of positive integers, and I^- the set of negative integers. Find each of the following sets.
 a. $W \cup I$ b. $W \cap I$
 c. $I^+ \cap I^-$ d. $W - I$
19. Place the integers $^-4, ^-3, ^-2, ^-1, 0, 1, 2, 3, 4$ in the grid to make a magic square.



20. Let $y = ^-x - 1$. Find the value of y (in parts (a)–(d)) when x has the following values.
 a. $^-1$ b. 100 c. $^-2$
 d. ^-a
 e. For which values of x will y be 3?
21. Determine the number of terms in the arithmetic and geometric sequences below.
 a. $^-40, ^-39, ^-38, \ldots, 37, 38, 39, 40$
 b. $2^5, 2^6, 2^7, \ldots, 4096$
22. Explain how you could use the Gauss method from Chapter 1 to find the sum of the first 100 negative integers.
23. Find the sum of the terms in the following arithmetic sequence: $^-40, ^-39, ^-38, \ldots, 37, 38, 39, 40$.
24. How could you explain the time change from the eastern coast of the United States to the western coast of the United States in terms of integers?
25. On an automobile trip, the speed limit on one stretch of the road was 55 mph. A driver was driving at the speed limit and then drove 10 mph over the limit, then decelerated 7 mph, and finally, decelerated 13 mph. Interpret the speed using integers and determine his speed when last reported.
26. Par in golf acts like zero in the set of integers. An eagle is 2 under par, a birdie is 1 under par, a bogey is one over par, and a double bogey is 2 over par. If Leonine Trees shot a bogey on hole 1, par on hole 2, an eagle on hole 3, a double bogey on hole 4, a birdie on hole 5 and a hole-in-one on par 4 hole 6, describe his total score in terms of integers and report his final score in relation to par.
27. Find all integers x, if there are any, such that the following are true.
 a. ^-x is positive.
 b. ^-x is negative.
 c. $^-x - 1$ is positive.
 d. $|x| = 2$

28. In each of the following equations, find all integers x satisfying the given equation.
a. $|x - 6| = 6$ **b.** $|x| + 2 = 10$
c. $|{-}x| = |x|$

29. An arithmetic sequence may have a positive or negative difference. In each of the following arithmetic sequences, find the difference and write the next two terms.
a. $0, {-}3, {-}6, {-}9$
b. $x + y, x, x - y$

30. Classify each of the following as true or false. If false, show a counterexample that makes it false.
a. $|{-}x| = |x|$
b. $|x - y| = |y - x|$
c. $|{-}x + {-}y| = |x + y|$

31. Solve the following equations.
a. $x + 7 = 3$ **b.** ${-}10 + x = {-}7$
c. ${-}x = 5$

32. Estimate each of the following and then compute the actual answer.
a. $327 + {-}52 - 398$
b. ${-}1772 + 2005 - 503$
c. $996 - {-}10{,}007 - 102$
d. ${-}303 - {-}1203 + 4997$

Assessment 5-1B

1. Find the additive inverse of each of the following integers. Write the answer in the simplest possible form.
a. 3 **b.** ${-}8$
c. q **d.** 6
e. ${-}n$ **f.** $3 + x$

2. Simplify each of the following expressions.
a. ${-}({-}5)$ **b.** ${-}({-}x)$

3. Evaluate each of the following expressions.
a. $|{-}3|$ **b.** $|15|$
c. ${-}|{-}3|$ **d.** ${-}|6|$

4. Demonstrate each of the following additions using the charged-field or chip model.
a. ${-}2 + 5$ **b.** ${-}5 + 2$
c. ${-}3 + {-}3$ **d.** $6 + {-}4$

5. Demonstrate each of the additions in exercise 4 using a number-line model.

6. Use the absolute value definition of addition to find the following sums.
a. $5 + {-}31$ **b.** ${-}5 + {-}31$

7. Compute each of the following using $a - b = a + {-}b$.
a. ${-}3 - 5$ **b.** $5 - ({-}3)$
c. ${-}2 - {-}3$

8. Answer each part of exercise 7 using the missing-addend approach.

9. Write an addition equation that corresponds to each of the following sentences and then answer the question.
a. A visitor in a Las Vegas casino lost \$200, won \$100, and then lost \$50. What is the change in the gambler's net worth?
b. In four downs, the football team lost 2 yd, gained 7 yd, gained 0 yd, and lost 8 yd. What is the total gain or loss?
c. In a game of Triominoes, Jack's scores in five successive turns are 17, ${-}8$, ${-}9$, 14, and 45. What is his total at the end of five turns?

10. **a.** The largest bubble chamber in the world is 15 ft in diameter and contains 7259 gal of liquid hydrogen at a temperature of ${-}247°$C. If the temperature is dropped by 11°C per hour for 2 consecutive hours, what is the new temperature?
b. The greatest recorded temperature ranges in the world are around the "cold pole" in Siberia. Temperatures in Verkhoyansk have varied from ${-}94°$F to 98°F. What is the difference between the high and low temperatures in Verkhoyansk?

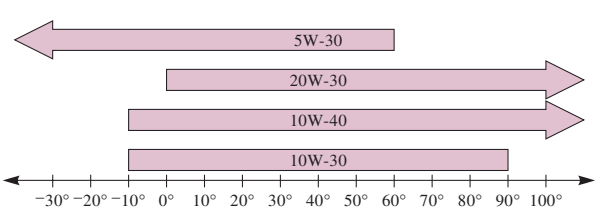
11. Use a number-line model to find the following differences.
a. ${-}3 - {-}2$ **b.** ${-}4 - 3$

12. Use patterns to show the following results.
a. ${-}2 - {-}3 = 1$ **b.** ${-}3 - 2 = {-}5$

13. Do exercise 12 using the charged-field model.

14. Compute each of the following.
a. ${-}2 - (7 + 10)$ **b.** $8 - 11 - 10$
c. ${-}2 - 7 + 3$

15. Motor oils protect car engines over a range of temperatures. These oils have names like 10W–40 or 5W–30. The following graph shows the temperatures, in degrees Fahrenheit, at which the engine is protected by a particular oil. Using the graph, find which oils can be used for the following temperatures.
a. Between ${-}5°$ and 90°
b. Below ${-}20°$
c. Between ${-}10°$ and 50°
d. From ${-}20°$ to over 100°
e. From ${-}8°$ to 90°



16. Simplify each of the following expressions as much as possible. Show all work.
a. $4x - 2 - 3x$ **b.** $4x - (2 - 3x)$

17. **a.** Prove that for all integers x and y, ${-}x - y = {-}y - x$.
b. Does part (a) imply that subtraction is commutative? Explain.

18. Let W stand for the set of whole numbers, I the set of integers, I^+ the set of positive integers, and I^- the set of negative integers. Find each of the following.
a. $W - I^+$ **b.** $W - I^-$
c. $I \cap I$ **d.** $I - W$

19. Complete the magic square using the following integers: ${-}13, {-}10, {-}7, {-}4, 2, 5, 8, 11$.

	${-}1$	

20. Let $y = {}^{-}3x - 2$. Find the value of y (in parts (a)–(d)) for each of the following values of x.
a. ${}^{-}1$ **b.** 100
c. ${}^{-}2$ **d.** ${}^{-}a$ in terms of a
e. For which values of x will y be ${}^{-}11$?

21. Determine the number of terms in the arithmetic and geometric sequences below.
a. ${}^{-}27, {}^{-}26, {}^{-}25, \ldots, 50, 51, 52$
b. $3^3, 3^4, 3^5, 3^6, \ldots, 3^{10}$

22. Find the sum of the following arithmetic sequence.
${}^{-}27, {}^{-}26, {}^{-}25, \ldots, 50, 51, 52.$

23. On *The Biggest Loser*, a person weighed 372 lbs, lost 9 lbs, gained 4 lbs, lost 13 lbs, and lost 24 lbs. Describe this weight fluctuation with a sum of integers and find the final weight of the contestant.

24. In swimming, a competitor at any level trains to better his best time. If Hamilton swam the 100-m backstroke in 65 sec in his last race and in the next training session bettered the time by 4 sec, express this discussion using a sum of integers and show what the time in the training sessions was.

25. When one downloads a file to a computer, memory space is taken up on the computer. When one deletes a file, one gains a certain amount of memory space on the computer. If Sarai downloaded a program of 3.6 MB to her computer and deleted a picture file of 7.2 MB, explain this in terms of integers and the resulting effect on her computer memory.

26. Find all integers x, if there are any, such that the following are true.
a. ${}^{-}|x| = 2$ **b.** ${}^{-}|x|$ is negative.
c. ${}^{-}|x|$ is positive. **d.** ${}^{-}x + 1$ is positive.
e. ${}^{-}x - 1$ is negative.

27. Let $y = |x - 5|$. Find the value of y (in parts (a)–(b)) for each of the following values of x.
a. 10 **b.** ${}^{-}1$
c. Find all the values of x for which y is 7.

28. By the definition of absolute value, the equation $y = |x|$ can be written as follows:

$$y = \begin{cases} x, & \text{if } x \geq 0 \\ {}^{-}x, & \text{if } x < 0 \end{cases}$$

Write $y = |x - 6|$ in a similar way without absolute value.

29. An arithmetic sequence may have a positive or negative difference. In each of the following arithmetic sequences, find the difference and write the next two terms:
a. $7, 3, {}^{-}1, {}^{-}5$
b. $1 - 3x, 1 - x, 1 + x$

30. Classify each of the following as true or false. If false, show a counterexample that makes it false.
a. $|x^2| = x^2$ **b.** $|x^3| = x^3$
c. $|x^3| = x^2|x|$

31. Solve the following equations:
a. ${}^{-}x + 5 = 7$ **b.** $1 - x = {}^{-}13$
c. ${}^{-}x - 8 = {}^{-}9$

32. Estimate each of the following.
a. $343 + {}^{-}42 - 402$
b. ${}^{-}1992 + 3005 - 497$
c. $992 - {}^{-}10{,}003 - 101$
d. ${}^{-}301 - {}^{-}1303 + 4993$

Mathematical Connections 5-1

Reasoning

1. A turnpike driver had car trouble. He knew that he had driven 12 mi from milepost 68 before the trouble started. Assuming he is confused and disoriented when he calls on his cellular phone for help, explain how he can determine his possible location.

2. Dolores claims that a way to understand that $a - b = a + {}^{-}b$, for all integers a and b, is to show that when you add b to each expression you get equal answers. Explain why Dolores's claim is true.

3. Explain why $b - a$ and $a - b$ are additive inverses of each other.

4. The absolute value of an integer is never negative. Does this contradict the fact that the absolute value of x could be equal to ${}^{-}x$? Explain why or why not.

5. If an integer a is pictured on the number line, then the distance from the point on the number line that represents the integer to the origin is $|a|$. Using this idea, explain why $|a - b|$ is the distance between the points that represent the integers a and b.

6. Recall the definition of less than for whole numbers using addition and define $a < b$ for any integers a and b. Use your definition to show that ${}^{-}8 < {}^{-}7$.

Open-Ended

7. Describe a realistic word problem that models ${}^{-}50 + {}^{-}85 - {}^{-}30$.

8. In a library some floors are below ground level and others are above ground level. If the ground-level floor is designated the zero floor, design a system to number the floors.

9. **a.** I choose an integer. I then subtract 10 from the integer, take the opposite of the result, add ${}^{-}3$, and find the opposite of the new result. My result is ${}^{-}3$. What is the original number?
b. Judy wants to do the activity in part (a) with her classmates. Each classmate probably chooses a different number, and Judy wants to tell each classmate quickly what number was chosen. Judy figures out that the only thing she needs to do is to add 7 to each answer she gets. Does this always work? Explain why or why not.
c. Come up with your own "trick" similar to the one in part (b) that works for each answer you get from your classmates.

10. Investigate how tides are measured and design an appropriate system to talk about negative, positive, and 0 tides.

Cooperative Learning

11. Look at several history of mathematics books and the Internet and report in your group on when and how negative integers were first introduced.

12. On 21 3 × 5 cards, write the integers ${}^{-}10$ to 10 using one integer per card. Lay the cards on the floor to form a

number line. Choose someone from your group to act like the hiker in integer addition and subtraction. Give directions to walk the number line to solve exercises 4 and 11 in Assessment 5-lA. Try other addition and subtraction problems to make sure that the number-line model is understood by everyone in your group and could be used in an elementary classroom.

Connecting Mathematics to the Classroom

13. A fourth-grade student devised the following algorithm for subtracting $84 - 27$:

4 minus 7 equals negative 3.

$$\begin{array}{r} 84 \\ -\ 27 \\ \hline {}^{-}3 \end{array}$$

80 minus 20 equals 60.

$$\begin{array}{r} 84 \\ -\ 27 \\ \hline {}^{-}3 \\ 60 \end{array}$$

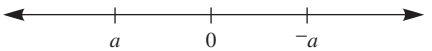
60 plus negative 3 equals 57.

$$\begin{array}{r} 84 \\ -\ 27 \\ \hline {}^{-}3 \\ +\ 60 \\ \hline 57 \end{array}$$

Thus, the answer is 57. How should you respond as a teacher? Will this technique always work?

14. An eighth-grade student claims she can prove that subtraction of integers is commutative. She points out that if a and b are integers, then $a - b = a + {}^{-}b$. Since addition is commutative, so is subtraction. What is your response?

15. A student had the following picture of an integer and its opposite. Other students in the class objected, saying that ${}^{-}a$ should be to the left of 0. How do you respond?



16. If the digits 1 through 9 are written in order, it is possible to place plus and minus signs between the numbers or to use no operation symbol at all to obtain a total of 100. For example, $1 + 2 + 3 - 4 + 5 + 6 + 78 + 9 = 100$. Can you obtain a total of 100 using fewer plus or minus signs than in the given example? Note that digits, such as 7 and 8 in the example, may be combined.

17. Mary wants to know more about absolute value. "Why can it be used to discuss distance? Is distance always positive?" she asks. How would you respond?

National Assessments

National Assessment of Educational Progress (NAEP) Questions

Paco had 32 trading cards. He gave N trading cards to his friend. Which expression tells how many trading cards Paco has now?
A. $32 + N$
B. $32 - N$
C. $N - 32$
D. $32 \div N$

NAEP, Grade 4, 2007

The lowest point of the St. Lawrence River is 294 feet below sea level. The top of Mt. Jacques Cartier is 1,277 feet above sea level. How many feet higher is the top of Mt. Jacques Cartier than the lowest point of the St. Lawrence River? Show your work.

NAEP, Grade 8, 1996

5-2 Multiplication and Division of Integers

5-2 Objectives

Students will be able to understand and explain

- Models for multiplication of integers.
- Properties of multiplication of integers.
- Integer division.
- Order of operations on integers.
- Inequalities with integers.

Multiplication of integers is approached through a variety of models: *patterns*, *charged-field*, *chip*, and *number-line*. Reasoning with these models is inductive and is used to motivate the operations.

Patterns Model for Multiplication of Integers

We may approach multiplication of integers using repeated addition. For example, if a running back lost 2 yd on each of three carries in a football game, then there was a net loss of ${}^{-}2 + {}^{-}2 + {}^{-}2$, or ${}^{-}6$, yards. Since ${}^{-}2 + {}^{-}2 + {}^{-}2$ can be written as $3({}^{-}2)$, we have $3({}^{-}2) = {}^{-}6$ by using repeated addition.

Consider $(^{-}2)3$. It is meaningless to say that there are $^{-}2$ threes in a sum. But if the commutative property of multiplication is to hold for all integers, then $(^{-}2)3 = 3(^{-}2) = {}^{-}6$.

To find $(^{-}3)(^{-}2)$, we develop the following pattern:

$$\begin{aligned} 3(^{-}2) &= {}^{-}6 \\ 2(^{-}2) &= {}^{-}4 \\ 1(^{-}2) &= {}^{-}2 \\ 0(^{-}2) &= 0 \\ {}^{-}1(^{-}2) &= ? \\ {}^{-}2(^{-}2) &= ? \\ {}^{-}3(^{-}2) &= ? \end{aligned}$$

The first four products, $^{-}6$, $^{-}4$, $^{-}2$, and 0, are terms in an arithmetic sequence with a fixed difference of 2. If the pattern continues, the next three terms in the sequence are 2, 4, and 6. Thus, it appears that $(^{-}3)(^{-}2) = 6$. Likewise, $(^{-}2)(^{-}3) = 6$. We used inductive reasoning to reach the results but will use deductive reasoning later to justify the generalized product.

Example 12

On a spreadsheet, in column A enter 5 as the first entry and then write a formula to add $^{-}1$ to 5 for the second entry. Then add $^{-}1$ to the second entry and fill down until there are 20 terms continuing the pattern. In column B, repeat the process. In column C, find the product of the respective entries in columns A and B. What patterns do you observe?

Solution

The described spreadsheet is seen in Figure 12.

	A	B	C
1	5	5	25
2	4	4	16
3	3	3	9
4	2	2	4
5	1	1	1
6	0	0	0
7	-1	-1	1
8	-2	-2	4
9	-3	-3	9
10	-4	-4	16
11	-5	-5	25
12	-6	-6	36
13	-7	-7	49
14	-8	-8	64
15	-9	-9	81
16	-10	-10	100
17	-11	-11	121
18	-12	-12	144
19	-13	-13	189
20	-14	-14	196

Figure 12

Patterns found may vary, but we see that any number squared is either positive or zero. Additionally, we have positive numbers times positive numbers having a positive product, and negative numbers times negative numbers having a positive product. The spreadsheet only depicts examples of the generalizations made and does not constitute a proof.

Multiplication of integers is approached using the chip model, the charged-field model, and the number-line model. In all of these models we start with 0, represented in various ways.

Chip Model and Charged-Field Model for Multiplication

The chip model and the charged-field model can both be used to illustrate multiplication of integers. Consider Figure 13(a), where $3(^{-}2)$ is pictured using a chip model. The product $3(^{-}2)$ is interpreted as putting in 3 groups of 2 red chips each. In Figure 13(b), $3(^{-}2)$ is pictured as 3 groups of 2 negative charges.

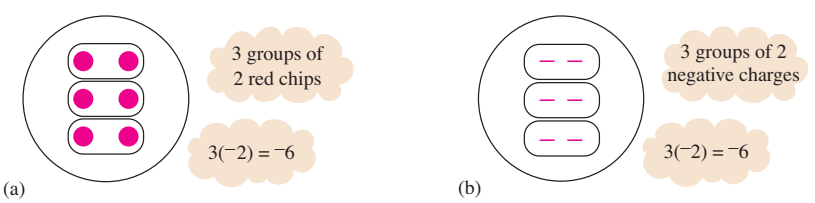


Figure 13

To find $(^{-}3)(^{-}2)$ using the chip model, we interpret the signs as follows: $^{-}3$ is taken to mean "*remove 3 groups of* "; and $^{-}2$ is taken to mean "*2 red chips*." To do this, we start with a value of 0 that includes at least 6 red chips, as shown in Figure 14(a). When we remove 6 red chips, we are left with 6 black chips. The result is a positive 6, so $(^{-}3)(^{-}2) = 6$. Similar reasoning can be used in Figure 14(b) with the charged-field model.

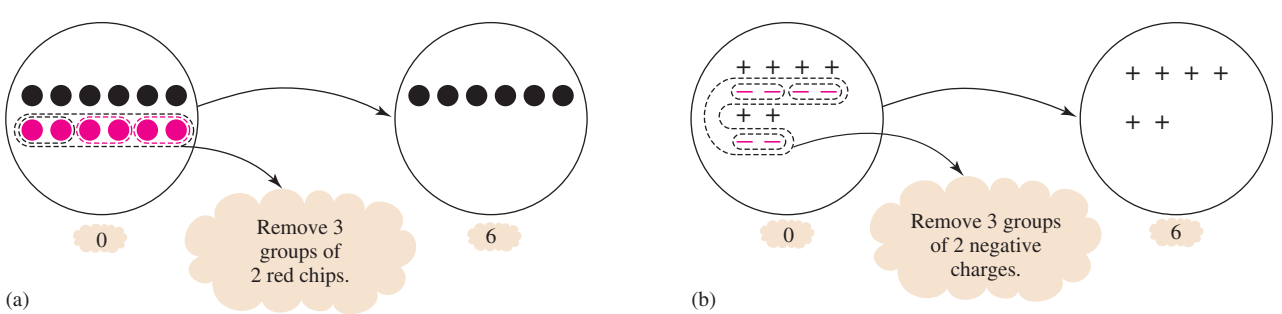


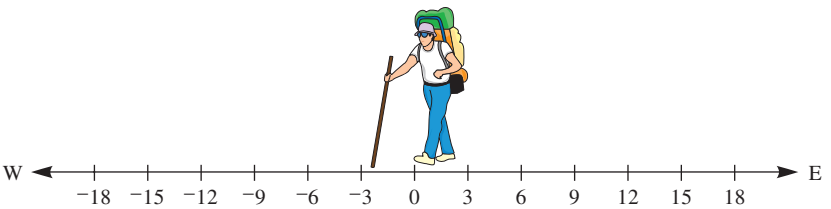
Figure 14

Number-Line Model

As with addition and subtraction, we demonstrate multiplication by using a hiker moving along a number line, according to the following rules:

1. Traveling to the left (west) means moving in the negative direction, and traveling to the right (east) means moving in the positive direction.
2. Time in the future is denoted by a positive value, and time in the past is denoted by a negative value.

Consider the number line shown in Figure 15. Various cases using this number line are given next.



1. If you are now at 0 and move east at 3 mph, where will you be 4 hr from now?

Move east at 3 mph. (3) × 4 hr from now (4) = You will be 12 mi east of 0. (12)

2. If you are now at 0, moving east at 3 mph, where were you 4 hr ago?

Moving east at 3 mph. (3) × 4 hr ago (−4) = You were 12 mi west of 0. (−12)

3. If you are now at 0 and move west at 3 mph, where will you be 4 hr from now?

Move west at 3 mph. (−3) × 4 hr from now (4) = You will be 12 mi west of 0. (−12)

4. If you are now at 0, moving west at 3 mph, where were you 4 hr ago?

Move west at 3 mph. (−3) × 4 hr ago (−4) = You were 12 mi east of 0. (12)

Figure 15



An alternative use of the number line to show multiplication of integers is seen on the following student shown on page 241. Observe how $^{-}3(^{-}2)$ is performed.

The models presented earlier in this section suggest ways to define the multiplication of integers. We give a formal definition using absolute values in the following:

Definition of Integer Multiplication

Let a and b be any integers.

a. If either $a \geq 0$ and $b \geq 0$, then a and b are whole numbers with product ab.
b. If $a \leq 0$ and $b \leq 0$, then $ab = |a||b|$.
c. If one of a or b is less than 0 while the other is greater than 0, then $ab = {}^{-}|a||b|$.

Properties of Integer Multiplication



In grade 7 *Common Core Standards*, students should

understand that multiplication is extended . . . by requiring that operations continue to satisfy the properties of operations [of addition and multiplication on whole numbers], particularly the distributive property leading to products such as $(^{-}1)(^{-}1) = 1$ and the rules for multiplying signed numbers. (p. 48)

School Book Page

Multiply and Divide Integers

What You'll Learn

To multiply and divide integers and to solve problems involving integers

Why Learn This?

Balloonists watch their altitude when they fly. You can multiply integers to find change in altitude.

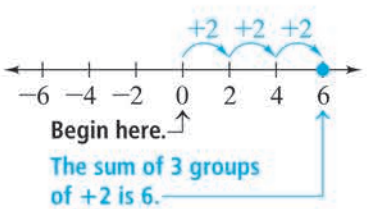


A balloon descends at a rate of 4 ft/min for 3 min. To multiply integers, think of multiplication as repeated addition.

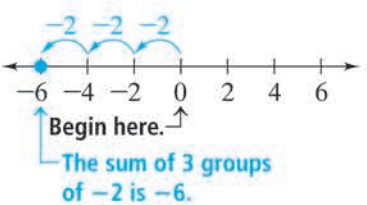
$3(-4) = (-4) + (-4) + (-4) = -12$ ← The balloon descends 12 ft.

You can use number lines to multiply integers.

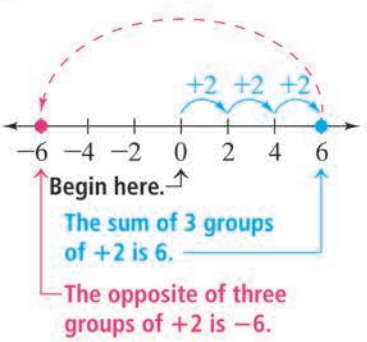
3(2) means three groups of 2.



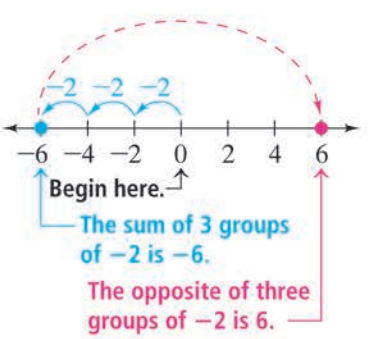
3(−2) means three groups of −2.



−3(2) is the opposite of three groups of 2.



−3(−2) is the opposite of three groups of −2.



This pattern suggests the rules for multiplying integers.

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The set of integers has properties under multiplication analogous to those of the set of whole numbers under multiplication, these properties are summarized in Theorem 5-5.

Theorem 5-5: Properties of Integer Multiplication

The set of integers I satisfies the following properties of multiplication for all integers $a, b, c \in I$:

Closure property of multiplication of integers ab is a unique integer.

Commutative property of multiplication of integers $ab = ba$.

Associative property of multiplication of integers $(ab)c = a(bc)$.

Identity property of multiplication of integers 1 is the unique integer such that for all integers a, $1 \cdot a = a = a \cdot 1$.

Distributive properties of multiplication over addition for integers $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Zero property of multiplication of integers 0 is the unique integer such that for all integers a, $a \cdot 0 = 0 = 0 \cdot a$.

Theorem 5-6 is another convenient theorem that follows from the definition of multiplication of integers and the properties in Theorem 5-5.

Theorem 5-6

For every integer a, $(^-1)a = {}^-a$.

It is important to keep in mind that $(^-1)a = {}^-a$ is true for all integers a. Thus, if we substitute $^-1$ for a, we get $(^-1)(^-1) = {}^-(^-1)$. Because $^-(^-1) = 1$, we have justification for the fact that $(^-1)(^-1) = 1$. Using this result, the preceding theorem, and the properties of integers listed earlier, we show that for all integers a and b, $(^-a)b = {}^-(ab)$ and $(^-a)(^-b) = ab$ as follows:

$$\begin{aligned}(^-a)b &= [(^-1)a]b \\ &= (^-1)(ab) \\ &= {}^-(ab)\end{aligned}$$

Also:

$$\begin{aligned}(^-a)(^-b) &= [(^-1)a][(^-1)b] \\ &= [(^-1)(^-1)](ab) \\ &= 1(ab) \\ &= ab\end{aligned}$$

We have established the following theorem:

Theorem 5-7

For all integers a and b,

a. $(^-a)b = {}^-(ab)$

b. $(^-a)(^-b) = ab$

The distributive property of multiplication over subtraction follows from the distributive property of multiplication over addition:

$$\begin{aligned}a(b - c) &= a(b + {}^-c) \\ &= ab + a(^-c) \\ &= ab + {}^-(ac) \\ &= ab - ac\end{aligned}$$

Consequently, $a(b - c) = ab - ac$. From this and the commutative property of multiplication, it follows that $(b - c)a = ba - ca$.

Theorem 5-8: Distributive Property of Multiplication over Subtraction for Integers
For all integers a, b, and c,
a. $a(b - c) = ab - ac$
b. $(b - c)a = ba - ca$

Example 13

Simplify each of the following expressions so there are no parentheses in the final answer.

a. $(^-3)(x - 2)$ **b.** $(a + b)(a - b)$

Solution

a. $(^-3)(x - 2) = (^-3)x - (^-3)(2) = {}^-3x - (^-6) = {}^-3x + {}^-(^-6) = {}^-3x + 6$

b.
$$\begin{aligned}(a + b)(a - b) &= (a + b)a - (a + b)b \\ &= (a^2 + ba) - (ab + b^2) \\ &= a^2 + ba + {}^-(ab + b^2) \\ &= a^2 + ab + {}^-(ab) + {}^-b^2 \quad (\text{Note: } {}^-b^2 \text{ means } {}^-(b^2).) \\ &= a^2 + 0 + {}^-b^2 \\ &= a^2 - b^2\end{aligned}$$

Thus, $(a + b)(a - b) = a^2 - b^2$.

The result $(a + b)(a - b) = a^2 - b^2$ in Example 13(b) is used so often that we call it the **difference-of-squares** formula and list it as Theorem 5-9.

Theorem 5-9: Difference of Squares Formula
For all expressions a and b, $(a + b)(a - b) = a^2 - b^2$.

Example 14

Use the difference-of-squares formula to simplify the following expressions.

a. $(4 + b)(4 - b)$ **b.** $(^-4 + b)(^-4 - b)$ **c.** $(x + 3)^2 - (x - 3)^2$

Solution

a. $(4 + b)(4 - b) = 4^2 - b^2 = 16 - b^2$

b. $(^-4 + b)(^-4 - b) = (^-4)^2 - b^2 = 16 - b^2$

c.
$$\begin{aligned}(x + 3)^2 - (x - 3)^2 &= [(x + 3) + (x - 3)][(x + 3) - (x - 3)] \\ &= 2x(x + 3 - x + 3) \\ &= 2x(x + 3 + {}^-x + 3) \\ &= 2x(6) \\ &= 12x\end{aligned}$$

When the distributive property of multiplication over subtraction is written in reverse order as

$$ab - ac = a(b - c) \quad \text{and} \quad ba - ca = (b - c)a$$

and similarly for addition, the expressions on the right of each equation are in *factored* form. We say that the common factor a has been *factored out*. Both the difference-of-squares formula and the distributive properties of multiplication over addition and subtraction can be used for factoring.

Example 15

Factor each of the following expressions completely.

a. $x^2 - 9$ **b.** $(x + y)^2 - z^2$ **c.** $^-3x + 5xy$ **d.** $3x - 6$ **e.** $5x^2 - 2x^2$

Solution

a. $x^2 - 9 = x^2 - 3^2 = (x + 3)(x - 3)$
b. $(x + y)^2 - z^2 = (x + y + z)(x + y - z)$
c. $^-3x + 5xy = {}^-3x + 5yx = (^-3 + 5y)x$
d. $3x - 6 = 3x - 3 \cdot 2 = 3(x - 2)$
e. $5x^2 - 2x^2 = (5 - 2)x^2 = 3x^2$

Problem Solving Barrel Problem

In the 1400s, European merchants used integers to label barrels of flour. For example, a barrel labeled $^+3$ meant the barrel was 3 lb overweight, while a barrel labeled $^-5$ meant the barrel was 5 lb underweight. A worker who had labeled 100-lb barrels turned in a weight sheet knowing that the total of the weight was off 53 lbs. He had used only 5s and 6s to label the barrels, but he forgot how many of the labels were positive and how many were negative. He was certain that there were fewer than 20 barrels. How might the barrels have been labeled?

Understanding the Problem In understanding the problem, one may start by explaining to oneself "the meaning of a problem and looking for entry points to its solution" as suggested in the *Common Core Standards* for mathematical practice. (p. 6) In this problem, we recognize that all barrels were labeled using 5s and 6s, but we do not know whether these were positive or negative. Thus, both will have to be considered. Additionally the total weight of all the barrels is off 53 lb, and there were fewer than 20 barrels total. All of these facts must be taken into account as a solution is sought.

Devising a Plan If we reason quantitatively about this problem, we see that we are looking for combinations of multiples of 5, 6, $^-5$, and $^-6$. As mentioned in the *Common Core Standards* for mathematical practice (p. 6), we must "attend to the meaning of the quantities," or in this case the "positiveness" and "negativeness" of the quantities as we look for combinations to find the desired weight of 53 with 20 or fewer combinations of multiples for each possible solution. One way to manage this search is to create a table of values showing the multiples of the possible used quantities.

Carrying Out the Plan The construction of a table may be aided by a spreadsheet, but construction can be achieved without it as seen in Table 4.

Table 4

Multiples of 6	Multiples of 5	Total Barrels	Total Weight
$8 \cdot 6 = 48$	$1 \cdot 5 = 5$	9	53
$3 \cdot 6 = 18$	$7 \cdot 5 = 35$	10	53
Multiples of $^-6$	**Multiples of 5**	**Total Barrels**	**Total Weight**
$2 \cdot {}^-6 = {}^-12$	$13 \cdot 5 = 65$	15	53
$13 \cdot {}^-6 = {}^-78$	$5 \cdot 5 = 25$	18	$^-53$
Multiples of 6	**Multiples of $^-5$**	**Total Barrels**	**Total Weight**
$2 \cdot 6 = 12$	$13 \cdot {}^-5 = {}^-65$	15	$^-53$
$13 \cdot 6 = 78$	$5 \cdot {}^-5 = {}^-25$	18	53
Multiples of $^-6$	**Multiples of $^-5$**	**Total Barrels**	**Total Weight**
$8 \cdot {}^-6 = {}^-48$	$1 \cdot {}^-5 = {}^-5$	9	$^-53$
$3 \cdot {}^-6 = {}^-18$	$7 \cdot {}^-5 = {}^-35$	10	$^-53$

Thus, there are eight possible ways that the barrels could have been labeled as seen above.

Looking Back There are many possible ways to change this problem. One is to use different sets of values of weights that the barrels could be off. A different way is to change the number of barrels used. The problem provides many different avenues for exploration.

Integer Division

In the set of whole numbers, where $b \neq 0$, $a \div b$ is the unique whole number c such that $a = bc$. If such a whole number c does not exist, then $a \div b$ is undefined. Division on the set of integers is defined analogously with division by 0 undefined.

Definition of Integer Division

For all integers a and b, $a \div b$ is the unique integer c, if it exists, such that $a = bc$.

Notice that $a \div b$, if it exists, is the solution of $a = bx$. If $a \div b$ is an integer, then we can extend the concept of divisibility introduced in Chapter 4 as follows:

Definition of Divisibility for Integers

For any integers a and b, if $a \div b$ is a unique integer, then a is **divisible** by b, or equivalently b **divides** a, written as $b \mid a$.

Example 16

Use the definition of integer division, if possible, to evaluate each of the following.

a. $12 \div (^-4)$ **b.** $^-12 \div 4$ **c.** $^-12 \div (^-4)$
d. $^-12 \div 5$ **e.** $(ab) \div b, b \neq 0$ **f.** $(ab) \div a, a \neq 0$

Solution

a. Let $12 \div (^-4) = c$. Then $12 = {}^-4c$ and consequently $c = {}^-3$. Thus, $12 \div (^-4) = {}^-3$.
b. Let $^-12 \div 4 = c$. Then $^-12 = 4c$ and therefore $c = {}^-3$. Thus, $^-12 \div 4 = {}^-3$.
c. Let $^-12 \div (^-4) = c$. Then $^-12 = {}^-4c$ and consequently $c = 3$. Thus, $^-12 \div (^-4) = 3$.
d. Let $^-12 \div 5 = c$. Then $^-12 = 5c$. Because no integer c exists to satisfy this equation (why?), we say that $^-12 \div 5$ is undefined over the set of integers.
e. Let $(ab) \div b = x$. Then $ab = bx$ and consequently $x = a$.
f. Let $(ab) \div a = x$. Then $ab = ax$ and hence $x = b$.

Example 16 suggests that *the quotient of two negative integers, if it exists, is a positive integer, and the quotient of a positive and a negative integer, if it exists, or of a negative and a positive integer, if it exists, is negative*.

Example 17

Division by 0 in the set of whole numbers is undefined. Use the definition of division for integers to show that dividing by 0 is not possible.

Solution

We know that $a \div 0 = x$ if, and only if, $0x = a$ and x is unique. Because $0x = 0$ for all integers, the equation has no solution if $a \neq 0$. If $a = 0$ then for all integers x, $0x = a$. Because the solution is not unique, $0 \div 0$ is not defined.

Algorithm for Division Extended

The division process for whole numbers can be extended to work with integers as seen in the following division: 12,985 ÷ 17.

$$
\begin{array}{r}
763 \\
3 \\
60 \\
{}^{-}200 \\
900 \\
\hline
17\overline{)12985} \\
-15300 \\
\hline
{}^{-}2315 \\
-({}^{-}3400) \\
\hline
1085 \\
-(1020) \\
\hline
65 \\
-51 \\
\hline
14
\end{array}
$$

Though not efficient as a standard division, the end result that $12{,}985 = 17 \cdot 763 + 14$ still holds with 763 as the quotient and 14 as the remainder.

Order of Operations on Integers

When addition, subtraction, multiplication, division, and exponentiation appear without parentheses, exponentiation is done first in order from right to left, then multiplications and divisions in the order of their appearance from left to right, and then additions and subtractions in the order of their appearance from left to right. Arithmetic operations that appear inside parentheses must be done first.

Example 18

Evaluate each of the following.

a. $2 - 5 \cdot 4 + 1$ **b.** $(2 - 5)4 + 1$ **c.** $2 - 3 \cdot 4 + 5 \cdot 2 - 1 + 5$
d. $2 + 16 \div 4 \cdot 2 + 8$ **e.** $({}^{-}3)^4$ **f.** ${}^{-}3^4$

Solution

a. $2 - 5 \cdot 4 + 1 = 2 - 20 + 1 = {}^{-}18 + 1 = {}^{-}17$
b. $(2 - 5)4 + 1 = ({}^{-}3)4 + 1 = {}^{-}12 + 1 = {}^{-}11$
c. $2 - 3 \cdot 4 + 5 \cdot 2 - 1 + 5 = 2 - 12 + 10 - 1 + 5 = 4$
d. $2 + 16 \div 4 \cdot 2 + 8 = 2 + 4 \cdot 2 + 8 = 2 + 8 + 8 = 10 + 8 = 18$
e. $({}^{-}3)^4 = ({}^{-}3)({}^{-}3)({}^{-}3)({}^{-}3) = 81$
f. ${}^{-}3^4 = {}^{-}(3^4) = {}^{-}(81) = {}^{-}81$

From Example 18(e) and (f), we have $({}^{-}3)^4 \neq {}^{-}3^4$. By convention, $({}^{-}x)^4$ means $({}^{-}x)({}^{-}x)({}^{-}x)({}^{-}x)$ and ${}^{-}x^4$ means ${}^{-}(x^4)$ and equals ${}^{-}(x \cdot x \cdot x \cdot x)$.

With the meaning of exponentiation given, the properties of exponents given in Chapter 3 hold for all integers when their exponents are whole numbers. Negative integer exponents will be explored in Chapter 6.

Ordering Integers

As with whole numbers, a number line as shown in Figure 16 can be used to describe greater than and less than relations for the set of integers. Because $^-5$ is to the left of $^-3$ on the number line, we say that "$^-5$ is less than $^-3$," and write $^-5 < {}^-3$. We also say that "$^-3$ is greater than $^-5$," and we write $^-3 > {}^-5$.

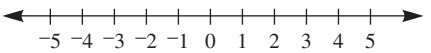


Figure 16

Notice that since $^-5$ is to the left of $^-3$, there is a positive integer that can be added to $^-5$ to get $^-3$, namely, 2. Thus, $^-5 < {}^-3$ because $^-5 + 2 = {}^-3$. The definition of *less than* for integers is similar to that for whole numbers.

Definition of Less Than for Integers

For all integers a and b, a is less than b, written $a < b$, if, and only if, there exists a positive integer k such that $a + k = b$.

The last equation implies that $k = b - a$. Thus, we have proved the following theorem:

Theorem 5-10

For integers a and b, $a < b$ (or equivalently, $b > a$) if, and only if, $b - a$ is equal to a positive integer; that is, $b - a > 0$, or equivalently $a - b < 0$.

Using Theorem 5-10, $^-5 < {}^-3$ because $^-3 - (^-5) = {}^-3 + {}^-(^-5) = {}^-3 + 5 = 2 > 0$. Theorem 5-10 can also be used to justify each of the following.

Theorem 5-11

Let x, y, n be any integers.

a. If $x < y$, then $x + n < y + n$.

b. If $x < y$, then $^-x > {}^-y$.

c. If $x < y$ and $n > 0$, then $nx < ny$.

d. If $x < y$ and $n < 0$, then $nx > ny$.

The justifications are given next.

a. To show that $x + n < y + n$, we need to show that $(y + n) - (x + n) > 0$. We have $y + n - (x + n) = y + n - x - n = y - x$. Because $x < y$, $y - x > 0$ and we have $y + n - (x + n) > 0$; hence $x + n < y + n$.

b. To show that $^-x > {}^-y$, we need to show that $^-x - (^-y) > 0$. We have $^-x - (^-y) = {}^-x + {}^-(^-y) = {}^-x + y = y + {}^-x = y - x$. Because $x < y$, $y - x > 0$ and we have $^-x - (^-y) > 0$; hence $^-x > {}^-y$.

c. To show that $nx < ny$, we need to show that $ny - nx > 0$. We have $ny - nx = n(y - x)$. Because n is a positive integer, $x < y$, and $y - x > 0$, $n(y - x)$ must also be positive. Because $ny - nx > 0$, we have $nx < ny$.

d. To show that $nx > ny$, we show that $nx - ny > 0$. We have $nx - ny = n(x - y)$. Since $y - x > 0$, $x - y < 0$ (why?). Because $n < 0$ and $x - y < 0$, $n(x - y)$ is positive. Thus, $nx - ny > 0$ and hence, $nx > ny$.

Example 19

Use the theorems developed above to find all integers x that satisfy the inequalities parts (a) and (b).

a. $x + 3 < {}^{-}2$
b. ${}^{-}x - 3 < 5$
c. If $x \leq {}^{-}2$, find the values of $5 - 3x$ when x is an integer.

Solution

a. If $x + 3 < {}^{-}2$, then by Theorem 5–11(a),

$$x + 3 + {}^{-}3 < {}^{-}2 + {}^{-}3$$
$$x < {}^{-}5, \quad x \text{ is an integer.}$$

We can also write the solution set (the set of all solutions) as $\{{}^{-}6, {}^{-}7, {}^{-}8, {}^{-}9, \ldots\}$.

Strictly speaking, we have only shown that every x that satisfies the first inequality also satisfies $x < {}^{-}5$. To be sure that $x < {}^{-}5$ represents all the solutions, we need to show the converse; that is, if $x < {}^{-}5$ then $x + 3 < {}^{-}2$. This can be easily done by adding 3 to both sides of $x < {}^{-}5$.

b. If ${}^{-}x - 3 < 5$, then

$${}^{-}x - 3 + 3 < 5 + 3$$
$${}^{-}x < 8$$
$${}^{-}({}^{-}x) > {}^{-}8 \quad \text{by Theorem 5–11(b)}$$
$$x > {}^{-}8, \quad x \text{ is an integer; that is, all the integers in the set } \{{}^{-}7, {}^{-}6, {}^{-}5, \ldots\}.$$

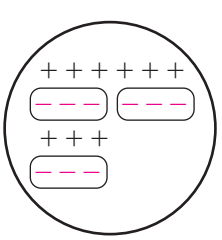
c. If $x \leq {}^{-}2$, then to find an inequality for $5 - 3x$, we proceed as follows.

$${}^{-}3x \geq {}^{-}3({}^{-}2)$$
$${}^{-}3x \geq 6$$
$$5 + {}^{-}3x \geq 5 + 6$$
$$5 - 3x \geq 5 + 6$$
$$5 - 3x \geq 11; \text{ that is, all integers in the set } \{11, 12, 13, 14, \ldots\}.$$

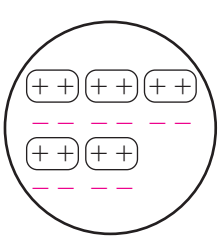
Assessment 5-2A

1. Use patterns to show that $({}^{-}1)({}^{-}1) = 1$.
2. Use the charged-field model to show that $({}^{-}4)({}^{-}2) = 8$.
3. Use the number-line model to show that $2({}^{-}4) = {}^{-}8$.
4. In each of the following charged-field models, the encircled charges are removed. Write the corresponding integer multiplication problem with its solution based on the model.

a.



b.



5. The number of students eating in the school cafeteria has been decreasing at the rate of 30 per year for many years. Assuming this trend continues, write a multiplication problem that describes the change in the number of students eating in the school cafeteria for each of the following situations.
a. The change over the next 4 years
b. The situation 4 years ago
c. The change over the next n years
d. The situation n years ago

6. Use the definition of division to find each quotient, if possible. If a quotient is not defined, explain why.
a. ${}^{-}40 \div {}^{-}5$
b. ${}^{-}143 \div 13$
c. ${}^{-}5 \div 0$

7. Evaluate each of the following expressions, if possible.
a. $({}^{-}10 \div {}^{-}2)({}^{-}2)$
b. $({}^{-}10 \cdot 5) \div 5$
c. ${}^{-}8 \div ({}^{-}8 + 8)$
d. $({}^{-}6 + 6) \div ({}^{-}2 + 2)$
e. $|{}^{-}24| \div [4(9 - 15)]$

8. Evaluate each of the following products and then, if possible, write two division statements that are equivalent to the given multiplication statement. If two division statements are not possible, explain why.
a. $({}^{-}6)5$
b. $({}^{-}5)({}^{-}4)$
c. $({}^{-}3)0$

9. In each of the following, x and y are integers; $y \neq 0$. Use the definition of division in terms of multiplication to perform the indicated operations. Write your answers in simplest form.
a. $(4x) \div 4$ **b.** $(^-xy) \div y$

10. In a lab, the temperature of various chemical reactions is changing by a fixed number of degrees per minute. Write a numeric expression that describes each of the following if all times are on the same day.
a. The temperature at 8:00 P.M. is 32°C. If it drops 1°C per minute, what will the temperature be at 8:30 P.M.?
b. The temperature at 8:20 P.M. is 0°C. If it has dropped 2°C per minute, what was the temperature at 7:55 P.M.?
c. The temperature at 8:00 P.M. is $^-20$°C. If it has dropped 2°C per minute, what was the temperature at 7:30 P.M.?
d. The temperature at 8:00 P.M. is 25°C. If it has been increasing every minute by 2°C, what was the temperature at 7:40 P.M.?

11. The farmland acreage lost to family dwellings over the next 8 years will be 12,000 acres per year. How much acreage would be lost to homes during this time period?

12. Illustrate the distributive property of multiplication over subtraction, $a(b - c) = ab - ac$, using $a = ^-1, b = ^-4$, and $c = ^-2$.

13. Compute each of the following.
a. $(^-2)^3$ **b.** $(^-2)^4$
c. $(^-10)^5 \div (^-10)^2$ **d.** $(^-3)^5 \div (^-3)$
e. $(^-1)^{50}$ **f.** $(^-1)^{151}$
g. $^-2 + 3 \cdot 5 - 1$ **h.** $10 - 3 \cdot 7 - 4(^-2) + 3$
i. $(^-2)^{64} - 2^{64}$ **j.** $^-2^8 + 2^8$

14. If x is an integer and $x \neq 0$, which of the following expressions are always positive and which are always negative?
a. $^-x^2$ **b.** x^2
c. $(^-x)^2$ **d.** $^-x^3$
e. $(^-x)^3$

15. Find all integer values of x for which the following are true.
a. $^-x^2 = x^2$ **b.** $^-x^3 = (^-x)^3$

16. If $48 \div x$ is an integer, what are the possible values of x?

17. Identify the property of integers being illustrated in each of the following.
a. $(^-3)(4 + 5) = (4 + 5)(^-3)$
b. $(^-4)(^-7) \in I$
c. $5[4(^-3)] = (5 \cdot 4)(^-3)$
d. $(^-9)[5 + (^-8)] = (^-9)5 + (^-9)(^-8)$

18. Simplify each of the following expressions.
a. $(^-x)(^-y)$ **b.** $^-2x(^-y)$
c. $^-2(^-x + y) + x + y$ **d.** ^-1x

19. Multiply each of the following and combine terms where possible.
a. $^-2(x - y)$ **b.** $x(x - y)$
c. $^-x(x - y)$ **d.** $^-2(x + y - z)$

20. Find all integers x (if any) that make each of the following true.
a. $^-3x = 6$
b. $^-2x = 0$
c. $x \div 3 = ^-12$
d. $x \div (^-3) = ^-2$
e. $x \div (^-x) = ^-1$
f. $^-3x - 8 = 7$
g. $^-2(5x - 3) = 26$
h. $3x - x - 2x = 3$
i. $^-2(5x - 6) - 30 = ^-x$
j. $x^2 = 4$
k. $(x - 1)^2 = 9$
l. $(x - 1)^2 = (x + 3)^2$
m. $(x - 1)(x + 3) = 0$

21. Use the difference-of-squares formula to simplify each of the following, if possible.
a. $52 \cdot 48$
b. $(5 - 100)(5 + 100)$
c. $(^-x - y)(^-x + y)$

22. Factor each of the following expressions completely.
a. $3x + 5x$
b. $xy + x$
c. $x^2 + xy$
d. $3xy + 2x - xz$
e. $abc + ab - a$
f. $16 - a^2$
g. $4x^2 - 25y^2$

23. Use the distributive property of multiplication over addition or over subtraction and other properties to show that

$$(a - b)^2 = a^2 - 2ab + b^2$$

24. In each of the following, find the next two terms. Assume each sequence is arithmetic or geometric, and find its common difference or ratio and the nth term.
a. $^-10, ^-7, ^-4, ^-1, 2, 5, \ldots$
b. $^-2, ^-4, ^-8, ^-16, ^-32, ^-64, \ldots$
c. $2, ^-2^2, 2^3, ^-2^4, 2^5, ^-2^6, \ldots$

25. Find the missing terms in the following arithmetic or geometric sequences.
a. $^-3, ^-9, _, _, ^-243$
b. $_, 32, ^-16, 8, _$

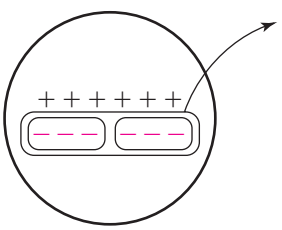
26. A hot air balloon descends at the rate of 5 ft per minute for 6 minutes and then rises at the rate of 3 ft per minute for 8 minutes. Write this information using integers and give the net result on balloon height.

27. A paper cup is 7 in. tall with 1 in. of that height being the rim. The cups nest together with the non-rim part (6″) fitting in a cup below. Write an expression for the total height of n nested cups using $^-6$ to represent each of the unseen nested portions of the cups.

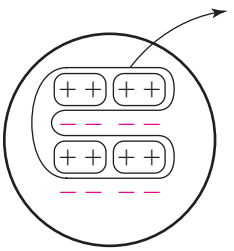
Assessment 5-2B

1. Use patterns to show that $(^-2)(^-2) = 4$.
2. Use the charged-field model to show that $(^-2)(^-3) = 6$.
3. Use the number-line model to show that $2(^-3) = ^-6$.
4. In each of the following charged-field models, the encircled charges are removed. Write the corresponding integer multiplication problem with its solution based on the model.

a.



b.



5. The number of students taking sophomore mathematics at Union College has been decreasing at the rate of 20 per year for many years. Assuming this trend continues, write a multiplication problem that describes the change in the number of sophomores taking mathematics at Union for each of the following situations.
 a. The change over the next 4 years
 b. The situation 5 years ago
 c. The change over the next n years
 d. The situation n years ago
6. Use the definition of division to find each quotient, if possible. If a quotient is not defined, explain why.
 a. $143 \div (^-10)$
 b. $0 \div (^-6)$
7. Evaluate each of the following, if possible.
 a. $(a \div b)b$ **b.** $(ab) \div b$
 c. $(^-8 + 8) \div 8$ **d.** $(^-23 - ^-7) \div 4$
 e. $|^-28| \div (2|^-7|)$
8. Evaluate each of the following products and then, if possible, write two division statements that are equivalent to the given multiplication statement. If two division statements are not possible, explain why.
 a. $(^-5)4$
 b. $(^-4)(^-3)$
 c. $0 \cdot 0$
9. In each of the following, x and y are integers. Use the definition of division in terms of multiplication to perform the indicated operations. Write your answers in simplest form.
 a. $(^-4x) \div x$
 b. $(^-10x + 5) \div 5$
10. In a lab, the temperature of various chemical reactions is changing by a fixed number of degrees per minute. Write a numeric expression that describes each of the following situations.
 a. The temperature at 8:00 A.M. is $^-5$°C. If it increases by d degrees per minute, what will the temperature be m minutes later?
 b. The temperature at 8:00 P.M. is 0°C. If it has been dropping by d degrees per minute, what was the temperature m minutes before?
 c. The temperature at 8:00 P.M. is 20°C. If it has been increasing every minute by d degrees, what was the temperature m minutes before?
11. **a.** On each of four consecutive plays in a football game, a team lost 11 yd. If lost yardage is interpreted as a negative integer, write the information as a product of integers and determine the total number of yards lost.
 b. If Jack Jones lost a total of 66 yd in 11 plays, how many yards, on the average, did he lose on each play?
12. Illustrate the distributive property of multiplication over subtraction, $a(b - c) = ab - ac$, using $a = ^-1, b = 4$, and $c = ^-2$.
13. Compute each of the following.
 a. $10 - 3 - 12$ **b.** $10 - (3 - 12)$
 c. $(^-3)^2$ **d.** $^-3^2$
 e. $^-5^2 + 3(^-2)^2$ **f.** $^-2^3$
 g. $(^-2)^5$ **h.** $^-2^4$
 i. $^-2^{63} + 2^{64}$ **j.** $7^5 + ^-7^5$
14. If x is an integer and $x \neq 0$, which of the following expressions are always positive and which are always negative?
 a. $^-x^4$ **b.** $(^-x)^4$
 c. x^4 **d.** x
 e. ^-x
15. Identify the property of integers being illustrated in each of the following.
 a. $(^-2)(3) \in I$
 b. $(^-4)0 = 0$
 c. $^-2(3 + 4) = ^-2(3) + (^-2)4$
 d. $(^-2)3 = 3(^-2)$
16. Simplify each of the following.
 a. $x - 2(^-y)$
 b. $a - (a - b)(^-1)$
 c. $y - (y - x)(^-2)$
 d. $^-1(x - y) + x$
17. Multiply each of the following and combine terms where possible.
 a. $^-x(x - y - 3)$
 b. $(^-5 - x)(5 + x)$
 c. $(x - y - 1)(x + y + 1)$
18. Find all integers x (if any) that make the statements in parts (a)–(k) true.
 a. $x^2 = 9$
 b. $x^2 = ^-9$
 c. $^-x \div ^-x = 1$
 d. $^-x^2$ is negative.
 e. $^-(1 - x) = x - 1$
 f. $x - 3x = ^-2x$
 g. $^-3(x + 2) = ^-3x + 6$
 h. $(2x - 1)^2 = (1 - 2x)^2$
 i. $x^3 = ^-2^9$
 j. $^-6x > ^-x + 20$
 k. $^-5(x - 3) > ^-5$
 l. If $x > ^-2$, find the values of $3 - 5x$.
 m. If $x < 0$, find the values of $2 - 7x$.
19. Use the difference-of-squares formula to simplify or compute each of the following, if possible.
 a. $(2 + 3x)(2 - 3x)$
 b. $(x - 1)(1 + x)$
 c. $213^2 - 13^2$

20. Factor each of the following expressions completely and then simplify, if possible.
- **a.** $ax + 2x$
- **b.** $ax - 2x$
- **c.** $3x - 4x + 7x$
- **d.** $3x^2 + xy - x$
- **e.** $(a + b)(c + 1) - (a + b)$
- **f.** $x^2 - 9y^2$
- **g.** $(x^2 - y^2) + x + y$

21. Show that $(a - 1)^2 = a^2 - 2a + 1$.

22. In each of the following, find the next two terms. Assume the sequence is arithmetic or geometric, and find its common difference or ratio and the nth term.
- **a.** $10, 7, 4, 1, {}^-2, {}^-5, \ldots$
- **b.** $^-2, 4, {}^-8, 16, {}^-32, 64, \ldots$

23. Find all integer values of x for which the following are true.
- **a.** ${}^-x = x^2$
- **b.** ${}^-x^2 = ({}^-x)^3$

24. If $54 \div x$ is an integer, what are the possible values of x?

25. Find the missing terms in the following arithmetic or geometric sequences.
- **a.** $^-4, {}^-16, _, _, {}^-1024$
- **b.** $_, 27, {}^-9, 3, _$

26. A hot air balloon descends at the rate of 4 ft per minute for 6 minutes and then rises at the rate of 2 ft per minute for 8 minutes. Write this information using integers and give the net result on balloon height.

27. A paper cup is 8 in. tall with 2 in. of that height being the rim. The cups nest together with the non-rim part (6″) fitting in a cup below. Write an expression for the total height of n nested cups using $^-6$ to represent each of the unseen nested portion of the cups.

Mathematical Connections 5-2

Reasoning

1. Explain whether $({}^-x - y)(x + y)$ can be multiplied by using the difference-of-squares formula.

2. We use the equation $(a + b)^2 = a^2 + 2ab + b^2$ to find a similar equation for $(a - b)^2$. Supply any missing steps or justifications.

$$\begin{aligned}(a - b)^2 &= [a + ({}^-b)]^2 \\ &= a^2 + 2a({}^-b) + ({}^-b)^2 \\ &= a^2 - 2ab + b^2\end{aligned}$$

3. Consider the argument to show that $({}^-a)b = {}^-(ab)$ for all integers a and b: I know that $({}^-1)a = {}^-a$ and a could be a variable to be replaced by ab so that $({}^-1)(ab) = {}^-(ab)$ and $({}^-1)(ab) = ({}^-1 \cdot a)b$ by the associative property. Also $({}^-1 \cdot a)b = ({}^-a)b$. Therefore, $({}^-a)b = {}^-(ab)$. If the argument is valid, give reasons for each step; if it is not valid, explain why not.

4. Evaluate the "proof" that $a - b = a + {}^-b$.

$$\begin{aligned}a - b &= a + {}^-b \\ (a - b) + b &= (a + {}^-b) + b \\ a &= a + ({}^-b + b) \\ a &= a + 0 \\ a &= a\end{aligned}$$

Since we obtain a true statement we have $a - b = a + {}^-b$.

5. Explain how to find the number of integers between x and y if $x < y$.

6. Explain why $^-500$ is less than 7.

7. The Swiss mathematician Leonhard Euler (1707–1783) argued that $({}^-1)({}^-1) = 1$ as follows: "The result must be either $^-1$ or 1. If it is $^-1$, then $({}^-1)({}^-1) = {}^-1$. Because ${}^-1 = ({}^-1)1$, we have $({}^-1)({}^-1) = ({}^-1)1$. Now dividing both sides of the last equation by $^-1$ we get ${}^-1 = 1$, which of course cannot be true. Hence $({}^-1)({}^-1)$ must be equal to 1."
- **a.** What is your reaction to this argument? Is it logical? Why or why not?
- **b.** Can Euler's approach be used to justify other properties of integers? Explain.

Open-Ended

8. On a national mathematics competition, scoring is accomplished using the formula 4 times the number answered correctly minus the number answered incorrectly. In this scheme, problems left blank are considered neither correct nor incorrect. Devise a scenario that would allow a student to have a negative score.

9. Choose two integers and illustrate the algorithm for division on page 246. Be careful with positive and negative signs in the computation.

10. Express each of the numbers from 1 through 10 using four 4s and any operations. For example,

$$\begin{aligned}1 &= 44 \div 44, \text{ or} \\ 1 &= (4 \div 4)^{44}, \text{ or} \\ 1 &= {}^-4 + 4 + (4 \div 4).\end{aligned}$$

11. In modeling the division algorithm, explain how any integer estimate may be used to find the first quotient and the resulting effect on the remainder of the division.

Cooperative Learning

12.
- **a.** How would you introduce multiplication of integers in a middle-school class and how would you explain that a product of two negative numbers is positive? Write a rationale for your approach.
- **b.** Present your answers and compare them to those of another class group and together decide the most appropriate way to introduce the concepts.

13. In your group, discuss each person's favorite approach to justify $({}^-1)({}^-1) = 1$.

Connecting Mathematics to the Classroom

14. A seventh-grade student does not believe that $^-5 < {}^-2$. The student argues that a debt of \$5 is greater than a debt of \$2. How do you respond?

15. A student computes $^-8 - 2({}^-3)$ by writing $^-10({}^-3) = 30$. How would you help this student?

16. A student says that his father showed him a very simple method for dealing with expressions like $^{-}(a - b + 1)$ and $x - (2x - 3)$. The rule is, if there is a negative sign before the parentheses, change the signs of the expressions inside the parentheses. Thus, $^{-}(a - b + 1) = {}^{-}a + b - 1$ and $x - (2x - 3) = x - 2x + 3$. What is your response?

17. Mariana felt that using absolute values with multiplications were too hard to be taught to middle school students. How would you respond based on the *Common Core Standards?*

Review Problems

18. Illustrate $^{-}8 + {}^{-}5$ on a number line.

19. Find the additive inverse of each of the following.
- **a.** $^{-}5$
- **b.** 7
- **c.** 0

20. Compute each of the following.
- **a.** $|{}^{-}14|$
- **b.** $|{}^{-}14| + 7$
- **c.** $8 - |{}^{-}12|$
- **d.** $|11| + |{}^{-}11|$

21. Solve for integers x in each of the following equations or inequalities.
- **a.** $|x - 7| = 0$
- **b.** $|x - 7| < 0$
- **c.** $|x - 7| > 0$

22. Is the set of integers closed under subtraction?

National Assessments

National Assessment of Educational Progress (NAEP) Questions

What are all values of n for which $^{-}2n \geq n + 6$?
A. $n \leq {}^{-}2$
B. $n \geq {}^{-}2$

NAEP, Grade 8, 2013

If $a > 0$ and $b < 0$, which of the following *must* be true?
A. $ab > 0$
B. $a - b > 0$
C. $b - a > 0$

NAEP, Grade 8, 2011

Hint for Preliminary Problem

One method of considering the issue is to find the differences in the first component of the ordered pair minus the second component of the ordered pair and look at those differences.

Chapter 5 Summary

KEY CONCEPTS	DEFINITIONS, DESCRIPTIONS, AND THEOREMS
Section 5-1	
Set of integers (p. 219)	$\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}$ Set of *positive integers:* $\{1, 2, 3, \ldots\}$ Set of *negative integers:* $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1\}$
Opposites (p. 220)	Pairs of numbers that are the same distance from 0 on the number line. The negative integers are the *opposites* of the positive integers, and vice versa.
Models for integer addition (p. 221)	• *Chip* • *Number-line* • *Absolute value* • *Charged-field* • *Pattern*
Absolute value (p. 224)	Absolute value of a number is its distance from 0 on a number line. For any integer x, $\lvert x\rvert = x$, if $x \geq 0$ and $\lvert x\rvert = {}^{-}x$, if $x < 0$.
Integer addition (p. 225)	Let m and n be any integers, as described. $0 + m = m = m + 0$ If $m \geq 0$ and $n \geq 0$, then $m + n$ is defined using addition of whole numbers. If $m \geq 0$ and $n \geq 0$, then ${}^{-}m + {}^{-}n = {}^{-}(m + n)$. If $n > 0$ and $m \geq n$, then $m + {}^{-}n = m - n$. If $n > m$ and $m \geq 0$, then $m + {}^{-}n = {}^{-}(n - m)$.
Integer addition using absolute values (p. 226)	Let m and n be any integers, as described. If m and n are non-negative integers, then $m + n$ is the sum of whole numbers m and n. If m and n are negative integers, then $m + n = {}^{-}(\lvert m\rvert + \lvert n\rvert)$. If m is a positive integer and n is a negative integer, then $m + n$ is defined as: If $\lvert m\rvert > \lvert n\rvert$, then $m + n = \lvert m\rvert - \lvert n\rvert$. If $\lvert m\rvert < \lvert n\rvert$, then $m + n = {}^{-}(\lvert n\rvert - \lvert m\rvert)$.
Addition properties for integers (p. 227)	• *Closure*—for all integers a and b, $a + b$ is a unique integer. • *Commutative*—for all integers a and b, $a + b = b + a$. • *Associative*—for all integers a, b, and c, $(a + b) + c = a + (b + c)$. • *Identity*—there is a unique integer 0, the *additive identity*, such that for all integers a, $0 + a = a = a + 0$. • *Additive inverse*—for every integer a, there is a unique integer ${}^{-}a$, the *additive inverse* of a, such that $a + {}^{-}a = 0 = {}^{-}a + a$. • *Theorem: Opposite of the opposite of an integer*—for every integer a, ${}^{-}({}^{-}a) = a$. • *Theorem: Addition property of equality*—for any integers a, b, and c, if $a = b$, then $a + c = b + c$. • *Theorem: Addition of opposites*—for any integers a and b, ${}^{-}a + {}^{-}b = {}^{-}(a + b)$.
Integer subtraction (p. 229)	For integers a and b, $a - b$ is the unique integer n such that $a = b + n$ or $b + n = a$. For all integers a and b, $a - b = a + {}^{-}b$.
Models for integer subtraction (p. 229)	• *Chip* • *Number-line* • *Missing-addend* • *Charged-field* • *Pattern* • *Adding the opposite*

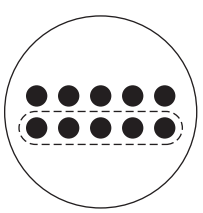
Section 5-2

Models for integer multiplication (p. 237)	• *Pattern* • *Chip* • *Charged-field* • *Number-line*
Integer multiplication (p. 240)	Let a and b be any integers. If either $a \geq 0$ and $b \geq 0$, then a and b are whole numbers with the product ab. If $a \leq 0$ and $b \leq 0$, then $ab = \lvert a \rvert \lvert b \rvert$. If one of a or b is less than 0 while the other is greater than 0, then $ab = {}^{-}\lvert a \rvert \lvert b \rvert$.
Multiplication properties for integers (p. 242)	• *Closure*—for all integers a and b, ab is a unique integer. • *Commutative*—for all integers a and b, $ab = ba$. • *Associative*—for all integers a, b, and c, $(ab)c = a(bc)$. • *Identity*—there is a unique integer 1, the *multiplicative identity*, such that for all integers a, $1 \cdot a = a = a \cdot 1$. • *Zero multiplication*—there is a unique integer 0 such that for every integer a, $0 \cdot a = 0 = a \cdot 0$.
Distributive property of multiplication over addition or subtraction (p. 242)	For any integers a, b, and c, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$. For any integers a, b, and c, $a(b - c) = ab - ac$.
Multiplication for integers theorems (p. 242)	*Theorem:* For every integer a, $({}^{-}1)(a) = {}^{-}a$. *Theorem:* For all integers a and b, $({}^{-}a)b = {}^{-}(ab)$. *Theorem:* For all integers a and b, $({}^{-}a)({}^{-}b) = ab$.
Difference of squares (p. 243)	For all integers a and b, $(a + b)(a - b) = a^2 - b^2$.
Integer division (p. 245)	For all integers a and b with $b \neq 0$, $a \div b$ is the unique integer c, if it exists, such that $a = bc$. For any integers a and b with $b \neq 0$, if $a \div b$ is a unique integer, then a *is divisible by* b, or equivalently b *divides* a, written as $b \mid a$.
Order of operations for integers (p. 246)	With no parentheses present, exponentiations are performed first in order from right to left, then multiplications and divisions are performed in the order they occur from left to right, and then the additions and subtractions are performed in the order they occur from left to right. Arithmetic operations inside parentheses must be performed before other operations.
Less than for integers (p. 247)	For any integers a and b, a is less than b, written $a < b$, if, and only if, there exists a positive integer k such that $a + k = b$.
Inequality for integers theorems (p. 247)	*Theorem:* For all integers a and b, $a < b$ (or equivalently $b > a$) if, and only if, $b - a$ is equal to a positive integer; that is, $b - a > 0$. *Theorem:* Let x, y, and n be any integers. If $x < y$, then $x + n < y + n$. If $x < y$, then ${}^{-}x > {}^{-}y$. If $x < y$ and $n > 0$, then $nx < ny$. If $x < y$ and $n < 0$, then $nx > ny$.

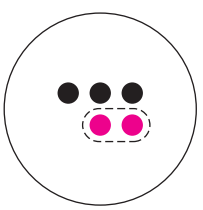
Chapter 5 Review

1. Find the additive inverse of each of the following.
 a. 3 b. ^-a
 c. $^-2 + 3$ d. $x + y$
 e. $^-x + y$ f. $^-x - y$
 g. $(^-2)^5$ h. $^-2^5$
2. Perform each of the following operations.
 a. $(^-2 + ^-8) + 3$ b. $^-2 - (^-5) + 5$
 c. $^-3(^-2) + 2$ d. $^-3(^-5 + 5)$
 e. $^-40 \div (^-5)$ f. $(^-25 \div 5)(^-3)$
3. For each of the following, find all possible integer values of x that make the given equation true.
 a. $^-x + 3 = 0$ b. $^-2x = 10$
 c. $0 \div (^-x) = 0$ d. $^-x \div 0 = ^-1$
 e. $3x - 1 = ^-124$ f. $^-2x + 3x = x$
4. Use a pattern approach to explain why $(^-2)(^-3) = 6$.
5. In each of the following chip models, the encircled chips are removed. Write the corresponding integer problem with its solution.

 a.



 b.



6. Simplify each of the following expressions.
 a. ^-1x
 b. $(^-1)(x - y)$
 c. $2x - (1 - x)$
 d. $(^-x)^2 + x^2$
 e. $(^-x)^3 + x^3$
 f. $(^-3 - x)(3 + x)$
 g. $(^-2 - x)(^-2 + x)$
7. Factor each of the following expressions and then simplify, if possible.
 a. $x - 3x$
 b. $x^2 + x$
 c. $x^2 - 36$
 d. $81y^4 - 16x^4$
 e. $5 + 5x$
 f. $(x - y)(x + 1) - (x - y)$
8. Classify each of the following statements as true or false (all letters represent integers). If false, tell why.
 a. $|x|$ always is positive.
 b. For all x and y, $|x + y| = |x| + |y|$.
 c. If $a < ^-b$, then $a < 0$.
 d. For all x and y, $(x - y)^2 = (y - x)^2$.
9. Find a counterexample to disprove each of the following properties on the set of integers.
 a. Commutative property for division
 b. Associative property for subtraction
 c. Closure for division
 d. Distributive property of division over subtraction
10. Solve each of the following for x, where x is an integer.
 a. $x + 3 = ^-x - 17$
 b. $2x = ^-2^{100}$
 c. $2^{10}x = 2^{99}$
 d. $^-x = x$
 e. $|^-x| = 3$
 f. $|x| = ^-x$
 g. $|x| > 3$
 h. $(x - 1)^2 = 100$
11. Write the first six terms of each of the sequences whose nth term is given.
 a. $(^-1)^n$
 b. $(^-2)^n$
 c. $^-2 - 3n$
12. In each part of exercise 11, if a sequence is arithmetic, find its difference, and if it is geometric, find its ratio.
13. Find the missing terms in the arithmetic, geometric, or Fibonacci-type sequences below.
 a. $^-1, 1, 0, _, _, _, 3$
 b. $^-49, 98, ^-196, _, _, _$
 c. $^-49, 98, 245, _, _, _$
14. Find the sum of each of the finite sequences in exercise 13.
15. How many terms are missing by the indicated ellipsis in each of the following arithmetic or geometric sequences?
 a. $101, 99, 97, \ldots, ^-101, ^-103$
 b. $5, ^-25, 125, \ldots, ^-15625$
16. Find the difference or ratio of each of the sequences in exercise 15.
17. On a college test, students receive 4 points for every question answered correctly and a penalty of 7 points when a question is answered incorrectly. On this particular test, Terry answered 87 questions correctly and 46 questions incorrectly. What is Terry's score for this test?
18. The following states' lowest temperatures on record are given below. Order the given states according to their lowest temperatures from lowest to highest.

State	Temperature in degree Celsius
Alabama	$^-33$
Arizona	$^-40$
California	$^-43$
Iowa	$^-44$
Nevada	$^-46$
Ohio	$^-39$
Oregon	$^-48$
West Virginia	$^-38$

19. In 1971, the lowest recorded temperature in the United States, $^-62$°C, was in Prospect Creek Camp, Alaska. That same year, Arizona recorded a low temperature of $^-40$°C at Hawley Lake. Alaska's temperature was recorded at an

elevation of 1100 ft above sea level while Arizona's was recorded at 8180 ft above sea level.

a. What is the low temperature difference between Alaska and Arizona?

b. What is the high elevation difference between Arizona and Alaska where the low temperatures were measured?

c. Hawaii's lowest temperature on record is $^-11°C$. What would you expect to be true about the elevation of the site where this temperature was measured?

20. Two states record their lowest points below sea level, one at $^-282$ ft and one at $^-8$ ft.

a. What is the difference in elevation between the lowest points of the two states?

b. Which two states do you think these are and why?

21. The average depth of the Pacific Ocean is 3963 m. Interpret this number in relation to sea level.

22. In the metric system there are two official temperature scales: degree Celsius and kelvin. The kelvin temperature scale is obtained by shifting the Celsius scale so that 0 kelvin corresponds to absolute zero, the absence of any heat whatsoever. An equation approximating the relationship between the two scales follows:

$$\text{kelvin} \approx \text{degree Celsius} + 273°$$

a. Write an equation to find degree Celsius in terms of kelvin.

b. If the boiling point of water in Celsius is 100°, what is the comparable temperature in kelvin?

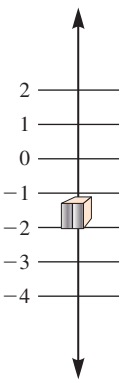
c. $^-40°$ is the point on a thermometer at which both degree Celsius and degree Fahrenheit are the same. What is this temperature in kelvin?

23. In the Battle of Gettysburg, it was estimated that there were 75,000 Confederate forces engaged and 82,289 Union forces engaged. The number of casualties were estimated at 23,049 for the Union and 28,063 for the Confederates.

a. How might positive and negative integers be used to describe these numbers?

b. Using positive and negative numbers, describe the total casualties.

24. The drawing below depicts an elevator. Explain what $^-2$ might mean as a floor number.



25. Find any common solutions to the following pairs of inequalities.

a. $^-2x < 14$ and $3x > {}^-15$

b. $x^2 < 9$ and $x^2 \leq 1$

c. $|x| > {}^-4$ and $|x| < {}^-1$

26. Joaquin said to his sister, "I'm thinking of your IQ and subtract 30 from it, take the opposite of the result, add $^-14$, and find the opposite of the new result. My final answer is 6." What is Joaquin suggesting is his sister's IQ?

27. Describe how you could use the difference of squares formula to compute $96 \cdot 104$.

28. Find the first five terms of the sequence whose nth term is given.

a. $n^2 - 4$

b. $^-5n - (^-3)$

c. $^-n^2 - 4$

29. Evaluate each expression.

a. $(^-1)^{2n}$

b. $(^-1)^{2n+1}$

30. Consider the sequence $1, ^-2, 3, ^-4, 5, ^-6, \ldots$.

a. Find the nth term of the sequence.

b. Find the sum of the first 100 terms of the sequence.

Chapter 6

Rational Numbers and Proportional Reasoning

6-1 The Set of Rational Numbers

6-2 Addition, Subtraction, and Estimation with Rational Numbers

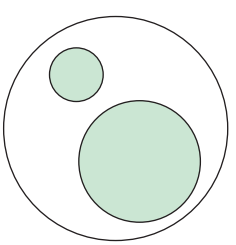
6-3 Multiplication, Division, and Estimation with Rational Numbers

6-4 Proportional Reasoning



Preliminary Problem

A special rubber washer is made with two holes cut out as pictured. The area of the smaller of the two holes is $\frac{1}{7}$ of the whole piece of rubber while the area of the larger hole is $\frac{1}{4}$ of the whole. If the area of the original piece of rubber was $1\frac{3}{8}$ in^2, what is the area of the finished washer?



If needed, see Hint on page 323.

Integers such as $^{-}5$ were invented to solve equations like $x + 5 = 0$. Similarly, a different type of number is needed to solve the equation $2x = 1$. We need notation for this new number. If multiplication is to work with this new type of number as with whole numbers, then $2x = x + x = 1$. In other words, the number $\frac{1}{2}$ (*one-half*) created to solve the equation must have the property that when added to itself, the result is 1. It is an element of the set of numbers of the form $\frac{a}{b}$, where $b \neq 0$ and a and b are integers. More generally, numbers of the form $\frac{a}{b}$ are solutions to equations of the form $bx = a$. This set Q of **rational numbers** is defined as follows:

$$Q = \left\{ \frac{a}{b} \mid a \text{ and } b \text{ are integers and } b \neq 0 \right\}$$

Each member of Q is a *fraction*. In general, fractions are of the form $\frac{a}{b}$, where $b \neq 0$ but a and b are not necessarily integers. Each element $\frac{a}{b}$ of set Q has a as the **numerator** and b as the **denominator**.

The English words used for denominators of rational numbers are similar to words to tell "order," for example, the *fourth* person in a line, and the glass is three-fourths full. In contrast, $\frac{3}{4}$ is read "out of four parts, (take) three" in Chinese. The Chinese model enforces the idea of partitioning quantities into equal parts and choosing some number of these parts. The concept of sharing quantities and comparing sizes of shares provides entry points to introduce students to rational numbers.



As early as grade 3 in the *Common Core Standards*, we find that students should "develop an understanding of fractions, beginning with unit fractions . . . view fractions as being built out of unit fractions . . . use fractions along with visual fraction models to represent parts of a whole." (p. 21) Additionally by grade 4, students should "understand a fraction as a number on the number line." (p. 24)

REMARK A unit fraction has a numerator of 1.

6-1 The Set of Rational Numbers

6-1 Objectives

Students will be able to understand and explain

- Different representations for rational numbers.
- Equal fractions, equivalent fractions, and the simplest form of fractions.
- Ordering of rational numbers.
- Denseness property of rational numbers.

The rational number $\frac{a}{b}$ may also be represented as a/b or $a \div b$. The word *fraction* is derived from the Latin word *fractus*, meaning "broken." The word *numerator* comes from a Latin word meaning "numberer," and *denominator* comes from a Latin word meaning "namer." Frequently it is only in the upper grades of middle school that students begin to use integers for the parts of rational numbers, but prospective teachers should know and recognize that rational numbers are negative as well as positive and zero. Some uses of rational numbers that will be considered in this chapter are seen in Table 1.

Table 1

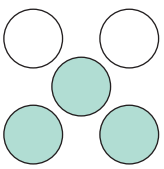
Use	Example
Division problem or solution to a multiplication problem	The solution to $2x = 3$ is $\frac{3}{2}$.
Portion, or part, of a whole	Joe received $\frac{1}{2}$ of Mary's salary each month for alimony.
Ratio	The ratio of Republicans to Democrats on a Senate committee is three to five.
Probability	When you toss a fair coin, the probability of getting heads is $\frac{1}{2}$.



(a) Bar model or area model



(b) Number-line model



(c) Set model

Figure 1

Figure 1 illustrates the use of rational numbers as equal-sized parts of a whole in part (a), a distance on a number line in part (b), and a part of a given set in part (c). The simplest representation is from part (a) where 1 part of 3 equal-sized parts is shaded. The fractional representation for this part is $\frac{1}{3}$ where *the entire bar represents 1 unit and the shaded part is* $\frac{1}{3}$ *of the unit whole.* [Later the bar model of part (a) will be extended to an area model where the shape may be different than the rectangular bar. Additionally, the bar model is helpful when we consider proportional reasoning later in the chapter.]

An extension of the thinking in the bar model is seen in the remaining parts of Figure 1. For example, part (b) could represent two one-thirds of the unit length, or two-thirds of the unit segment. Part (c) could represent three one-fifths of the whole set, or three-fifths of the whole set.

Early student exposure to rational numbers as fractions usually takes the form of description rather than mathematical notation. They hear phrases such as "one-half of a pizza," "one-third of a cake," or "three-fourths of a pie." They encounter such questions as "If three identical fruit bars are distributed equally among four friends, how much does each receive?" The answer is that each receives $\frac{3}{4}$ of a bar.

When rational numbers are introduced as fractions that represent a part of a whole, we must pay attention to the whole from which a rational number is derived. For example, if we talk about $\frac{3}{4}$ of a pizza, then the amount of pizza depends on the size of the pizza, for example, 10" or 12", and the fractional part, $\frac{3}{4}$.

To understand the meaning of any fraction, $\frac{a}{b}$, where $a, b \in W$ and $b \neq 0$, using the parts-to-whole model, we must consider each of the following:

IMAP Video
Watch three students explore part-to-whole relationships.

1. The *whole* being considered.
2. The number b of equal-size parts into which the whole has been divided.
3. The number a of parts of the whole that are selected.

A fraction $\frac{a}{b}$, where $0 \leq a < b$, is a **proper fraction**. A proper fraction is less than 1. For example, $\frac{4}{7}$ is a proper fraction, but $\frac{7}{4}$, $\frac{4}{4}$, and $\frac{9}{7}$ are not; $\frac{7}{4}$ is an **improper fraction**. In general $\frac{a}{b}$ is an improper fraction if $a \geq b > 0$. An improper fraction is greater than or equal to 1.

Historical Note



Egyptian Symbol for 1/3

The early Egyptian numeration system had symbols for fractions with numerators of 1 (unit fractions). Most fractions with other numerators were expressed as a sum of unit fractions, for example, $\frac{7}{12} = \frac{1}{3} + \frac{1}{4}$.

Fractions with denominator 60 or powers of 60 were seen in Babylon about 2000 BCE, where 12,35 meant $12 + \frac{35}{60}$. This usage was adopted by the Greek astronomer Ptolemy (approximately 125 CE), was used in Islamic and European countries, and is presently used in the measurements of angles.

The modern notation for fractions—a bar between numerator and denominator—is of Hindu origin. It came into general use in Europe in sixteenth-century books. ●

Other meanings of fractions can be considered using *whole-to-part* and *part-to-part* references. For example whole-to-part might give us an improper fraction and part-to-part allows us to write, for example, the ratio of the number of band students in the school to the number of non-band students in the school.

The *Common Core Standards* state that grade 3 students should "express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers." (p. 24)

Later students learn that every integer n can be represented as a rational number because $n = \frac{nk}{k}$, where k is any nonzero integer. In particular, $0 = \frac{0 \cdot k}{k} = \frac{0}{k}$.

Rational Numbers on a Number Line

In the grade 3 *Common Core Standards*, we find the following standard:

Represent a fraction $\frac{a}{b}$ on a number line by marking off a lengths of $\frac{1}{b}$ from 0. Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the number $\frac{a}{b}$ on the number line. (p. 24)

Once the integers 0 and 1 are assigned to points on a line, the unit segment is defined and every other rational number is assigned to a specific point. For example, to represent $\frac{3}{4}$ on the number line, we divide the segment from 0 to 1 into 4 segments of equal length and mark the line accordingly. Then, starting from 0, we count 3 of these segments and stop at the mark corresponding to the right endpoint of the third segment to obtain the point assigned to the rational number $\frac{3}{4}$.

The *Common Core Standards* for grade 3 talk about a lengths of $\frac{1}{b}$, where a and b are both positive (or a could be 0), but we also use integers as numerators or denominators of rational numbers, though negative integers are not used to talk about lengths. We think of the positive fractions described in the *Common Core Standards* as marked on a number line on the right side, and as with integers, we can consider the opposites of those fractions reflected over 0 to the left side of the number line as seen in Figure 2. We adopt two conventions for negative fractions, either $\frac{-a}{b}$ or $-\frac{a}{b}$.

Figure 2 shows the points that correspond to $^-2$, $-\frac{5}{4} = \frac{^-5}{4}$, $^-1$, $-\frac{3}{4} = \frac{^-3}{4}$, 0, $\frac{3}{4}$, 1, $\frac{5}{4}$, and 2.

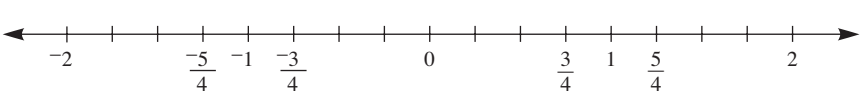


Figure 2

Example 1

Describe how to locate the following numbers on the number line of Figure 3: $\frac{1}{2}$, $-\frac{1}{2}$, $\frac{7}{4}$, and $-\frac{7}{4}$.

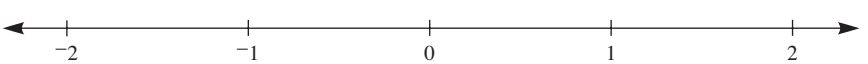


Figure 3

Solution To decide how to find the point on the number line representing $\frac{1}{2}$, we consider the unit length 1. We find the point that would be the rightmost endpoint of the segment starting at 0 and ending at the point marking the middle of the unit segment. This is seen in Figure 4.

To find the point on the number line representing $-\frac{1}{2}$, we find the mirror image of $\frac{1}{2}$ on the left side of the number line as seen in Figure 4 when it is reflected in 0.

To find the location of $-\frac{7}{4}$, we first find the image or $\frac{7}{4}$ on the right side of 0 by marking the unit length in four parts, duplicating the four parts to mark points between 1 and 2, and then counting 7 of those parts starting at 0. Once $\frac{7}{4}$ is found on the right side of 0, then its reflection image in 0 gives the point where $-\frac{7}{4} = \frac{-7}{4}$ should be marked. This is seen in Figure 4.

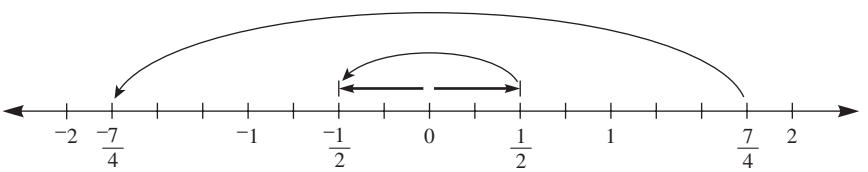


Figure 4

Equivalent or Equal Fractions

CCSS **The grade 4 *Common Core Standards* state that students should be able to:**

Explain why a fraction $\frac{a}{b}$ is equivalent to a fraction $\frac{na}{nb}$ by using visual fraction models with attention to how the number and the size of the parts differ even though the two fractions themselves are the same size. Use the principle to recognize and generate equivalent fractions. (p. 30)

Fractions may be introduced in the classroom through a concrete activity such as paperfolding. In Figure 5(a), 1 of 3 congruent parts, or $\frac{1}{3}$, is shaded. In this case, the whole is the rectangle. In Figure 5(b), each of the thirds has been folded in half so that now we have 6 sections, and 2 of 6 congruent parts, or $\frac{2}{6}$, are shaded. Thus, both $\frac{1}{3}$ and $\frac{2}{6}$ represent exactly the same shaded portion. Although the symbols $\frac{1}{3}$ and $\frac{2}{6}$ do not look alike, they represent the same rational number and are **equivalent fractions**, or **equal fractions**. *Equivalent fractions are numbers that represent the same point on a number line*. Because they represent equal amounts, we write $\frac{1}{3} = \frac{2}{6}$ and say that "$\frac{1}{3}$ equals $\frac{2}{6}$."

E-Manipulative Activity
For practice with the concept of fractions, see the *Naming Fractions* and *Visualizing Fractions* modules on the E-manipulatives disc.

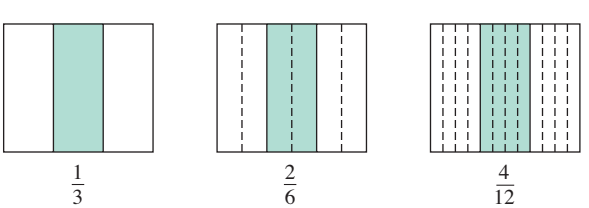


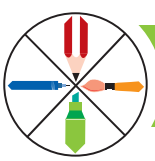
Figure 5

Figure 5(c) shows the rectangle with each of the original thirds folded into 4 equal parts with 4 of the 12 parts now shaded. Thus, $\frac{1}{3}$ is equal to $\frac{4}{12}$ because the same portion of the model is shaded. Similarly, we could illustrate that $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \frac{5}{15}, \ldots$ are all equal.

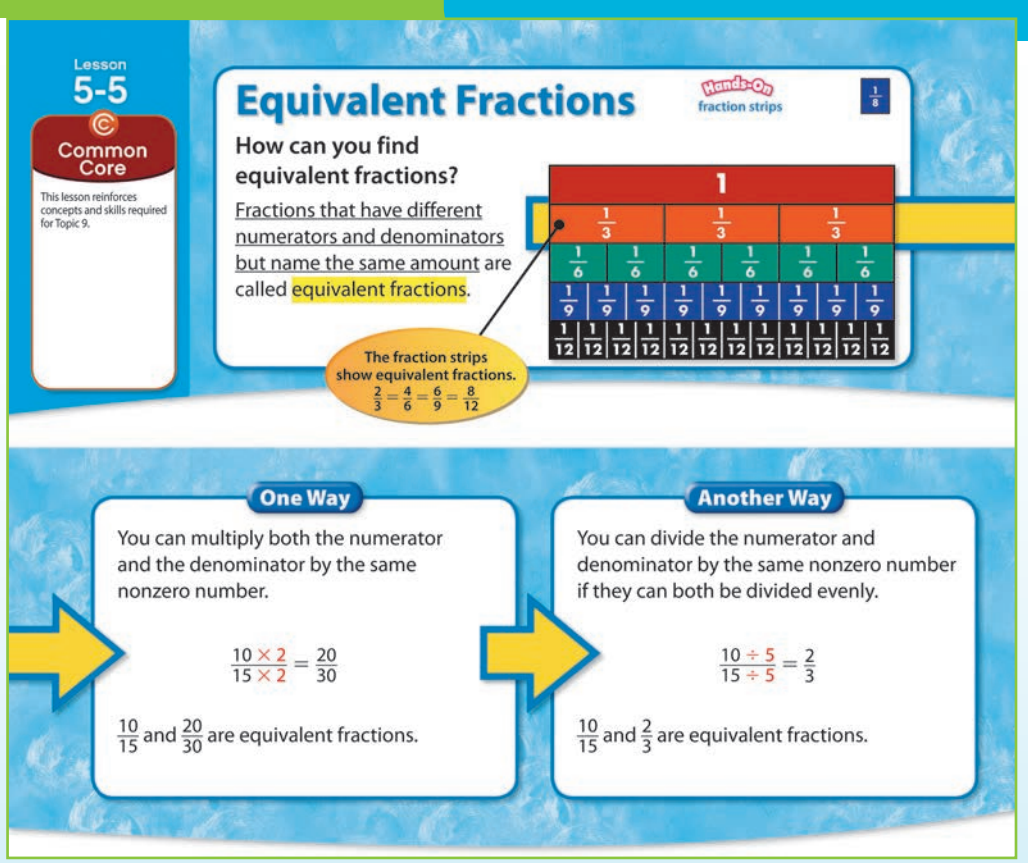
Fraction strips can be used for generating equivalent fractions, as seen on the student page below. This technique makes use of the Fundamental Law of Fractions, which can be stated as follows: *The value of a fraction does not change if its numerator and denominator are multiplied by the same nonzero integer.* Under certain assumptions this Law of Fractions can be proved and is stated as a theorem.

Theorem 6-1: Fundamental Law of Fractions

If $\frac{a}{b}$ is a fraction and n a nonzero number, then $\frac{a}{b} = \frac{an}{bn}$.



School Book Page — Equivalent Fractions



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From the Fundamental Law of Fractions, $\frac{7}{^-15} = \frac{^-7}{15}$ because $\frac{7}{^-15} = \frac{7(^-1)}{^-15(^-1)} = \frac{^-7}{15}$. Similarly, $\frac{a}{^-b} = \frac{^-a}{b}$. *The form* $\frac{^-a}{b}$, *where b is a positive number, is usually preferred.*

Simplifying Fractions

Theorem 6-1 implies that if d is a common factor of a and b, then $\frac{a}{b} = \frac{a \div d}{b \div d}$. Because d is a divisor of both a and b, we know that $a \div d = m$ is a nonzero integer. Also $b \div d = n$ is a nonzero integer. Thus, we have the following:

$$\frac{a}{b} = \frac{a \div d}{b \div d} = \frac{m}{n}$$

Hence, we have two equivalent forms of the same rational number. A numerical example is given next.

$$\frac{60}{210} = \frac{60 \div 10}{210 \div 10} = \frac{6}{21}$$

This process is referred to as **simplifying a fraction**. A slightly different look is seen below.

$$\frac{60}{210} = \frac{6 \cdot 10}{21 \cdot 10} = \frac{6}{21}$$

Also,

$$\frac{6}{21} = \frac{2 \cdot 3}{7 \cdot 3} = \frac{2}{7}.$$

We can simplify $\frac{60}{210}$ because the numerator and denominator have a common factor of 10. We can simplify $\frac{6}{21}$ because 6 and 21 have a common factor of 3. However, we cannot simplify $\frac{2}{7}$ because 2 and 7 have no positive common factor other than 1. We could also simplify $\frac{60}{210}$ in one step: $\frac{60}{210} = \frac{2 \cdot 30}{7 \cdot 30} = \frac{2}{7}$. Notice that $\frac{2}{7}$ is the **simplest form** of $\frac{60}{210}$ because both 60 and 210 have been divided by their greatest common divisor, 30.

Suppose we wanted to simplify the rational number $-\frac{60}{210}$ or its equivalent $\frac{^-60}{210}$. The problem could be addressed by simplifying $\frac{60}{210}$ as above and then taking the opposite of the result. This could be thought of as considering $-\frac{60}{210} = -\left|\frac{60}{210}\right| = -\frac{2}{7}$. Note that the same result could have been found using $-\frac{60}{210} = -\frac{|60|}{|210|} = -\frac{2}{7}$.

Definition of Simplest Form

A rational number $\frac{a}{b}$ is in **simplest form** or **lowest terms** if, and only if, $\text{GCD}(a, b) = 1$, that is, if a and b have no common factor greater than 1.



Scientific/fraction calculators can simplify fractions. For example, to simplify $\frac{6}{12}$, we enter $\boxed{6}\ \boxed{/}\ \boxed{1}\ \boxed{2}$ and press $\boxed{\text{SIMP}}\ \boxed{=}$, and $\frac{3}{6}$ appears on the screen. At this point, an indicator tells us that this is not in simplest form, so we press $\boxed{\text{SIMP}}\ \boxed{=}$ again to obtain $\frac{1}{2}$. At any time, we can view the factor that was removed by pressing the $\boxed{x \bigcirc y}$ key.

Example 2

Find a value for x such that $\frac{12}{42} = \frac{x}{210}$.

Solution We use the mathematical practice of reasoning to see that because $210 \div 42 = 5$, we can use the Fundamental Law of Fractions to obtain $\frac{12}{42} = \frac{12 \cdot 5}{42 \cdot 5} = \frac{60}{210}$. Hence, $\frac{x}{210} = \frac{60}{210}$, and $x = 60$.

Alternative approach: $\frac{12}{42} = \frac{2 \cdot 6}{7 \cdot 6} = \frac{2}{7} = \frac{2 \cdot 30}{7 \cdot 30} = \frac{60}{210}$. Therefore $x = 60$.

Example 3

Write each of the following fractions in simplest form if they are not already.

a. $\frac{28ab^2}{42a^2b^2}$ **b.** $\frac{(a+b)^2}{3a+3b}$ **c.** $\frac{x^2+x}{x+1}$ **d.** $\frac{3+x^2}{3x^2}$

e. $\frac{3+3x^2}{3x^2}$ **f.** $\frac{a^2-b^2}{a-b}$ **g.** $\frac{a^2+b^2}{a+b}$

Solution

a. $\frac{28ab^2}{42a^2b^2} = \frac{2(14ab^2)}{3a(14ab^2)} = \frac{2}{3a}$

b. $\frac{(a+b)^2}{3a+3b} = \frac{(a+b)(a+b)}{3(a+b)} = \frac{a+b}{3}$

c. $\frac{x^2+x}{x+1} = \frac{x(x+1)}{x+1} = \frac{x(x+1)}{1(x+1)} = \frac{x}{1} = x$

d. $\frac{3+x^2}{3x^2}$ cannot be simplified because $3 + x^2$ and $3x^2$ have no factors in common except 1.

e. $\frac{3+3x^2}{3x^2} = \frac{3(1+x^2)}{3x^2} = \frac{1+x^2}{x^2}$

f. Recall the difference of squares formula from Chapter 5: $a^2 - b^2 = (a-b)(a+b)$. Thus,

$$\frac{a^2-b^2}{a-b} = \frac{(a-b)(a+b)}{(a-b)1} = \frac{a+b}{1} = a+b.$$

g. The fraction is already in simplest form because $a^2 + b^2$ does not have $(a+b)$ as a factor. Notice that $a^2 + b^2 \neq (a+b)^2$.

When an algebraic expression is written as a fraction, the denominator may not be 0. Thus, when the fraction is simplified, this restriction has to be maintained. For example, in part (c) of Example 3, $\frac{x^2 + x}{x + 1} = x$ if $x \neq {}^{-}1$, and in part (f) the result holds if $a - b \neq 0$, that is, if $a \neq b$.

Some students think of the Fundamental Law of Fractions as a *cancellation property* and "simplify" an expression like $\frac{6 + a^2}{3a}$ by thinking of it as $\frac{2 \cdot 3 + a \cdot a}{3a}$ and "canceling" equal numbers in the products to obtain $2 + a$ as the answer. Emphasizing the factor approach that neither 3 nor a is a factor of $6 + a^2$ may help to avoid such mistakes.

Equality of Fractions

We use three equivalent methods to show that two fractions, such as $\frac{12}{42}$ and $\frac{10}{35}$, are equal.

1. Simplify both fractions to simplest form.

$$\frac{12}{42} = \frac{2^2 \cdot 3}{2 \cdot 3 \cdot 7} = \frac{2}{7} \quad \text{and} \quad \frac{10}{35} = \frac{5 \cdot 2}{5 \cdot 7} = \frac{2}{7}$$

 Thus,

$$\frac{12}{42} = \frac{10}{35}.$$

2. Rewrite both fractions with the same least common denominator. Since $\text{LCM}(42, 35) = 210$, then

$$\frac{12}{42} = \frac{60}{210} \quad \text{and} \quad \frac{10}{35} = \frac{60}{210}.$$

 Thus,

$$\frac{12}{42} = \frac{10}{35}.$$

3. Rewrite both fractions with a common denominator (not necessarily the least). A common multiple of 42 and 35 may be found by finding the product $42 \cdot 35 = 1470$.

$$\frac{12}{42} = \frac{420}{1470} \quad \text{and} \quad \frac{10}{35} = \frac{420}{1470}$$

 Hence,

$$\frac{12}{42} = \frac{10}{35}.$$

E-Manipulative Activity
Additional practice with ordering fractions is available in the *Ranking Fractions* module on the E-Manipulative disc.

The third method suggests a general algorithm for determining whether two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal. Rewrite both fractions with common denominator bd; that is,

$$\frac{a}{b} = \frac{ad}{bd} \quad \text{and} \quad \frac{c}{d} = \frac{bc}{bd}.$$

Because the denominators are the same, $\frac{ad}{bd} = \frac{bc}{bd}$ if, and only if, $ad = bc$. For example, $\frac{24}{36} = \frac{6}{9}$ because $24 \cdot 9 = 216 = 36 \cdot 6$. In general, the following theorem holds.

Theorem 6-2: Equality of Fractions

Two fractions $\frac{a}{b}$ and $\frac{c}{d}$, with $b \neq 0$ and $d \neq 0$, are equal if, and only if, $ad = bc$.



Using a calculator, we determine whether two fractions are equal by using Theorem 6-2. Since both 2 × 2 1 9 6 = and 4 × 1 0 9 8 = yield a display of 4392, we see that $\frac{2}{4} = \frac{1098}{2196}$.

Ordering Rational Numbers



The grade 4 *Common Core Standards* state that a student should compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $\frac{1}{2}$. Recognize that comparisons are valid only when two fractions refer to the same whole. (p. 30)

Example 4

Jim claims that $\frac{1}{3} > \frac{1}{2}$ because in Figure 6 the shaded portion for $\frac{1}{3}$ is larger than the shaded portion for $\frac{1}{2}$. How would you help him?



Solution Jim needs to understand as noted in the *Common Core Standards* that comparisons of two fractions are valid only when they refer to the same whole. In Figure 6, the circle is clearly larger than the square so the two wholes are not the same.

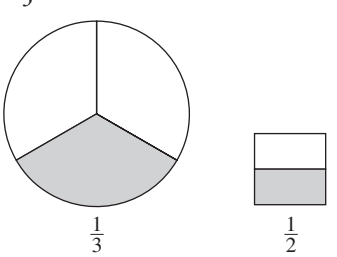


Figure 6

In order to compare two fractions referring to the same whole, it is easiest to compare fractions with like denominators. Children know that $\frac{7}{8} > \frac{5}{8}$ because if a pizza is divided into 8 parts of equal size, then 7 parts of the pizza is more than 5 parts. Similarly, $\frac{3}{7} < \frac{4}{7}$. Thus, given two fractions with common positive denominators, the one with the greater numerator is the greater fraction. To make ordering of rational numbers consistent with the ordering of whole numbers and integers we have the following definition.

Definition of Greater Than for Rational Numbers with Like Denominators

If a, b, and c are integers and $b > 0$, then $\frac{a}{b} > \frac{c}{b}$ if, and only if, $a > c$.

IMAP Video

Watch Jace compare $\frac{1}{5}$ and $\frac{1}{8}$.

To compare fractions with unlike denominators, some students may incorrectly reason that $\frac{1}{8} > \frac{1}{7}$ because 8 is greater than 7. In another case, they might falsely believe that $\frac{6}{7}$ is equal to $\frac{7}{8}$ because in both fractions the difference between the numerator and the denominator is 1. Comparing positive fractions with unlike denominators may be aided by using fraction strips to compare the fractions visually. For example, consider the fractions $\frac{4}{5}$ and $\frac{11}{12}$ shown in Figure 7.

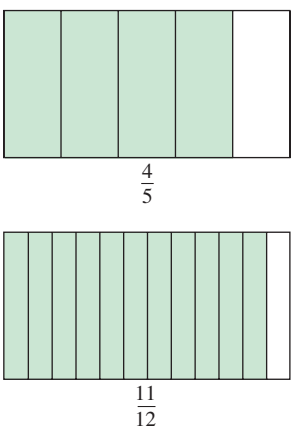


Figure 7

IMAP Video

Watch Jacky use fraction strips to compare fractions and a practicing teacher explain Jacky's misconceptions.

From Figure 7, students see that each fraction is one piece less than the same-size whole unit. However, they see that the missing piece for $\frac{11}{12}$ is smaller than the missing piece for $\frac{4}{5}$, so $\frac{11}{12}$ must be greater than $\frac{4}{5}$.

Comparing any fractions with unlike denominators can be accomplished by rewriting the fractions with the same positive common denominator. Using the common denominator bd, we can write the fractions $\frac{a}{b}$ and $\frac{c}{d}$ as $\frac{ad}{bd}$ and $\frac{bc}{bd}$. Because $b > 0$ and $d > 0$, then $bd > 0$; and we have the following:

$$\frac{a}{b} > \frac{c}{d} \text{ if, and only if, } \frac{ad}{bd} > \frac{bc}{bd}; \text{ and } \frac{ad}{bd} > \frac{bc}{bd} \text{ if, and only if, } ad > bc.$$

Therefore, we have the following theorem.

Theorem 6-3

If a, b, c, and d are integers with $b > 0$ and $d > 0$, then $\frac{a}{b} > \frac{c}{d}$ if, and only if, $ad > bc$.

Next consider two fractions with both numerators and denominators positive and with numerators that are the same. For example, consider $\frac{3}{4}$ and $\frac{3}{10}$. If the whole is the same for both fractions, this means that we have three $\frac{1}{4}$s and three $\frac{1}{10}$s. Because $\frac{1}{4}$ is greater than $\frac{1}{10}$, three of the larger parts is greater than three of the smaller parts. Thus, $\frac{3}{4} > \frac{3}{10}$.

Denseness of Rational Numbers

The set of rational numbers has a property unlike the set of whole numbers and the set of integers. Consider $\frac{1}{2}$ and $\frac{2}{3}$. To find a rational number between $\frac{1}{2}$ and $\frac{2}{3}$, we first rewrite the fractions with a common denominator, as $\frac{3}{6}$ and $\frac{4}{6}$. Because there is no whole number between the numerators 3 and 4, we next find two fractions equal, respectively, to $\frac{1}{2}$ and $\frac{2}{3}$ with greater denominators. For example, $\frac{1}{2} = \frac{6}{12}$ and $\frac{2}{3} = \frac{8}{12}$, and $\frac{7}{12}$ is between the two fractions $\frac{6}{12}$ and $\frac{8}{12}$. So $\frac{7}{12}$ is between $\frac{1}{2}$ and $\frac{2}{3}$. This property is generalized as follows and stated as a theorem.

Theorem 6-4: Denseness Property for Rational Numbers

Given any two different rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, there is another rational number between these two numbers.

NOW TRY THIS 1

Explain why there are infinitely many rational numbers between any two rational numbers.

Example 5

a. Find two fractions between $\frac{7}{18}$ and $\frac{1}{2}$.

b. Show that the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1} \ldots$, where $n \in N$, is an *increasing sequence;* that is, that each term starting from the second term is greater than the preceding term.

Solution

a. Because $\frac{1}{2} = \frac{1 \cdot 9}{2 \cdot 9} = \frac{9}{18}$, we see that $\frac{8}{18}$, or $\frac{4}{9}$, is between $\frac{7}{18}$ and $\frac{9}{18}$. To find another fraction between the given fractions, we find two fractions equal to $\frac{7}{18}$ and $\frac{9}{18}$, respectively, but with greater denominators; for example, $\frac{7}{18} = \frac{14}{36}$ and $\frac{9}{18} = \frac{18}{36}$.

We now see that $\frac{15}{36}$, $\frac{16}{36}$, and $\frac{17}{36}$ are all between $\frac{14}{36}$ and $\frac{18}{36}$ and thus between $\frac{7}{18}$ and $\frac{1}{2}$.

b. Because the nth term of the sequence is $\frac{n}{n+1}$, the next term is $\frac{n+1}{(n+1)+1}$, or $\frac{n+1}{n+2}$. We need to show that for all positive integers n, $\frac{n+1}{n+2} > \frac{n}{n+1}$.

The terms of the sequence are positive. The inequality will be true if, and only if, $(n+1)(n+1) > n(n+2)$. This inequality is equivalent to

$$\begin{aligned} n^2 + 2n + 1 &> n^2 + 2n \\ 2n + 1 &> 2n \\ 1 &> 0, \text{ which is true.} \end{aligned}$$

Therefore we have an increasing sequence.

Another way to find a number between any two rational numbers involves adding numerators and adding denominators. In Example 5(a), to find a number between $\frac{7}{18}$ and $\frac{1}{2}$ we could add the numerators and add the denominators to produce $\frac{7+1}{18+2} = \frac{8}{20}$. We see that $\frac{7}{18} < \frac{8}{20}$ because $140 < 144$. Also, $\frac{8}{20} < \frac{1}{2}$ because $16 < 20$. The general property is stated in the following theorem and explored in Now Try This 2.

Theorem 6-5

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers with positive denominators such that $\frac{a}{b} < \frac{c}{d}$. Then,

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

NOW TRY THIS 2

Prove Theorem 6-5. $\left(\textit{Hint:}\ \text{Prove that } \frac{a}{b} < \frac{a+c}{b+d} \text{ and } \frac{a+c}{b+d} < \frac{c}{d}.\right)$

The proof of Theorem 6-5 suggested in Now Try This 2 also proves Theorem 6-4.

Assessment 6-1A

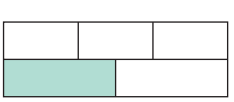
1. Write a sentence that illustrates the use of $\frac{7}{8}$ in each of the following ways.
- **a.** As a division problem
- **b.** As part of a whole
- **c.** As a ratio

2. For each of the following, write a fraction to approximate the shaded portion as part of the whole.

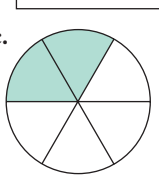
a.



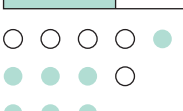
b.



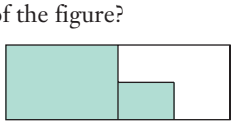
c.



d.

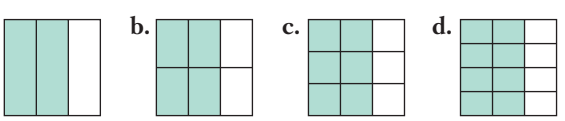


3. If the entire rectangle is a whole, what fraction represents the shaded portion of the figure?



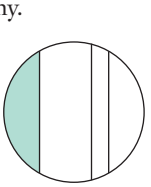
4. For each of the following four squares, write a fraction to describe the shaded portion. What property of fractions does the diagram illustrate?

a. **b.** **c.** **d.**



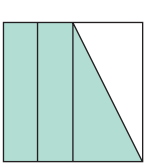
5. Based on your observations, could the shaded portions in the following figures represent the indicated fractions? If not, tell why.

a.



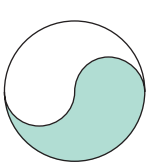
$\frac{1}{4}$

b.



$\frac{3}{4}$

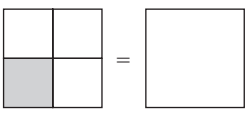
c.



$\frac{1}{2}$

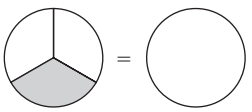
6. In each case, subdivide the *whole* shown on the right to show the equivalent fraction.

a.



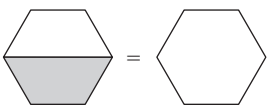
$\frac{1}{4} = \frac{2}{8}$

b.



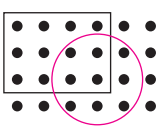
$\frac{1}{3} = \frac{3}{9}$

c.



$\frac{1}{2} = \frac{3}{6}$

7. Referring to the figure, represent each of the following quantities as a fraction.



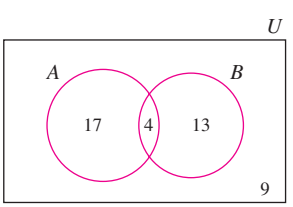
a. The dots in the interior of the circle as a part of all the dots

b. The dots in the interior of the rectangle as a part of all the dots

c. The dots in the intersection of the interiors of the rectangle and the circle as a part of all the dots

d. The dots outside the circular region but inside the rectangular region as part of all the dots

8. Use the Venn diagram pictured with A representing the people in an arts class and B representing the people in a botany class to express the fraction of elements in the indicated sets as a part of the universal set.



a. $A \cap B$ **b.** $A - B$

c. U **d.** $\overline{A \cap B}$

9. For each of the following fractions, write three equal fractions.

a. $\frac{2}{9}$ **b.** $\frac{^-2}{5}$

c. $\frac{0}{3}$ **d.** $\frac{a}{2}$

10. Find the simplest form for each of the following fractions.

a. $\frac{156}{93}$ **b.** $\frac{27}{45}$ **c.** $\frac{^-65}{91}$

11. For each of the following fractions, choose the expression in parentheses that equals or describes best the given fraction.

a. $\frac{0}{0}$ (1, undefined, 0)

b. $\frac{5}{0}$ (undefined, 5, 0)

c. $\frac{0}{5}$ (undefined, 5, 0)

d. $\frac{2+a}{a}$ (2, 3, cannot be simplified)

e. $\frac{15+x}{3x}$ $\left(\frac{5+x}{x}, 5, \text{cannot be simplified}\right)$

12. Find the simplest form for each of the following fractions.

a. $\frac{a^2 - b^2}{3a + 3b}$ **b.** $\frac{14x^2y}{63xy^2}$

13. Determine whether the following pairs are equal.

a. $\frac{3}{8}$ and $\frac{375}{1000}$ **b.** $\frac{18}{54}$ and $\frac{23}{69}$

14. Determine whether the following pairs are equal by changing both to have the same denominator.

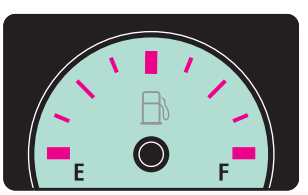
a. $\frac{10}{16}$ and $\frac{12}{18}$ **b.** $\frac{^-21}{86}$ and $\frac{^-51}{215}$

15. Draw an area model to show that $\frac{3}{4} = \frac{6}{8}$.

16. If a fraction is equal to $\frac{3}{4}$ and the sum of the numerator and denominator is 84, what is the fraction?

17. Mr. Gomez filled his car's 16 gal gas tank. He took a trip and used $\frac{7}{8}$ of the gas.

a. Draw an arrow in the following figure to show what his gas gauge looked like after the trip:



b. How many gallons of gas were used?

18. Solve for x in each of the following.

a. $\frac{2}{3} = \frac{x}{16}$ **b.** $\frac{3}{4} = \frac{^-27}{x}$

19. For each of the following pairs of fractions, replace the comma with the correct symbol ($<$, $=$, $>$) to make a true statement:

a. $\frac{7}{8}, \frac{5}{6}$ **b.** $\frac{^-7}{8}, \frac{^-4}{5}$

20. Arrange each of the following in decreasing order.

a. $\frac{11}{22}, \frac{11}{16}, \frac{11}{13}$

b. $\frac{^-1}{5}, \frac{^-19}{36}, \frac{^-17}{30}$

21. For each of the following, find two rational numbers between the given fractions.

a. $\frac{3}{7}$ and $\frac{4}{7}$

b. $\frac{^-7}{9}$ and $\frac{^-8}{9}$

22. **a.** 6 oz is what part of a pound? A ton?

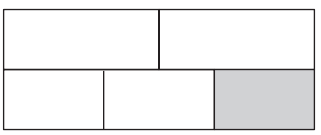
b. A dime is what fraction of a dollar?

c. 15 min is what fraction of an hour?

d. 8 hr is what fraction of a day?

23. Determine whether the following is true: If a, b, c are integers and $b < 0$, then $\frac{a}{b} > \frac{c}{b}$ if, and only if, $a > c$.

24. Based on your visual observation write a fraction to represent the shaded portion.



25. Fill in missing numbers for x, y, z, and w to create a sequence with each term being greater than the preceding term.

a. $\frac{1}{3}, \frac{x}{4}, \frac{y}{5}, \frac{z}{6}$ **b.** $\frac{-3}{x}, \frac{-4}{y}, \frac{-5}{z}, \frac{-6}{w}$

26. Explain why in the Fundamental Law of Fractions (Theorem 6-1), n must be nonzero.

27. Prove that any integer n can be written as a fraction.

28. A typical English ruler is marked in sixteenths of an inch. Sketch a ruler marking $\frac{3}{8}$ in.

29. Ten light bulbs were in a chandelier. One-fifth of the bulbs were not shining. How many light bulbs were not shining?

30. At a party, there were 35 guests. Two-fifths of the guests were men. What fraction of the guests were women?

Assessment 6-1B

1. Write a sentence that illustrates the use of $\frac{7}{10}$ in each of the following ways:

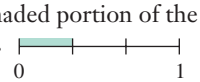
a. As a division problem

b. As part of a whole

c. As a ratio

2. For each of the following, write a fraction to approximate the shaded portion of the whole.

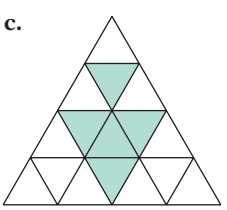
a.



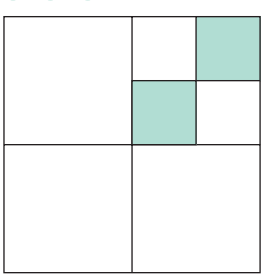
b.



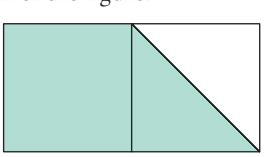
c.



d.



3. If the entire rectangle is a whole, what fraction represents the shaded portion of the figure?



4. Complete each of the following figures so that it illustrates $\frac{3}{5}$.

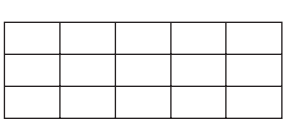
a.



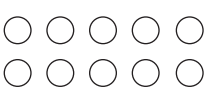
b.



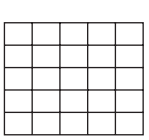
c.



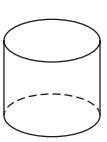
d.



e.

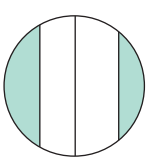


f.



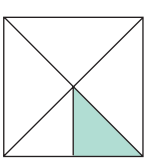
5. Based on your observations, could the shaded portions in the following figures represent the indicated fractions? Tell why.

a.



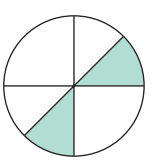
$\frac{1}{2}$

b.



$\frac{1}{8}$

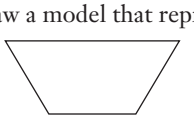
c.



$\frac{2}{8}$

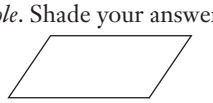
6. If each of the following models represents the given fraction, draw a model that represents the *whole*. Shade your answer.

a.



$\frac{3}{4}$

b.



$\frac{4}{3}$

c.



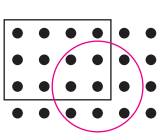
$\frac{1}{5}$

d.



$\frac{1}{4}$

7. Referring to the figure, represent each of the following quantities as a fraction.



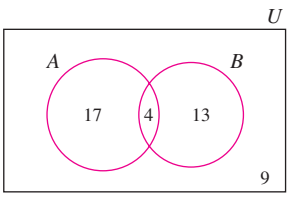
a. The dots outside the circular region as a part of all the dots

b. The dots outside the rectangular region as a part of all the dots

c. The dots in the union of the rectangular and the circular regions as a part of all the dots

d. The dots inside the circular region but outside the rectangular region as a part of all the dots.

8. Use the Venn diagram pictured with A representing the people in an algebra class and B representing the people in a biology class to express the fraction of elements in the indicated sets as a part of the universal set.



a. $A \cup B$ **b.** $A - \overline{B}$
c. $\varnothing$ **d.** $\overline{A \cup B}$

9. For each of the following, write three fractions equal to the given fractions.

a. $\frac{1}{3}$ **b.** $\frac{4}{5}$
c. $\frac{-3}{7}$ **d.** $\frac{a}{3}$

10. Find the simplest form for each of the following fractions.

a. $\frac{0}{68}$ **b.** $\frac{84^2}{91^2}$ **c.** $\frac{662}{703}$

11. For each of the following, choose the expression in parentheses that equals or describes best the given fraction.

a. $\frac{6+x}{3x}$ $\left(\frac{2+x}{x}, 3, \text{cannot be simplified}\right)$

b. $\frac{2^6+2^5}{2^4+2^7}$ $\left(1, \frac{2}{3}, \text{cannot be simplified}\right)$

c. $\frac{2^{100}+2^{98}}{2^{100}-2^{98}}$ $\left(2^{196}, \frac{5}{3}, \text{too large to simplify}\right)$

12. Find the simplest form for each of the following fractions.

a. $\frac{a^2+ab}{a+b}$ **b.** $\frac{a}{3a+ab}$

13. Determine whether the following pairs are equal.

a. $\frac{6}{16}$ and $\frac{3{,}750}{10{,}000}$ **b.** $\frac{17}{27}$ and $\frac{25}{45}$

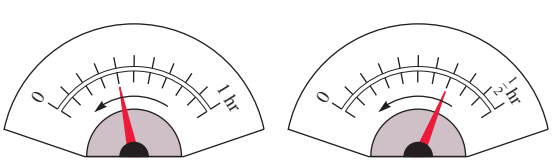
14. Determine whether the following pairs are equal by changing both to the same denominator.

a. $\frac{3}{-12}$ and $\frac{-36}{144}$ **b.** $\frac{-21}{430}$ and $\frac{-51}{215}$

15. Draw an area model to show that $\frac{2}{3} = \frac{6}{9}$.

16. A board is needed that is exactly $\frac{11}{32}$ in. wide to fit a hole. Can a board that is $\frac{3}{8}$ in. be shaved down to fit the hole? If so, how much must be shaved from the board?

17. The following two parking meters are next to each other with the times left as shown. Which meter has more time left on it? How much more?



Meter A Meter B

18. Solve for x in each of the following.

a. $\frac{2}{3} = \frac{x}{18}$ **b.** $\frac{3}{x} = \frac{3x}{x^2}$

19. For each of the following pairs of fractions, replace the comma with the correct symbol ($<$, $=$, $>$) to make a true statement.

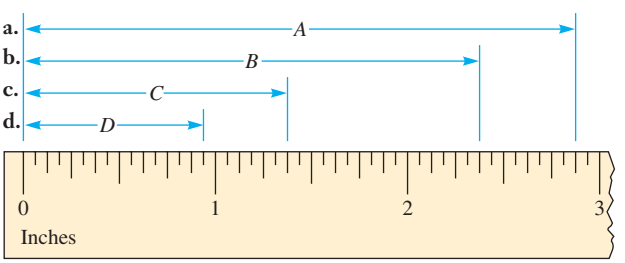
a. $\frac{1}{-7}, \frac{1}{-8}$ **b.** $\frac{2}{5}, \frac{4}{10}$ **c.** $\frac{0}{7}, \frac{0}{17}$

20. For each of the following, find two rational numbers between the given fractions.

a. $\frac{5}{6}$ and $\frac{83}{100}$ **b.** $\frac{-1}{3}$ and $\frac{3}{4}$

21. **a.** 12 oz is what part of a pound?
b. A nickel is what fraction of a dollar?
c. 25 min is what fraction of an hour?
d. 16 hr is what fraction of a 24-hr day?

22. Read each measurement as shown on the following ruler.



23. Fill in missing numbers for x, y, z, and w to create a sequence with each term being less than the preceding term.

a. $\frac{1}{3}, \frac{x}{4}, \frac{y}{5}, \frac{z}{6}$ **b.** $\frac{-3}{x}, \frac{-4}{y}, \frac{-5}{z}, \frac{-6}{w}$

24. Use the Fundamental Law of Fractions (Theorem 6-1) to show that 0 could be written in infinitely many ways.

25. Prove that a negative integer n can be written as a fraction with a positive denominator.

26. A typical metric ruler is marked in millimeters (mm) where $1 \text{ mm} = \frac{1}{1000}$ m (where m is the designation of meter). How many millimeters would 5 meters be?

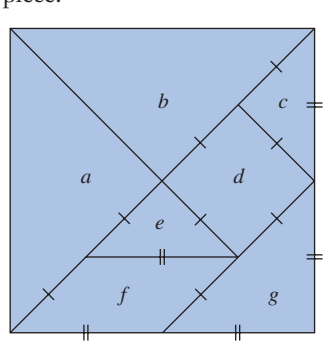
27. Ten light bulbs were in a chandelier. Three-fifth of the bulbs were shining. What fraction of the light bulbs were not shining?

28. At a party, there were 40 guests. One-fifth of the guests were men. What fraction of the guests were women?

29. Answer each of the following.

a. If the area of the entire square is 1 square unit, find the area of each tangram piece.

b. If the area of piece a is 1 square unit, find the area of each tangram piece.



Mathematical Connections 6-1

Reasoning

1. Explain why 25 cents is one-fourth of a dollar, yet 15 minutes is one-fourth of an hour. Why should these one-fourths not be equal?
2. In each of two different fourth-grade classes, $\frac{1}{3}$ of the members are girls. Does each class have the same number of girls? Explain your answer.
3. Consider the set of all fractions equal to $\frac{1}{2}$. If you take any 10 of those fractions, add their numerators to obtain the numerator of a new fraction and add their denominators to obtain the denominator of a new fraction, how does the new fraction relate to $\frac{1}{2}$? Generalize what you found and explain.
4. Draw a Venn diagram showing the relationship among natural numbers, whole numbers, integers, and rational numbers. Use subset relations to explain your Venn diagram.

Open-Ended

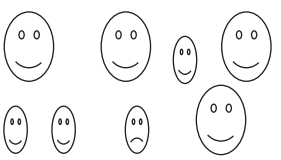
5. Make three statements about yourself or your environment and use fractions in each. Explain why your statements are true. For example, your parents have three children, two of whom live at home; hence $\frac{2}{3}$ of their children live at home.
6. Consider the demographics of your class including gender and ethnicity. Write fractions to describe the demographics of the class.
7. Sketch four different windows having different numbers of window panes in each. Shade $\frac{3}{4}$ of the panes in each window.

Cooperative Learning

8. Assume that the shortest person in your group is 1 unit tall and do the following:
 a. Find rational numbers to approximately represent the heights of other members of the group.
 b. Make a number line and plot the rational number for each person ordered according to height.
9. Assume the tallest person in your group is 1 unit tall and do the following.
 a. Find rational numbers to approximately represent the heights of other members of the group.
 b. Make a number line and plot the rational number for each person ordered according to height.

Connecting Mathematics to the Classroom

10. A student asks if $\frac{0}{6}$ is in its simplest form. How do you respond?
11. A student writes $\frac{15}{53} < \frac{1}{3}$ because $3 \cdot 15 < 53 \cdot 1$. Another student writes $\frac{1\cancel{5}}{\cancel{5}3} = \frac{1}{3}$. Where is the fallacy?
12. A student claims that there are no numbers between $\frac{999}{1000}$ and 1 because they are so close together. What is your response?
13. A student argued that a pizza cut into 12 pieces was more than a pizza cut into 6 pieces. How would you respond?
14. Ann claims that she cannot show $\frac{3}{4}$ of the following faces because some are big and some are small. What do you tell her?



15. How would you respond to each of the following students?
 a. Iris claims that if we have two positive rational numbers, the one with the greater numerator is the greater.
 b. Shirley claims that if we have two positive rational numbers, the one with the greater denominator is the lesser.
16. Steve claims that the shaded circles below cannot represent $\frac{2}{3}$ since there are 10 circles shaded and $\frac{2}{3}$ is less than 1. How do you respond?

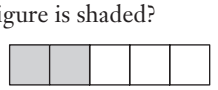


17. Carl says that $\frac{3}{8} > \frac{2}{3}$ because $3 > 2$ and $8 > 3$. How would you help Carl?
18. Mr. Jimenez and Ms. Cortez gave the same test. In Mr. Jimenez's class, 20 out of 25 students passed, and in Ms. Cortez's class, 24 of 30 passed. One of Ms. Cortez's students claimed that the classes did equally well. How could you explain the student's reasoning?

National Assessments

National Assessment of Educational Progress (NAEP) Questions

What fraction of the figure is shaded?



NAEP, Grade 4, 2007

In which of the following are the three fractions arranged from least to greatest?

A. $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$ B. $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$ C. $\frac{1}{2}, \frac{5}{9}, \frac{2}{7}$

D. $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$ E. $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$

NAEP, Grade 8, 2007

Kim, Les, Mario, and Nina each had a string 10 feet long.
Kim cut hers into fifths.
Les cut his into fourths.
Mario cut his into sixths.
Nina cut hers into thirds.
After the cuts were made, who had the longest pieces of string?
A. Kim
B. **Les**
C. Mario
D. **Nina**

NAEP, Grade 4, 2013

Which fraction has a value closest to $\frac{1}{2}$?

A. $\frac{5}{8}$

B. $\frac{1}{6}$

C. $\frac{2}{2}$

D. $\frac{1}{5}$

NAEP, Grade 4, 2011

6-2 Addition, Subtraction, and Estimation with Rational Numbers

6-2 Objectives

Students will be able to understand and explain

- Addition of rational numbers with like and unlike denominators.
- Rational numbers as mixed numbers.
- Subtraction of rational numbers with like and unlike denominators.
- Properties of addition and subtraction of rational numbers.
- Addition properties of equality.
- Estimation with rational numbers.

Addition and subtraction of rational numbers is very much like addition and subtraction of whole numbers and integers. We first demonstrate the addition of two rational numbers with like denominators, $\frac{2}{5} + \frac{1}{5}$, using an area model in Figure 8(a) and a number line model in Figure 8(b).

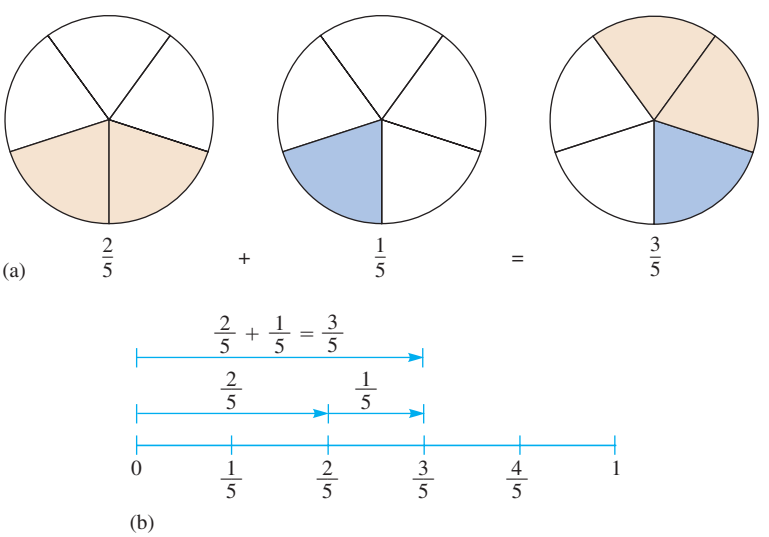


Figure 8

IMAP Video
Watch Felisha use an area model to add rational numbers.

Why does the area model in Figure 8(a) make sense? Suppose that someone gives us $\frac{2}{5}$ of a pie initially and then gives us another $\frac{1}{5}$ of the pie. In Figure 8(a), $\frac{2}{5}$ is represented by 2 pieces when the pie is cut into 5 equal-size pieces, and $\frac{1}{5}$ is represented by 1 piece of the 5 equal-size pieces. So

you have $2 + 1 = 3$ pieces of the 5 equal-size pieces, or $\frac{3}{5}$ of the total (whole) pie. The number line model in Figure 8(b) works the same as the number line model for whole numbers.

Using a bar model, the addition $\frac{3}{5} + \frac{4}{5}$ is depicted in Figure 9.

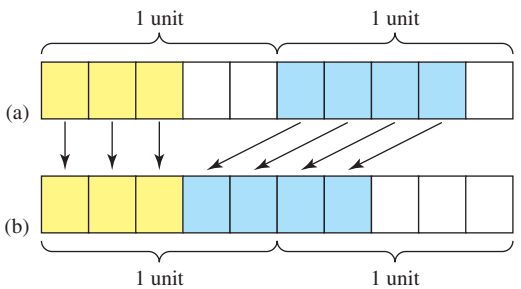


Figure 9

In Figure 9(a), we have $\frac{3}{5}$ and $\frac{4}{5}$ pictured as parts of two equal-sized wholes where each whole consists of five parts. In Figure 9(b), we have $\frac{3}{5} + \frac{4}{5}$ again pictured as parts of two equal-sized wholes, where each whole consists of five parts. We see that there is one whole and two parts of another whole shaded. Additionally, we see that there are seven of the one-fifth parts shaded. Thus, we could say that $\frac{3}{5} + \frac{4}{5} = \frac{7}{5}$. Also $\frac{3}{5} + \frac{4}{5} = 1 + \frac{2}{5}$, which could be written as $1\frac{2}{5}$. The latter representation is a *mixed number*, which is discussed later in the section.

The ideas illustrated in Figures 8 and 9 are summarized in the following definition.

Definition of Addition of Rational Numbers with Like Denominators

If $\frac{a}{b}$ and $\frac{c}{b}$ are rational numbers, then $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$.



In the grade 5 *Common Core Standards*, we find that students should be able to:

Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. (p. 36)



We use the four-step process and *Common Core Standards* practices to develop this concept.

Problem Solving **Adding Rational Numbers Problem**

Determine how to add the rational numbers $\frac{2}{3}$ and $\frac{1}{4}$.

Understanding the Problem We model $\frac{2}{3}$ and $\frac{1}{4}$ as parts of the same-sized whole, as seen in Figure 10, but we need a way to combine the two drawings to find the sum.

IMAP Video
Watch Felisha use an area model to solve a story problem.

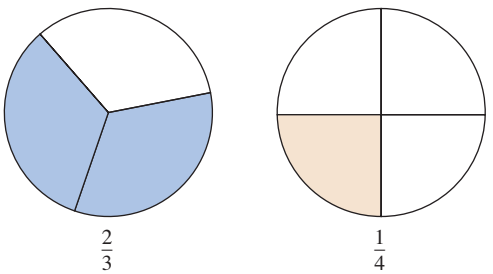


Figure 10

Devising a Plan We use the strategy of *solving a related problem:* adding rational numbers with the same denominators. We can find the sum using reasoning to write each fraction with a common denominator and then complete the computation.

Carrying Out the Plan We know that $\frac{2}{3}$ has infinitely many representations, including $\frac{4}{6}, \frac{6}{9}, \frac{8}{12}$, and so on. Also $\frac{1}{4}$ has infinitely many representations, including $\frac{2}{8}, \frac{3}{12}, \frac{4}{16}$, and so on. We see that $\frac{8}{12}$ and $\frac{3}{12}$ have the same denominator. One is 8 parts of 12 equal parts, while the other is 3 parts of 12 equal parts. Consequently, the sum is $\frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12}$. Figure 11 illustrates the addition.

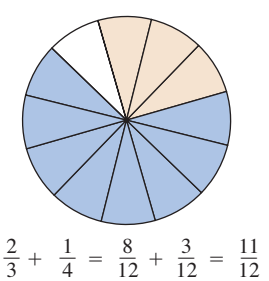


Figure 11

Looking Back To add two rational numbers with unlike denominators, we considered equal rational numbers with like denominators. A common denominator for $\frac{2}{3}$ and $\frac{1}{4}$ is 12. This is also the least common denominator, or the LCM, of 3 and 4. To add two fractions with unequal denominators such as $\frac{5}{12}$ and $\frac{7}{18}$, we could find equal fractions with LCM$(12, 18) = 36$ as the denominator. However, any common denominator will work as well, for example, 72 or even $12 \cdot 18$.

By considering the sum $\frac{2}{3} + \frac{1}{4} = \frac{2 \cdot 4}{3 \cdot 4} + \frac{1 \cdot 3}{4 \cdot 3} = \frac{8}{12} + \frac{3}{12} = \frac{8 + 3}{12} = \frac{11}{12}$, we can generalize to find the sum of two rational numbers with unlike denominators, as in the following.

Alternate Definition of Addition of Rational Numbers with Unlike Denominators

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

The definition of addition of rational numbers with unlike denominators can be applied to like denominators as seen below.

$$\frac{a}{b} + \frac{c}{b} = \frac{ab + cb}{b \cdot b} = \frac{(a + c)b}{b \cdot b} = \frac{a + c}{b}$$

Example 6

Find each of the following sums.

a. $\frac{2}{15} + \frac{4}{21}$ **b.** $\frac{2}{-3} + \frac{1}{5}$ **c.** $\left(\frac{3}{4} + \frac{1}{5}\right) + \frac{1}{6}$ **d.** $\frac{3}{x} + \frac{4}{y}$ **e.** $\frac{2}{a^2b} + \frac{3}{ab^2}$

Solution

a. $\text{LCM}(15, 21) = 3 \cdot 5 \cdot 7$; $\frac{2}{15} + \frac{4}{21} = \frac{2 \cdot 7}{15 \cdot 7} + \frac{4 \cdot 5}{21 \cdot 5} = \frac{14}{105} + \frac{20}{105} = \frac{34}{105}$

b. $\frac{2}{-3} + \frac{1}{5} = \frac{(2)(5) + (-3)(1)}{(-3)(5)} = \frac{10 + -3}{-15} = \frac{7}{-15} = \frac{7(-1)}{-15(-1)} = \frac{-7}{15}$

c. $\frac{3}{4} + \frac{1}{5} = \frac{3 \cdot 5 + 4 \cdot 1}{4 \cdot 5} = \frac{19}{20}$;

$\left(\frac{3}{4} + \frac{1}{5}\right) + \frac{1}{6} = \frac{19}{20} + \frac{1}{6} = \frac{19 \cdot 6 + 20 \cdot 1}{20 \cdot 6} = \frac{134}{120} = \frac{67}{60}$

d. $\frac{3}{x} + \frac{4}{y} = \frac{3y}{xy} + \frac{4x}{xy} = \frac{3y + 4x}{xy}$

e. $\text{LCM}(a^2b, ab^2) = a^2b^2$; $\frac{2}{a^2b} + \frac{3}{ab^2} = \frac{2b}{(a^2b)b} + \frac{3a}{a(ab^2)} = \frac{2b + 3a}{a^2b^2}$

Mixed Numbers

In everyday life, we often use **mixed numbers**, that is, numbers that are made up of an integer and a proper fraction. Figure 12 shows a nail that is $2\frac{3}{4}$ in. long. The mixed number $2\frac{3}{4}$ means $2 + \frac{3}{4}$. Students may infer that $2\frac{3}{4}$ means 2 times $\frac{3}{4}$, since xy means $x \cdot y$, but this is not correct. Also, the number $-4\frac{3}{4}$ means $-\left(4\frac{3}{4}\right) = -4 - \frac{3}{4}$, not $-4 + \frac{3}{4}$.

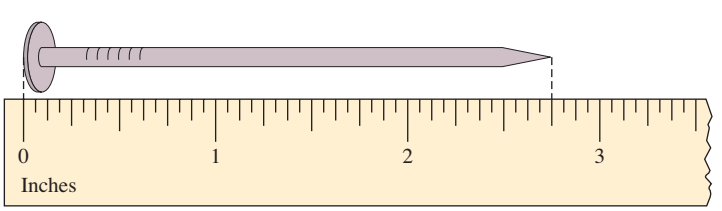


Figure 12

IMAP Video

Watch Rachel correct her own misconception about converting mixed numbers to improper fractions using an area model.

A mixed number is a rational number because it can always be written in the form $\frac{a}{b}$. For example,

$$2\frac{3}{4} = 2 + \frac{3}{4} = \frac{2}{1} + \frac{3}{4} = \frac{2 \cdot 4 + 1 \cdot 3}{1 \cdot 4} = \frac{8 + 3}{4} = \frac{11}{4}.$$

Example 7

Change each of the following mixed numbers to the form $\frac{a}{b}$, where a and b are integers:

a. $4\frac{1}{3}$ **b.** $^{-}3\frac{2}{5}$

Solution

a. $4\frac{1}{3} = 4 + \frac{1}{3} = \frac{4}{1} + \frac{1}{3} = \frac{4 \cdot 3 + 1 \cdot 1}{1 \cdot 3} = \frac{12 + 1}{3} = \frac{13}{3}$

b. $^{-}3\frac{2}{5} = {}^{-}\left(3 + \frac{2}{5}\right) = {}^{-}\left(\frac{3}{1} + \frac{2}{5}\right) = {}^{-}\left(\frac{3 \cdot 5 + 1 \cdot 2}{1 \cdot 5}\right) = {}^{-}\left(\frac{17}{5}\right) = \frac{^{-}17}{5}$

Example 8

Change $\frac{39}{5}$ to a mixed number.

Solution We divide 39 by 5 and use the division algorithm as follows:

$$\frac{39}{5} = \frac{7 \cdot 5 + 4}{5} = \frac{7 \cdot 5}{5} + \frac{4}{5} = 7 + \frac{4}{5} = 7\frac{4}{5}$$

In elementary schools, problems like Example 8 are usually computed using division, as follows:

$$\begin{array}{r} 5 \\ 5\overline{)29} \\ \underline{25} \\ 4 \end{array}$$

Hence, $\frac{29}{5} = 5 + \frac{4}{5} = 5\frac{4}{5}$. The remainder of 4 in the division actually represents $\frac{4}{5}$ of a unit when put in context.



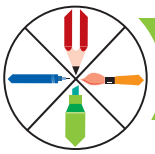
Scientific/fraction calculators can change improper fractions to mixed numbers. For example, if we enter [2] [9] [/] [5] and press [*Ab/c*], then 5 ⌴ 4/5 appears, which means $5\frac{4}{5}$.

We can also use scientific/fraction calculators to add mixed numbers. For example, to add $2\frac{4}{5} + 3\frac{5}{6}$, we enter [2] [Unit] [4] [/] [5] [+] [3] [Unit] [5] [/] [6] [=], and the display reads 5 ⌴ 49/30. We then press [*Ab/c*] to obtain 6 ⌴ 19/30, which means $6\frac{19}{30}$.

Adding Mixed Numbers

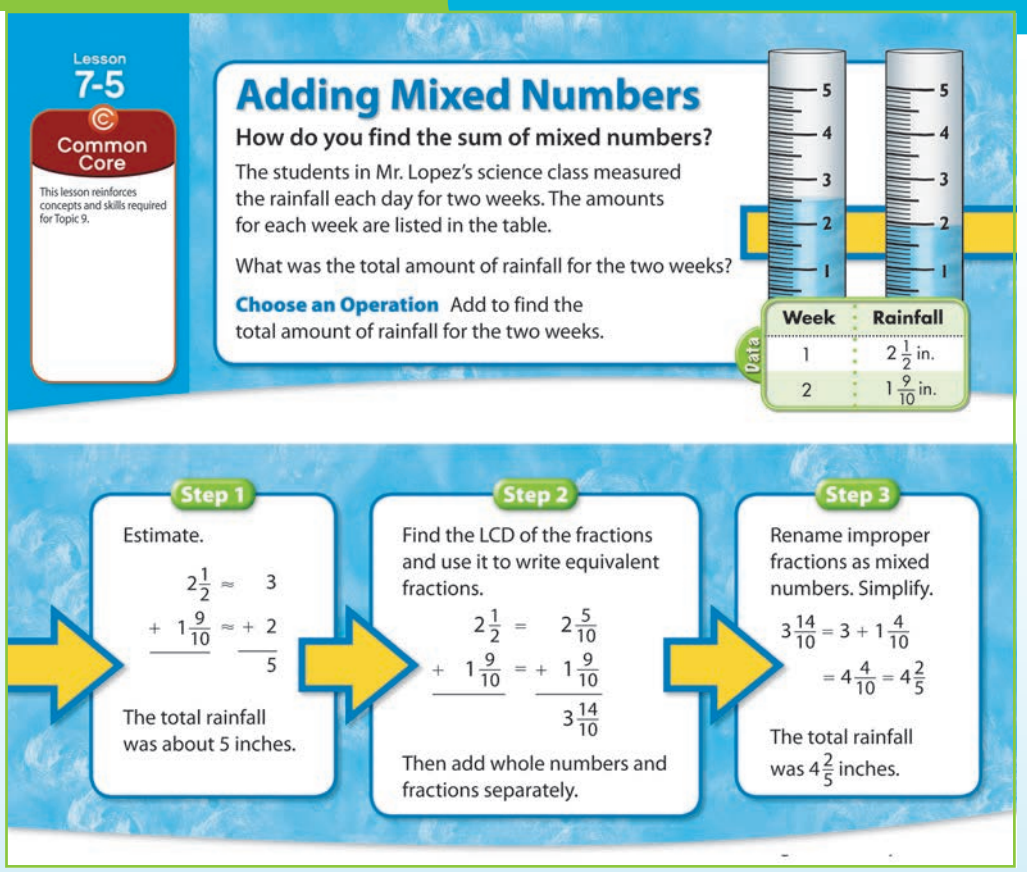


Because mixed numbers are rational numbers, the method of adding rationals can be used to include mixed numbers. The student page shown on page 279 shows a method for computing sums of mixed numbers that uses the commutative and associative properties discussed later in the next section.



School Book Page

Equivalent Fractions



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Properties of Addition for Rational Numbers

Rational numbers have the following properties for addition: *closure*, *commutative*, *associative*, *additive identity*, and *additive inverse*. To emphasize the additive inverse property of rational numbers, we state it explicitly, as follows.

Theorem 6-6: Additive Inverse Property of Rational Numbers

For any rational number $\frac{a}{b}$, there exists a unique rational number $-\frac{a}{b}$, the additive inverse of $\frac{a}{b}$, such that

$$\frac{a}{b} + \left(-\frac{a}{b}\right) = 0 = \left(-\frac{a}{b}\right) + \frac{a}{b}.$$

As mentioned earlier, another form of $-\frac{a}{b}$ can be found by considering the sum $\frac{a}{b} + \frac{^-a}{b}$. Because

$$\frac{a}{b} + \frac{^-a}{b} = \frac{a + {}^-a}{b} = \frac{0}{b} = 0,$$

it follows that $-\frac{a}{b}$ and $\frac{^-a}{b}$ are both additive inverses of $\frac{a}{b}$, so $-\frac{a}{b} = \frac{^-a}{b}$.

Example 9

Find the additive inverses for each of the following:

a. $\frac{3}{5}$ **b.** $\frac{^-5}{11}$ **c.** $4\frac{1}{2}$

Solution

a. $-\frac{3}{5}$ or $\frac{^-3}{5}$ **b.** $-\left(\frac{^-5}{11}\right) = \frac{^-(^-5)}{11} = \frac{5}{11}$ **c.** $^-4\frac{1}{2}$, or $\frac{^-9}{2}$

Properties of the additive inverse for rational numbers are analogous to those of the additive inverse for integers, as shown in Table 2.

Table 2

Integers	Rational Numbers
1. $^-(^-a) = a$	1. $^-\left(\frac{^-a}{b}\right) = \frac{a}{b}$
2. $^-(a + b) = {}^-a + {}^-b$	2. $^-\left(\frac{a}{b} + \frac{c}{d}\right) = \frac{^-a}{b} + \frac{^-c}{d}$

The set of rational numbers has the addition property of equality, which says that the same number can be added to both sides of an equation.

Theorem 6-7: Addition Property of Equality of Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers such that $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f}$ is any rational number, then

$$\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}.$$

IMAP Video

Watch Francisco use the take-away model with rational numbers.

Subtraction of Rational Numbers

In elementary school, subtraction of rational numbers is usually introduced by using a take-away model. If we have $\frac{6}{7}$ of a pizza and $\frac{2}{7}$ of the original pizza is taken away, $\frac{4}{7}$ of the pizza remains; that is, $\frac{6}{7} - \frac{2}{7} = \frac{6-2}{7} = \frac{4}{7}$. In general, subtraction of rational numbers with like denominators is determined as follows:

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

As with integers, a number line can be used to model subtraction of nonnegative rational numbers. If a line is marked off in units of length $\frac{1}{b}$ and $a \geq c$, then $\frac{a}{b} - \frac{c}{b}$ is equal to $(a - c)$ units of length $\frac{1}{b}$, which implies that $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$.

When the denominators are not the same, we can perform the subtraction by finding a common denominator. For example,

$$\frac{3}{4} - \frac{2}{3} = \frac{3 \cdot 3}{4 \cdot 3} - \frac{2 \cdot 4}{3 \cdot 4} = \frac{9}{12} - \frac{8}{12} = \frac{9 - 8}{12} = \frac{1}{12}.$$

Subtraction of rational numbers, like subtraction of integers, can be defined in terms of addition as follows.

Definition of Subtraction of Rational Numbers in Terms of Addition

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d}$ is the unique rational number $\frac{e}{f}$ such that $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$.

As with integers, we can see that subtraction of rational numbers can be performed by adding the additive inverses as stated in the following theorem.

Theorem 6-8

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{\bar{}c}{d}$.

Now, using Theorem 6-8, we obtain the following.

$$\begin{aligned} \frac{a}{b} - \frac{c}{d} &= \frac{a}{b} + \frac{\bar{}c}{d} \\ &= \frac{ad + b(\bar{}c)}{bd} \\ &= \frac{ad + \bar{}(bc)}{bd} \\ &= \frac{ad - bc}{bd} \end{aligned}$$

We proved the following theorem, which is sometimes given as a definition of subtraction.

Theorem 6-9

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.

Example 10

Find each difference.

a. $\frac{5}{8} - \frac{1}{4}$ **b.** $5\frac{1}{3} - 2\frac{3}{4}$

Solution

a. One approach is to find the LCM for the denominators. LCM$(8, 4) = 8$.

$$\frac{5}{8} - \frac{1}{4} = \frac{5}{8} - \frac{2}{8} = \frac{3}{8}$$

An alternative approach follows.

$$\frac{5}{8} - \frac{1}{4} = \frac{5 \cdot 4 - 8 \cdot 1}{8 \cdot 4} = \frac{20 - 8}{32} = \frac{12}{32} = \frac{3}{8}$$

b. Two methods of solution are given.

$$\begin{array}{rrrr} 5\frac{1}{3} = & 5\frac{4}{12} = & 4 + 1\frac{4}{12} = & 4\frac{16}{12} \\ \underline{-2\frac{3}{4}} = & \underline{-2\frac{9}{12}} = & \underline{-2\frac{9}{12}} \quad = & \underline{-2\frac{9}{12}} \\ & & & 2\frac{7}{12} \end{array}$$

$$\begin{aligned} 5\frac{1}{3} - 2\frac{3}{4} &= \frac{16}{3} - \frac{11}{4} \\ &= \frac{16 \cdot 4 - 3 \cdot 11}{3 \cdot 4} \\ &= \frac{64 - 33}{12} \\ &= \frac{31}{12} = 2\frac{7}{12} \end{aligned}$$

The following examples show the use of fractions in algebra.

Example 11

Add or subtract, writing your answer in simplest form.

a. $\frac{x}{2} + \frac{x}{3}$ **b.** $\frac{2 - x}{6 - 3x} + \frac{4 - 2x}{3x - 6}$

c. $\frac{2}{a + b} - \frac{2}{a - b}$ **d.** $\frac{1}{x} - \frac{1}{2x^2}$

Solution

a.
$$\begin{aligned} \frac{x}{2} + \frac{x}{3} &= \frac{3x}{3 \cdot 2} + \frac{2x}{2 \cdot 3} \\ &= \frac{3x + 2x}{6} = \frac{5x}{6} \end{aligned}$$

b. We first write each fraction in simplest form.

$$\frac{2 - x}{6 - 3x} = \frac{2 - x}{3(2 - x)} = \frac{1(2 - x)}{3(2 - x)} = \frac{1}{3}$$

$$\frac{4 - 2x}{3x - 6} = \frac{^-2(x - 2)}{3(x - 2)} = \frac{^-2}{3}$$

Thus, the sum is $\frac{1}{3} + \frac{^-2}{3} = \frac{^-1}{3}$.

c. We use Theorem 6-9.

$$\frac{2}{a+b} - \frac{2}{a-b} = \frac{2(a-b) - 2(a+b)}{(a+b)(a-b)}$$
$$= \frac{2a - 2b - 2a - 2b}{(a+b)(a-b)}$$
$$= \frac{^{-}4b}{(a+b)(a-b)} = \frac{^{-}4b}{a^2 - b^2}$$

d. $\frac{1}{x} - \frac{1}{2x^2} = \frac{2x \cdot 1}{2x \cdot x} - \frac{1}{2x^2}$

$$= \frac{2x}{2x^2} - \frac{1}{2x^2}$$
$$= \frac{2x - 1}{2x^2}$$

Estimation with Rational Numbers

Estimation helps us make practical decisions in our everyday lives. For example, suppose we need to double a recipe that calls for $\frac{3}{4}$ of a cup of flour. Will we need more or less than a cup of flour? Many of the estimation and mental math techniques that we learned to use with whole numbers also work with rational numbers.

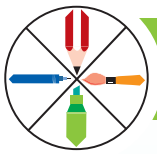
The grade 5 *Common Core Standards* calls for students to "use benchmark fraction and number sense of fractions to estimate mentally and assess the reasonableness of answers." (p. 36) Estimation plays an important role in judging the reasonableness of computations.

NOW TRY THIS 3

A student added $\frac{3}{4} + \frac{1}{2}$ and obtained $\frac{4}{6}$. How would you use estimation to show that this answer could not be correct?

Sometimes to obtain an estimate it is desirable to round fractions to a ***convenient*** or ***benchmark*** fraction, such as $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{3}, \frac{3}{4}$, or 1. For example, if a student had 59 correct answers out of 80 questions, the student answered $\frac{59}{80}$ of the questions correctly, which is approximately $\frac{60}{80}$, or $\frac{3}{4}$. We know $\frac{60}{80}$ is greater than $\frac{59}{80}$. The estimate $\frac{3}{4}$ for $\frac{59}{80}$ is a high estimate. In a similar way, we can estimate $\frac{31}{90}$ by $\frac{30}{90}$, or $\frac{1}{3}$. In this case, the estimate of $\frac{1}{3}$ is a low estimate of $\frac{31}{90}$.

An example of benchmark estimation is seen on the student page shown on page 284.



School Book Page

Another Example How can you use benchmark fractions such as $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and $\frac{3}{4}$ to estimate?

Estimate $\frac{5}{8} - \frac{3}{16}$.

$\frac{5}{8}$ is close to $\frac{6}{8}$, and $\frac{6}{8} = \frac{3}{4}$.

$\frac{3}{16}$ is close to $\frac{4}{16}$, and $\frac{4}{16} = \frac{1}{4}$.

So, $\frac{5}{8} - \frac{3}{16}$ is close to $\frac{3}{4} - \frac{1}{4}$.

$\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$ or $\frac{1}{2}$

So, $\frac{5}{8} - \frac{3}{16} \approx \frac{1}{2}$.

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Example 12

A sixth-grade class is collecting cans to take to the recycling center. Becky's group brought the following amounts (in pounds). About how many pounds does her group have all together?

$$1\frac{1}{8}, 3\frac{4}{10}, 5\frac{7}{8}, \frac{6}{10}$$

Solution We can estimate the amount by using front-end estimation with the adjustment made by using 0 which is close to $\frac{1}{8}, \frac{1}{2}$ which is close to $\frac{4}{10}$ and $\frac{6}{10}$, and 1 which is close to $\frac{7}{8}$ as benchmark fractions. The front-end estimate is $1 + 3 + 5 = 9$. The adjustment is $0 + \frac{1}{2} + 1 + \frac{1}{2}$, or 2. An adjusted estimate would be $9 + 2 = 11$ lb.

Example 13

Estimate each of the following additions.

a. $\frac{27}{13} + \frac{10}{9}$ **b.** $3\frac{9}{10} + 2\frac{7}{8} + \frac{11}{12}$ **c.** $3\frac{7}{8} + 11\frac{1}{2} + 2\frac{2}{5} + 5\frac{1}{16}$

Solution

a. Because $\frac{27}{13}$ is slightly more than 2 and $\frac{10}{9}$ is slightly more than 1, an estimate might be 3. We know the estimate is low.

b. We first add the whole-number parts to obtain $3 + 2 = 5$. Because each of the fractions, $\frac{9}{10}, \frac{7}{8}$, and $\frac{11}{12}$, is close to but less than 1, their sum is close to but less than 3. The approximate answer is $5 + 3 = 8$. The estimate is high.

c. Using *grouping to nice numbers*, we group $\left(3\frac{7}{8} + 5\frac{1}{16}\right)$ and $\left(11\frac{1}{2} + 2\frac{2}{5}\right)$ to obtain approximately $9 + 14 = 23$. The estimate is high. (Why?)

Assessment 6-2A

1. Perform the following additions or subtractions.

a. $\frac{1}{2} + \frac{2}{3}$

b. $\frac{4}{12} - \frac{2}{3}$

c. $\frac{5}{x} + \frac{^-3}{y}$

d. $\frac{^-3}{2x^2y} + \frac{5}{2xy^2} + \frac{7}{x^2}$

e. $\frac{5}{6} + 2\frac{1}{8}$

f. $^-4\frac{1}{2} - 3\frac{1}{6}$

g. $7\frac{1}{4} + 3\frac{5}{12} - 2\frac{1}{3}$

2. Change each of the following fractions to a mixed number.

a. $\frac{56}{3}$ **b.** $-\frac{293}{100}$

3. Change each of the following mixed numbers to a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

a. $6\frac{3}{4}$ **b.** $^-3\frac{5}{8}$

4. Approximate each of the following situations with a benchmark fraction. Tell whether your estimate is high or low.

a. Giorgio had 15 base hits out of 46 times at bat.

b. Ruth made 7 goals out of 41 shots.

c. Laura answered 62 problems correctly out of 80.

d. Jonathan made 9 baskets out of 19.

5. Use the information in the table to answer each of the following questions.

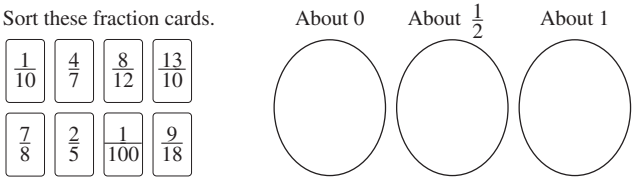
Team	Games Played	Games Won
Ducks	22	10
Beavers	19	10
Tigers	28	9
Bears	23	8
Lions	27	7
Wildcats	25	6
Badgers	21	5

a. Which team won more than $\frac{1}{2}$ of its games and was closest to winning $\frac{1}{2}$ of its games?

b. Which team won less than $\frac{1}{2}$ of its games and was closest to winning $\frac{1}{2}$ of its games?

c. Which team won more than $\frac{1}{3}$ of its games and was closest to winning $\frac{1}{3}$ of its games?

6. Sort the following fraction cards into the ovals by estimating in which oval the fraction belongs.



7. Approximate each of the following fractions by 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, or 1. Tell whether your estimate is high or low.

a. $\frac{19}{39}$ **b.** $\frac{3}{197}$

c. $\frac{150}{201}$ **d.** $\frac{8}{9}$

8. Without actually finding the exact answer, state which of the numbers given in parentheses is the best approximation for the given sum.

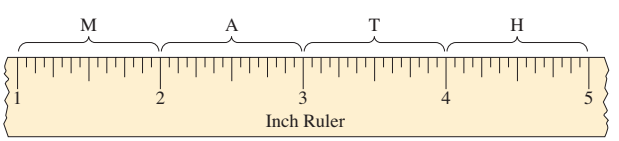
a. $\frac{6}{13} + \frac{7}{15} + \frac{11}{23} + \frac{17}{35}$ $\left(1, 2, 3, 3\frac{1}{2}\right)$

b. $\frac{30}{41} + \frac{1}{1000} + \frac{3}{2000}$ $\left(\frac{3}{8}, \frac{3}{4}, 1, 2\right)$

9. Compute each of the following mentally.

a. $1 - \frac{3}{4}$ **b.** $3\frac{3}{8} + 2\frac{1}{4} - 5\frac{5}{8}$

10. The following ruler has regions marked M, A, T, H.



Use estimation to determine into which region on the ruler each of the following measurements falls. For example, $\frac{12}{5}$ in. falls into region A.

a. $\frac{20}{8}$ in. **b.** $\frac{36}{8}$ in.

c. $\frac{60}{16}$ in. **d.** $\frac{18}{4}$ in.

11. Use *clustering* to estimate the following sum.

$$3\frac{1}{3} + 3\frac{1}{5} + 2\frac{7}{8} + 2\frac{7}{9}$$

12. A class consists of $\frac{2}{5}$ freshmen, $\frac{1}{4}$ sophomores, and $\frac{1}{10}$ juniors; the rest are seniors. What fraction of the class is seniors?

13. A clerk sold three pieces of one type of ribbon to different customers. One piece was $\frac{1}{3}$ yd long, another was $2\frac{3}{4}$ yd long, and the third was $3\frac{1}{2}$ yd long. What was the total length of that type of ribbon sold?

14. Martine bought $8\frac{3}{4}$ yd of fabric. She wants to make a skirt using $1\frac{7}{8}$ yd, pants using $2\frac{3}{8}$ yd, and a vest using $1\frac{2}{3}$ yd. How much fabric will be left over?

15. Give an example illustrating each of the following properties of rational number addition.

a. Closure **b.** Commutative **c.** Associative

16. Insert five fractions between the numbers 1 and 2 so that the seven numbers (including 1 and 2) constitute part of an arithmetic sequence.

17. a. Check that each of the following statements is true.

i. $\frac{1}{3} = \frac{1}{4} + \frac{1}{3 \cdot 4}$

ii. $\frac{1}{4} = \frac{1}{5} + \frac{1}{4 \cdot 5}$

iii. $\frac{1}{5} = \frac{1}{6} + \frac{1}{5 \cdot 6}$

b. Based on the examples in (a), write $\frac{1}{n}$ as a sum of two unit fractions; that is, as a sum of fractions with numerator 1.

18. Solve for x.

a. $x + 2\frac{1}{2} = 3\frac{1}{3}$

b. $x - 2\frac{2}{3} = \frac{5}{6}$

19. Al runs $\frac{5}{8}$ mi in 10 min. Bill runs $\frac{7}{8}$ mi in 10 min. If both runners continue to run at the same rate, how much farther can Bill run than Al in 20 min?

20. One recipe calls for $1\frac{3}{4}$ cups of milk and a second recipe calls for $1\frac{1}{2}$ cups of milk. If you only have 3 cups of milk, can you make both recipes? Why?

21. The table below shows census data from the state of Pennsylvania for 2011.

Population	2011
Total 8-yr. old	150,155
Boys	76,432
Girls	73,723
Total 9-yr. old	149,904
Boys	76,503
Girls	73,401
Total 10-yr. old	154,144
Boys	78,672
Girls	75,472

Data from 2011 Census taken from http://www.census.gov/schools/facts/pennsylvania.html

a. About what part of the total population is 8- to 10-year-old boys?

b. About what part of the total population is 8- to 10-year-old girls?

c. If the population of the entire state of Pennsylvania in 2011 was 12,742,886, about what part of the population was boys of ages 8- to 10-year-olds?

22. According to the US Census Bureau, in October 2012, there were about 633,000 3- and 4-year-olds enrolled in schools. Additionally there were about 1,186,000 16- and 17-year-olds enrolled in schools. In the respective ages there were about 1,193,000 and 1,259,000 total. Which age group had the greater fraction in school?

23. There are 360° in a circle graph. If 40° of the graph represents rent and 5° of the graph represents savings, what fractional portion of the whole graph is represented by rent and savings?

24. The energy sources for the US in 2012 included $\frac{37}{100}$ coal, $\frac{3}{10}$ natural gas, $\frac{19}{100}$ nuclear, and $\frac{7}{100}$ hydropower. How much is unaccounted for in this list?

Assessment 6-2B

1. Perform the following additions or subtractions.

a. $\frac{^-1}{2} + \frac{2}{3}$

b. $\frac{5}{12} - \frac{2}{3}$

c. $\frac{5}{4x} + \frac{^-3}{2y}$

d. $\frac{^-3}{2x^2y^2} + \frac{5}{2xy^2} + \frac{7}{x^2y}$

e. $\frac{5}{6} - 2\frac{1}{8}$

f. $^-4\frac{1}{2} + 3\frac{1}{6}$

g. $5\frac{1}{3} + 5\frac{5}{6} - 3\frac{1}{9}$

2. Change each of the following fractions to a mixed number.

a. $\frac{14}{5}$ **b.** $-\frac{47}{8}$

3. Change each of the following mixed numbers to a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

a. $7\frac{1}{2}$ **b.** $^-4\frac{2}{3}$

4. Place the numbers 2, 5, 6, and 8 in the following boxes to make the equation true.

$$\frac{\square}{\square} + \frac{\square}{\square} = \frac{23}{24}$$

5. Use the information in the table to answer each of the following questions.

Team	Games Played	Games Won
Ducks	22	10
Beavers	19	10
Tigers	28	9
Bears	23	8
Lions	27	7
Wildcats	25	6
Badgers	21	5

a. Which team won less than $\frac{1}{3}$ of its games and was closest to winning $\frac{1}{3}$ of its games?

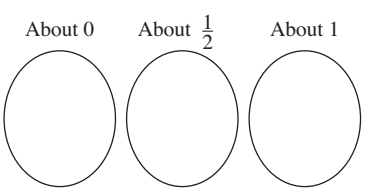
b. Which team won more than $\frac{1}{4}$ of its games and was closest to winning of $\frac{1}{4}$ of its games?

c. Which teams won less than $\frac{1}{4}$ of their games?

6. Sort the following fraction cards into the ovals by estimating in which oval the fraction belongs.

Sort these fraction cards.

$\frac{14}{16}$ $\frac{1}{100}$ $\frac{36}{70}$ $\frac{19}{36}$ $\frac{1}{30}$ $\frac{7}{800}$



7. Approximate each of the following fractions by 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, or 1. Tell whether your estimate is high or low.

a. $\frac{113}{100}$ **b.** $\frac{3}{1978}$

c. $\frac{150}{198}$ **d.** $\frac{8}{9}$

8. Without actually finding the exact answer, state which of the numbers given in parentheses is the best approximation for the given sum.

a. $\frac{2}{13} + \frac{7}{15} + \frac{12}{23} + \frac{33}{35}$ $\left(1, 2, 3, 3\frac{1}{2}\right)$

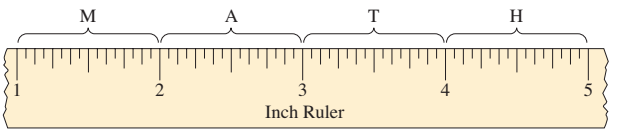
b. $\frac{30}{41} + \frac{220}{1000} + \frac{5}{2000}$ $\left(\frac{3}{8}, \frac{3}{4}, 1, 2\right)$

9. Compute each of the following mentally.

a. $6 - \frac{7}{8}$

b. $2\frac{3}{5} + 4\frac{1}{10} + 3\frac{3}{10}$

10. The following ruler has regions marked M, A, T, H.



Use mental mathematics and estimation to determine into which region on the ruler each of the following measurements falls. For example, $\frac{12}{5}$ in. falls into region A.

a. $\frac{9}{8}$ in. **b.** $\frac{18}{8}$ in.

c. $\frac{50}{16}$ in. **d.** $\frac{17}{4}$ in.

11. A class consists of $\frac{1}{4}$ freshmen, $\frac{1}{5}$ sophomores, and $\frac{1}{10}$ juniors; the rest are seniors. What fraction of the class is seniors?

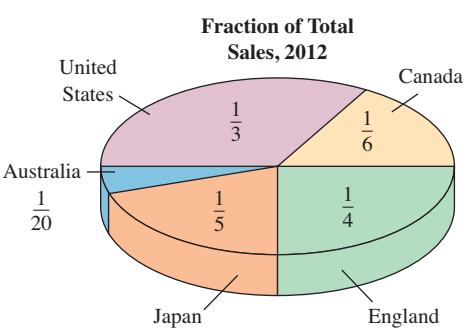
12. The Naturals Company sells its products in many countries. The following two circle graphs show the fractions of the company's earnings for 2012 and 2014. Based on this information, answer the following questions.

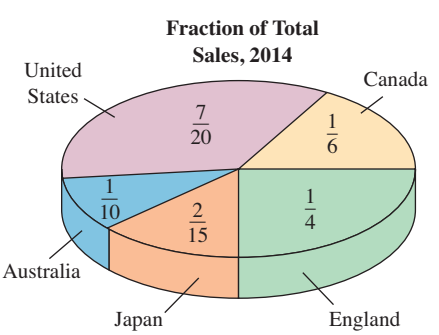
a. In 2012, how much greater was the fraction of sales for Japan than for Canada?

b. In 2014, how much less was the fraction of sales for England than for the United States?

c. How much greater was the fraction of total sales for the United States in 2014 than in 2012?

d. Is it true that the amount of sales in dollars in Australia was less in 2012 than in 2014? Why?





13. A recipe requires $3\frac{1}{2}$ cups of milk. Ran put in $1\frac{3}{4}$ cups in a bowl. How much more milk does he need?

14. A $15\frac{3}{4}$ in. board is cut in a single cut from a $38\frac{1}{4}$ in. board. The saw cut takes $\frac{3}{8}$ in. How much of the $38\frac{1}{4}$ in. board is left after cutting?

15. Students from Rattlesnake School formed four teams to collect cans for recycling during the months of April and May. A record of their efforts follows.

Number of Pounds Collected

	Team 1	Team 2	Team 3	Team 4
April	$28\frac{3}{4}$	$32\frac{7}{8}$	$28\frac{1}{2}$	$35\frac{3}{16}$
May	$33\frac{1}{3}$	$28\frac{5}{12}$	$25\frac{3}{4}$	$41\frac{1}{2}$

a. Which team collected the most for the 2-month period? How much did they collect?

b. What was the difference between the total pounds collected in April and the total pounds collected in May?

16. Insert four fractions between the numbers 1 and 3 so that the six numbers (including 1 and 3) constitute part of an arithmetic sequence.

17. Solve for x in each of the following.

a. $x - \frac{5}{6} = \frac{2}{3}$

b. $x - \frac{7}{2^3 \cdot 3^2} = \frac{5}{2^2 \cdot 3^2}$

18. Joe has $\frac{3}{4}$ cup of paint in a container. He uses $\frac{1}{3}$ cup on a project and then adds another $\frac{1}{2}$ cup. How much paint does he have now?

19. **a.** Find $\frac{1}{2} + \frac{1}{4}$.

b. Find $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$.

c. Find $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$.

d. If you continue in this pattern with powers of 2 in the denominator, will the sum ever become greater than 1? Why?

20. The table below shows census data from the state of Pennsylvania for 2011.

Population	2011
Total 8-yr. old	150,155
Boys	76,432
Girls	73,723
Total 9-yr. old	149,904
Boys	76,503
Girls	73,401
Total 10-yr. old	154,144
Boys	78,672
Girls	75,472

Data from 2011 Census taken from http://www.census.gov/schools/facts/pennsylvania.html

a. About what fraction more of girls is there in the 8-year-olds than in the 10-year-olds?

b. About what part of the total population of 8- to 9-year-olds are girls?

c. If the population of the entire state of Pennsylvania in 2011 was 12,742,886, about what part of the population was girls of ages 8–10?

21. According to the US Census Bureau in October 2012, there were about 633,000 3- and 4-year olds enrolled in schools. Additionally there were about 1,186,000 16- and 17-year-olds enrolled in schools. In the respective ages there were about 1,193,000 and 1,259,000 total. Which of these age groups had the lesser fraction in school?

22. There are 360° in a circle graph. If 50° of the graph represents rent and 7° of the graph represents savings, what fractional portion of the whole graph is not represented by rent and savings?

23. The energy sources for the United States in 2012 included $\frac{37}{100}$ coal, $\frac{3}{10}$ natural gas, $\frac{19}{100}$ nuclear, and $\frac{7}{100}$ hydropower. What fraction more coal than natural gas is used?

Mathematical Connections 6-2

Reasoning

1. Suppose a large pizza is divided into 3 equal-size pieces and a small pizza is divided into 4 equal-size pieces and you get 1 piece from each pizza. Does $\frac{1}{3} + \frac{1}{4}$ represent the amount that you received? Explain why or why not.

2. **a.** When we add two fractions with unlike denominators and convert them to fractions with the same denominator, must we use the least common denominator? What are the advantages of using the least common denominator?

b. When the least common denominator is used in adding or subtracting fractions, is the result always a fraction in simplest form?

3. Explain why we can do the following to convert $5\frac{3}{4}$ to a mixed number.

$$\frac{5 \cdot 4 + 3}{4} = \frac{23}{4}$$

4. Kara spent $\frac{1}{2}$ of her allowance on Saturday and $\frac{1}{3}$ of what she had left on Sunday. Can this situation be modeled as $\frac{1}{2} - \frac{1}{3}$? Explain why or why not.

5. Compute $3\frac{3}{4} + 5\frac{1}{3}$ in two different ways and leave your answer as a mixed number. Tell which way you prefer and why.

6. Explain whether each of the following properties holds for subtraction of rational numbers.

a. Closure **b.** Commutative
c. Associative **d.** Identity
e. Inverse

Open-Ended

7. Write a story problem for $\frac{2}{3} - \frac{1}{4}$.

8. a. Write two fractions whose sum is 1. If one of the fractions is $\frac{a}{b}$, what is the other?

b. Write three fractions whose sum is 1.

c. Write two fractions whose difference is very close to 1 but not exactly 1.

9. a. With the exception of $\frac{2}{3}$, the Egyptians used only unit fractions (fractions that have numerators of 1). Every unit fraction can be expressed as the sum of two unit fractions in more than one way, for example, $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$ and $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$. Find at least two different unit fraction representations for each of the following.

i. $\frac{1}{3}$ **ii.** $\frac{1}{7}$

b. Show that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

c. Rewrite the equation in part (b) as a sum and then use the sum to answer part (a).

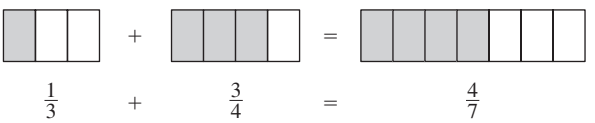
d. Write $\frac{1}{17}$ as a sum of two different unit fractions.

Cooperative Learning

10. Interview 10 people and ask them if and when they add and subtract fractions in their lives. Combine their responses with those of the rest of the class to get a view of how "ordinary" people use computation of rational numbers in their daily lives.

Connecting Mathematics to the Classroom

11. Kendra showed that $\frac{1}{3} + \frac{3}{4} = \frac{4}{7}$ by using the following figure. How would you help her?



12. To show $2\frac{3}{4} = \frac{11}{4}$, the teacher drew the following picture.

Ken said this shows a picture of $\frac{11}{12}$, not $\frac{11}{4}$. What is Ken thinking and how should the teacher respond?



13. Jill claims that for positive fractions, $\frac{a}{b} + \frac{a}{c} = \frac{a}{b+c}$ because the fractions have a common numerator. How do you respond?

14. Explain the error pattern on Jon's test.

a. $\frac{13}{35} = \frac{1}{5}$, $\frac{27}{73} = \frac{2}{3}$, $\frac{16}{64} = \frac{1}{4}$

b. $\frac{4}{5} + \frac{2}{3} = \frac{6}{8}$, $\frac{2}{5} + \frac{3}{4} = \frac{5}{9}$, $\frac{7}{8} + \frac{1}{3} = \frac{8}{11}$

c. $8\frac{3}{8} - 6\frac{1}{4} = 2\frac{2}{4}$, $5\frac{3}{8} - 2\frac{2}{3} = 3\frac{1}{5}$, $2\frac{2}{7} - 1\frac{1}{3} = 1\frac{1}{4}$

Review Problems

15. Simplify each rational number if possible.

a. $\frac{14}{21}$ **b.** $\frac{117}{153}$
c. $\frac{5^2}{7^2}$ **d.** $\frac{a^2 + a}{1 + a}$
e. $\frac{a^2 + 1}{a + 1}$ **f.** $\frac{a^2 - b^2}{a - b}$

16. Determine whether the fractions in each of the following pairs are equal.

a. $\frac{a^2}{b}$ and $\frac{a^2 b}{b^2}$

b. $\frac{377}{400}$ and $\frac{378}{401}$

c. $\frac{0}{10}$ and $\frac{0}{-10}$

d. $\frac{a}{b}$ and $\frac{a+1}{b+1}$, where $a \neq b$

17. There are 206 bones in the body. Can the fraction $\frac{27}{103}$ represent the number of bones in both hands as a part of the total number of bones in the body? Explain your answer.

18. Explain why there are infinitely many fractions equivalent to $\frac{3}{5}$.

19. Mary ate $\frac{3}{5}$ of the cookies and left the rest for Suzanne. What fraction of the cookies are left for Suzanne?

20. On a number line, explain why $\frac{-1}{100}$ is greater than $\frac{-1}{10}$.

National Assessments

National Assessment of Educational Progress (NAEP) Questions

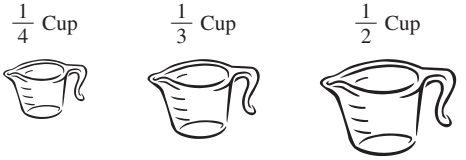
$$\frac{1}{20}, \frac{4}{20}, \frac{7}{20}, \frac{10}{20}, \frac{13}{20}, \ldots$$

If the pattern shown continues, what is the first fraction in the pattern that will be greater than 1?

A. $\frac{20}{20}$ B. $\frac{21}{20}$

C. $\frac{22}{20}$ D. $\frac{25}{20}$

NAEP, Grade 4, 2013



A recipe requires $1\frac{1}{3}$ cups of sugar. Which of the following ways describes how the measuring cups shown can be used to measure $1\frac{1}{3}$ cups of sugar accurately?

A. Use the $\frac{1}{2}$ cup three times.

B. Use the $\frac{1}{4}$ cup three times.

C. Use the $\frac{1}{2}$ cup twice and the $\frac{1}{3}$ cup once.

D. Use the $\frac{1}{3}$ cup twice and the $\frac{1}{2}$ cup once.

E. Use the $\frac{1}{4}$ cup once, the $\frac{1}{3}$ cup once, and the $\frac{1}{2}$ cup once.

NAEP, Grade 8, 2011

6-3 Multiplication, Division, and Estimation with Rational Numbers

6-3 Objectives

Students will be able to understand and explain

- Multiplication and division of rational numbers.
- Properties of multiplication and division of rational numbers.
- Estimation of multiplication and division with rational numbers.
- Extension of exponents to include negative integers.

Multiplication of Rational Numbers

To motivate the definition of multiplication of rational numbers, we use the interpretation of multiplication as repeated addition. Using repeated addition, we interpret $3\left(\frac{3}{4}\right)$ as follows:

$$3\left(\frac{3}{4}\right) = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{9}{4} = 2\frac{1}{4}$$

The area model in Figure 13 is another way to calculate this product.

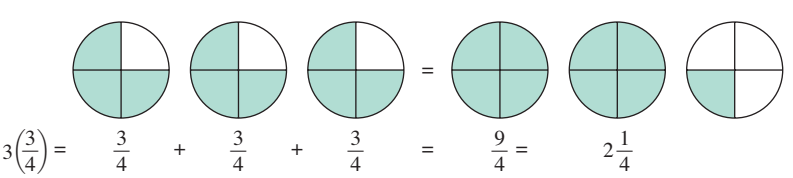


Figure 13

We next consider $\left(\frac{3}{4}\right)3$. How should this product be interpreted? If the commutative property of multiplication of rational numbers is to hold, then $\left(\frac{3}{4}\right)3 = 3\left(\frac{3}{4}\right) = \frac{9}{4}$.

IMAP Video

Watch Felisha find $\frac{3}{4}$ of 2 using the area model.

Next, we consider another interpretation of multiplication. What is $\frac{3}{4}$ of 3? In Figure 14(a) consider the 1-unit bar broken into fourths. Thus, there are four equal parts of the 1 unit, each of length $\frac{1}{4}$ unit. In Figure 14(b) suppose the length of the bar was 3 units. We want the same type of action to occur so it is divided into three-fourths using the same strategy. But analogously, each part would be 3 times the length of the bars in Figure 13(a). If this strategy is used then $\frac{3}{4}$ of 3 is seen as the shaded portion and $\frac{3}{4} \cdot 3 = \frac{9}{4}$ of 1 or simply $\frac{9}{4}$ is $\frac{3}{4}$ of 3.

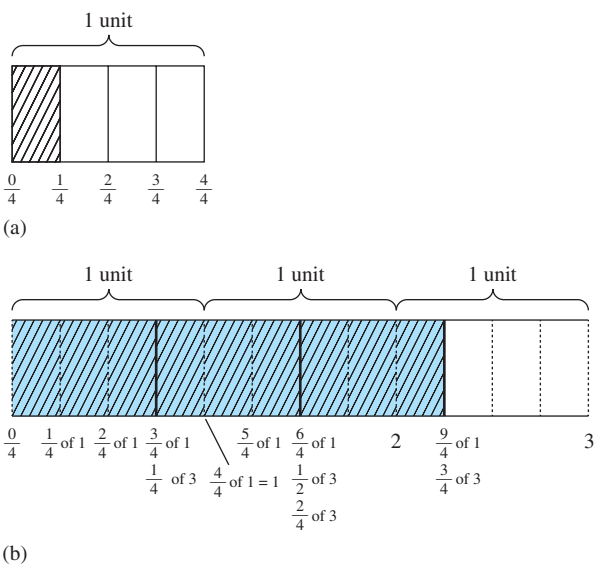


Figure 14

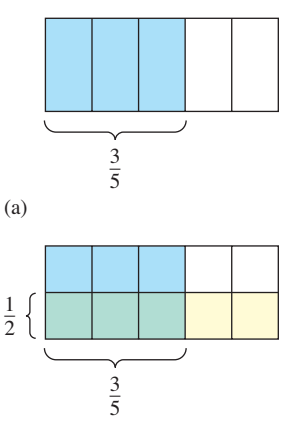


Figure 15

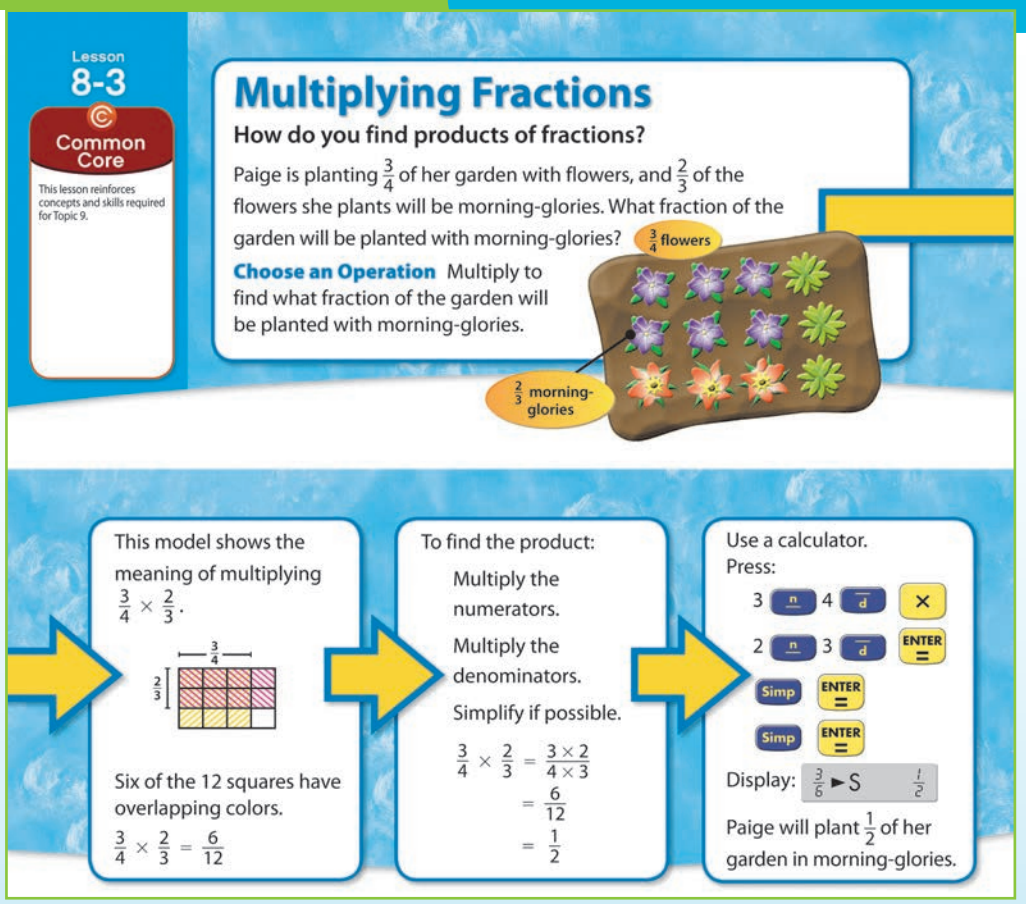
If forests once covered about $\frac{3}{5}$ of Earth's land and only about $\frac{1}{2}$ of these forests remain, what fraction of Earth is covered with forests today? We need to find $\frac{1}{2}$ of $\frac{3}{5}$, and can use an area model to find the answer.

Figure 15(a) shows a rectangle representing the *whole* separated into fifths, with $\frac{3}{5}$ shaded. To find $\frac{1}{2}$ of $\frac{3}{5}$, we divide the shaded portion of the rectangle in Figure 15(a) into two congruent parts and take one of those parts. The result would be the green portion of Figure 15(b). However, the green portion represents 3 parts out of 10, or $\frac{3}{10}$, of the whole. Thus,

$$\frac{1}{2} \text{ of } \frac{3}{5} = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10} = \frac{1 \cdot 3}{2 \cdot 5}.$$

An area model like the one in Figure 14 is used on the student page shown on page 292.

School Book Page — Multiplying Fractions



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This discussion leads to the following definition of multiplication for rational numbers.

Definition of Multiplication of Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Example 14

If $\frac{5}{6}$ of the population of a certain city are college graduates and $\frac{7}{11}$ of the city's college graduates are female, what fraction of the population of that city is female college graduates?

Solution The fraction should be $\frac{7}{11}$ of $\frac{5}{6}$, or $\frac{7}{11} \cdot \frac{5}{6} = \frac{7 \cdot 5}{11 \cdot 6} = \frac{35}{66}$.

The fraction of the population who are female college graduates is $\frac{35}{66}$.

Properties of Multiplication of Rational Numbers

Multiplication of rational numbers has properties analogous to the properties of multiplication of integers. These include the following properties for multiplication: closure, commutative, associative, and multiplicative identity. When we expand from the set of integers to the set of rationals, we pick up an additional property; that is, the multiplicative inverse property. For emphasis, we list the last two properties.

Theorem 6-10: Identity and Inverse Properties of Multiplication of Rational Numbers

Multiplicative Identity Property of Rational Numbers

The rational number 1 is the unique number such that for every rational number $\frac{a}{b}$,

$$1 \cdot \frac{a}{b} = \frac{a}{b} = \frac{a}{b} \cdot 1.$$

Multiplicative Inverse Property of Rational Numbers

For any nonzero rational number $\frac{a}{b}$, the multiplicative inverse (**reciprocal**) is the unique rational number $\frac{b}{a}$ such that

$$\frac{a}{b} \cdot \frac{b}{a} = 1 = \frac{b}{a} \cdot \frac{a}{b}.$$

Example 15

Find the multiplicative inverse, if possible, for each of the following rational numbers.

a. $\frac{2}{3}$ **b.** $\frac{-2}{5}$ **c.** 4 **d.** 0 **e.** $6\frac{1}{2}$

Solution

a. $\frac{3}{2}$

b. $\frac{5}{-2}$, or $\frac{-5}{2}$

c. Because $4 = \frac{4}{1}$, the multiplicative inverse of 4 is $\frac{1}{4}$.

d. Even though $0 = \frac{0}{1}, \frac{1}{0}$ is undefined; there is no multiplicative inverse of 0.

e. Because $6\frac{1}{2} = \frac{13}{2}$, the multiplicative inverse of $6\frac{1}{2}$ is $\frac{2}{13}$.

Multiplication and addition are connected through the distributive property of multiplication over addition. Also, there are multiplication properties of equality and inequality for rational numbers and a multiplication property of zero similar to those for whole numbers and integers. These properties can be proved using the definition of operations on rational numbers and properties of integers. They are stated in the following theorem.

Theorem 6-11: Properties of Rational Number Operations

Distributive Properties of Multiplication Over Addition and Subtraction for Rational Numbers

Let $\frac{a}{b}, \frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers. Then

$$\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\cdot\frac{e}{f}; \text{ and } \frac{a}{b}\left(\frac{c}{d}-\frac{e}{f}\right)=\frac{a}{b}\cdot\frac{c}{d}-\frac{a}{b}\cdot\frac{e}{f}.$$

Multiplication Property of Equality for Rational Numbers

Let $\frac{a}{b}, \frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers such that $\frac{a}{b}=\frac{c}{d}$; then $\frac{a}{b}\cdot\frac{e}{f}=\frac{c}{d}\cdot\frac{e}{f}$.

Multiplication Properties of Inequality for Rational Numbers

Let $\frac{a}{b}, \frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers. Then

a. $\frac{a}{b}>\frac{c}{d}$ and $\frac{e}{f}>0$, then $\frac{a}{b}\cdot\frac{e}{f}>\frac{c}{d}\cdot\frac{e}{f}$.

b. $\frac{a}{b}>\frac{c}{d}$ and $\frac{e}{f}<0$, then $\frac{a}{b}\cdot\frac{e}{f}<\frac{c}{d}\cdot\frac{e}{f}$.

Multiplication Property of Zero for Rational Numbers

Let $\frac{a}{b}$ be any rational number; then $\frac{a}{b}\cdot 0=0=0\cdot\frac{a}{b}$.

Example 16

A bicycle is on sale at $\frac{3}{4}$ of its original price. If the sale price is \$330, what was the original price?

Solution Let x be the original price. Then $\frac{3}{4}$ of the original price is $\frac{3}{4}x$. Because the sale price is \$330, we have $\frac{3}{4}x=330$. Solving for x gives

$$\begin{aligned}\frac{4}{3}\cdot\frac{3}{4}x&=\frac{4}{3}\cdot 330\\ 1\cdot x&=440\\ x&=440.\end{aligned}$$

Thus, the original price was \$440.

An alternative approach follows. Suppose the bar in Figure 16 represents the original price. We know that $\frac{3}{4}$ of the original price is the sale price \$330 as shown. We have 3 of the $\frac{1}{4}$ parts of the original price is \$330, and one part, $\frac{1}{4}$ of the original price, must be \$110. Thus, the original price is \$330 + \$110 = \$440.

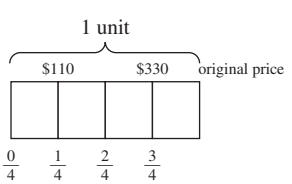


Figure 16

Multiplication with Mixed Numbers

In Figure 17, Johnny just figured out how to multiply mixed numbers while his colleagues seem to be struggling. How might we help them?

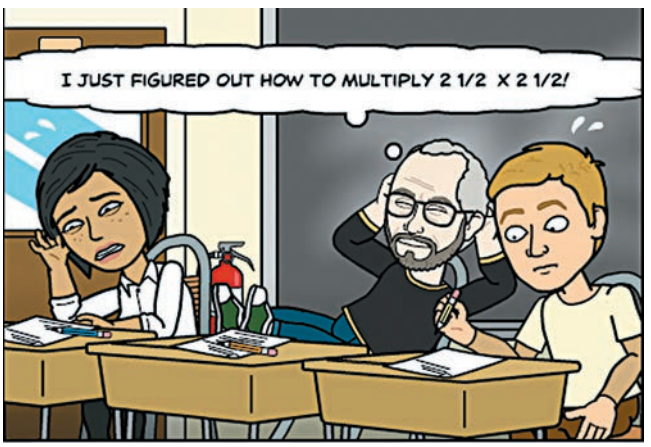


Figure 17

In Figure 17, Johnny may have decided one way to multiply $2\frac{1}{2} \cdot 2\frac{1}{2}$ is to change the mixed numbers to improper fractions and use the definition of multiplication as shown.

$$2\frac{1}{2} \cdot 2\frac{1}{2} = \frac{5}{2} \cdot \frac{5}{2} = \frac{25}{4}$$

We could then change $\frac{25}{4}$ to the mixed number $6\frac{1}{4}$.

Another way to multiply mixed numbers uses the distributive property of multiplication over addition, as seen below.

$$\begin{aligned} 2\frac{1}{2} \cdot 2\frac{1}{2} &= \left(2 + \frac{1}{2}\right)\left(2 + \frac{1}{2}\right) \\ &= \left(2 + \frac{1}{2}\right)2 + \left(2 + \frac{1}{2}\right)\frac{1}{2} \\ &= 2 \cdot 2 + \frac{1}{2} \cdot 2 + 2 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \\ &= 4 + 1 + 1 + \frac{1}{4} \\ &= 6 + \frac{1}{4} \\ &= 6\frac{1}{4} \end{aligned}$$

Multiplication of fractions enables us to obtain equivalent fractions, to perform addition and subtraction of fractions, as well as to solve equations in a different way, as shown in the following example.

Example 17

Use the definition of multiplication of fractions and its properties to justify the following.

a. The Fundamental Law of Fractions: $\frac{a}{b} = \frac{an}{bn}$ if $b \neq 0, n \neq 0$.

b. Addition of fractions using a common denominator.

Solution

a. $\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{n}{n} = \frac{an}{bn}$

b. $\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b}$

$= \frac{ad}{bd} + \frac{bc}{bd}$

$= \frac{ad + bc}{bd}$

Division of Rational Numbers

CCSS **In the grade 6 *Common Core Standards*, we find the following concerning division of rational numbers:**

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. **Interpret and compute quotients of fractions and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. (p. 42)**

IMAP Video

Watch Trina to see why it is important to build a conceptual understanding as well as a procedural understanding of mathematical concepts.

We apply and extend division of whole numbers by recalling that $6 \div 3$ means "How many 3s are there in 6?" We found that $6 \div 3 = 2$ because $3 \cdot 2 = 6$ and, in general, when $a, b, c \in W$, $a \div b = c$ if, and only if, c is the unique whole number such that $bc = a$. Consider $3 \div \left(\frac{1}{2}\right)$, which is equivalent to finding how many halves there are in 3. We see from the area model in Figure 18 that there are 6 half pieces in the 3 whole pieces. We record this as $3 \div \left(\frac{1}{2}\right) = 6$. This is true because $\left(\frac{1}{2}\right)6 = 3$.

IMAP Video

Watch Shelby explain how she determined 12 divided by $\frac{1}{2}$.

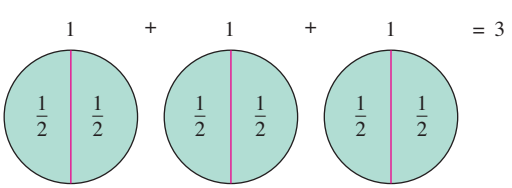


Figure 18

Another way to show that $3 \div \left(\frac{1}{2}\right) = 6$ is on a ruler. In Figure 19 we see that there are six $\frac{1}{2}$s in 3.

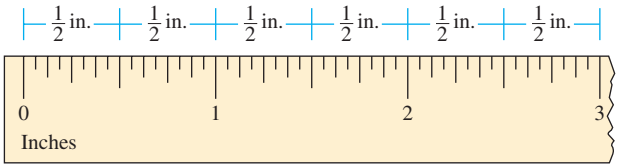


Figure 19

With whole numbers, one way to think about division is in terms of *repeated subtraction*. We found that $6 \div 2 = 3$ because 2 could be subtracted from 6 three times; that is, $6 - 3 \cdot 2 = 0$. Similarly, with $3 \div \frac{1}{2}$, we want to know how many halves can be subtracted from 3. Because $3 - 6\left(\frac{1}{2}\right) = 0$, we know that $3 \div \frac{1}{2} = 6$.

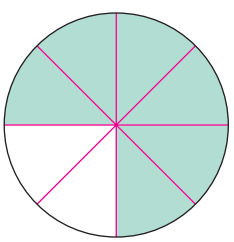


Figure 20

Next, consider $\frac{3}{4} \div \frac{1}{8}$. This means "How many $\frac{1}{8}$s are in $\frac{3}{4}$?" Figure 20 shows that there are six $\frac{1}{8}$s in the shaded portion, which represents $\frac{3}{4}$ of the whole. Therefore, $\frac{3}{4} \div \frac{1}{8} = 6$. This is true because $\left(\frac{1}{8}\right)6 = \frac{3}{4}$.

The measurement, or number-line, model may be used to understand division of fractions. For example, consider $\frac{7}{8} \div \frac{3}{4}$. First we draw a number line divided into eighths, as shown in Figure 21. Next we want to know how many $\frac{3}{4}$s there are in $\frac{7}{8}$. The bar of length $\frac{3}{4}$ is made up of 6 equal-size pieces of length $\frac{1}{8}$. We see that there is at least one length of $\frac{3}{4}$ in $\frac{7}{8}$. If we put another bar of length $\frac{3}{4}$ on the number line, we see there is 1 more of the 6 equal-length segments needed to make $\frac{7}{8}$. Therefore, the answer is $1\frac{1}{6}$, or $\frac{7}{6}$.

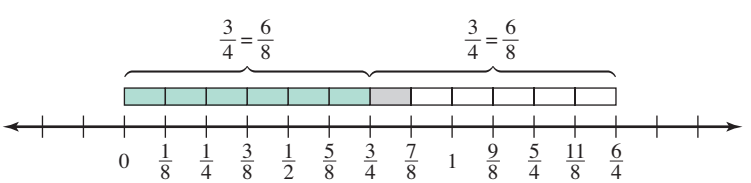


Figure 21

Additionally we know that $\frac{3}{4} \cdot \frac{7}{6} = \frac{21}{24} = \frac{7}{8}$, so the answer is correct.

In the previous examples, we saw a relationship between division and multiplication of rational numbers. We can define division for rational numbers formally in terms of multiplication in the same way that we defined division for whole numbers.

Definition of Division of Rational Numbers

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be rational numbers such that $\frac{c}{d} \neq 0$. Then $\frac{a}{b} \div \frac{c}{d} = \frac{e}{f}$ if, and only if, $\frac{e}{f}$ is the unique rational number such that $\frac{c}{d} \cdot \frac{e}{f} = \frac{a}{b}$.

In the above definition of division, $\frac{c}{d} \neq 0$ because division by 0 is not defined.

NOW TRY THIS 4

Students often confuse finding half of a number with dividing by one-half. Notice that

$a \div 2 = \frac{a}{2} = \frac{1}{2}a$, but $a \div \frac{1}{2} = x$ if, and only if, $\frac{1}{2}x = a$ which implies $2\left(\frac{1}{2}x\right) = 2a$ and $x = 2a$.

Write a real-life story that will help students see the difference between finding half of a number and division by $\frac{1}{2}$.

The partial student page below illustrates a method for dividing called "invert and multiply."

Algorithm for Division of Rational Numbers

Does the *invert-and-multiply* method make sense based on what we know about rational numbers? Consider what such a division might mean. For example, using the definition of division of rational numbers,

$$\frac{2}{3} \div \frac{5}{7} = x \quad \text{if, and only if,} \quad \frac{2}{3} = \frac{5}{7}x.$$

To solve for x, we multiply both sides of the equation by $\frac{7}{5}$, the reciprocal of $\frac{5}{7}$. Thus,

$$\frac{7}{5} \cdot \frac{2}{3} = \frac{7}{5}\left(\frac{5}{7}x\right) = \left(\frac{7}{5} \cdot \frac{5}{7}\right)x = 1 \cdot x = x.$$

Therefore, $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5}$. This illustrates the "invert-and-multiply" method, or the "use the reciprocal and multiply" method.



School Book Page

To divide by a fraction, multiply by the reciprocal of the fraction. You can remember this by thinking "invert and multiply."

KEY CONCEPTS **Dividing Fractions**

Arithmetic	Algebra
$\frac{3}{5} \div \frac{1}{3} = \frac{3}{5} \cdot \frac{3}{1}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, where b, c, and d are not 0.

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A traditional justification of this rule follows. The algorithm for division of fractions is usually justified in the middle grades by using the Fundamental Law of Fractions, $\frac{a}{b} = \frac{ac}{bc}$, where a, b, and c are fractions, or equivalently, the identity property of multiplication. For example,

$$\frac{2}{3} \div \frac{5}{7} = \frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3}}{\frac{5}{7}} \cdot 1 = \frac{\frac{2}{3}}{\frac{5}{7}} \cdot \frac{\frac{7}{5}}{\frac{7}{5}} = \frac{\frac{2}{3} \cdot \frac{7}{5}}{1} = \frac{2}{3} \cdot \frac{7}{5}.$$

Theorem 6-12: Algorithm for Division of Fractions

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers with $\frac{c}{d} \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Alternative Algorithm for Division of Rational Numbers

An alternative algorithm for division of fractions can be found by first dividing fractions that have equal denominators. For example, $\frac{9}{10} \div \frac{3}{10} = \frac{9}{10} \cdot \frac{10}{3} = \frac{9}{3}$ and $\frac{15}{23} \div \frac{5}{23} = \frac{15}{23} \cdot \frac{23}{5} = \frac{15}{5}$. These examples suggest that when two fractions with the same denominator are divided, the result can be obtained by dividing the numerator of the first fraction by the numerator of the second; that is, $\frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$. To divide fractions with different denominators, we rename the fractions so that the denominators are equal. Thus,

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bd} \cdot \frac{bd}{bc} = \frac{ad}{bc}.$$

NOW TRY THIS 5

Show that $\frac{a}{b} \div \frac{c}{d}$ and $\frac{a \div c}{b \div d}$ are equal.

The next three examples illustrate the use of division of rational numbers.

Example 18

A radio station provides 36 min for public service announcements for every 24 hr of broadcasting.

a. What part of the 24-hr broadcasting day is allotted to public service announcements?

b. How many $\frac{3}{4}$-min public service announcements can be broadcast in the 36 min?

Solution

a. There are 60 min in an hour and $60 \cdot 24$ min in the broadcasting day. Thus, $36 \div (60 \cdot 24)$, or $\frac{1}{40}$, of the day is allotted for the announcements.

b. $36 \div \left(\frac{3}{4}\right) = 36\left(\frac{4}{3}\right) = 48$ announcements are broadcast.

Example 19

We have $35\frac{1}{2}$ yd of material available to make towels. Each towel requires $\frac{3}{8}$ yd of material.

a. How many towels can be made?
b. How much material will remain?

Solution

a. We need to find the integer part of the answer to $35\frac{1}{2} \div \frac{3}{8}$.

$$35\frac{1}{2} \div \frac{3}{8} = \frac{71}{2} \cdot \frac{8}{3} = \frac{284}{3} = 94\frac{2}{3}$$

Thus, we can make 94 towels.

b. Because the division in part (a) was by $\frac{3}{8}$, the amount of material remaining is $\frac{2}{3}$ of $\frac{3}{8}$, or $\frac{2}{3} \cdot \frac{3}{8} = \frac{1}{4}$ yd.

Example 20

A bookstore has a shelf that is $37\frac{1}{2}$ in. long. Each book that is to be placed on the shelf is $1\frac{1}{4}$ in. thick. How many books can be placed on the shelf?

Solution We need to find how many $1\frac{1}{4}$s there are in $37\frac{1}{2}$.

$$\frac{37\frac{1}{2}}{1\frac{1}{4}} = \frac{75}{2} \div \frac{5}{4}$$

$$= \frac{75}{2} \cdot \frac{4}{5}$$

$$= \frac{300}{10}$$

$$= 30$$

Note that we could compute $\frac{75}{2} \cdot \frac{4}{5}$ by first eliminating common factors; that is, $\frac{75}{2} \cdot \frac{4}{5} = \frac{15 \cdot 2}{1 \cdot 1} = \frac{30}{1} = 30$.

Therefore, 30 books can be placed on the shelf.

Mental Math and Estimation with Rational Numbers

Mental math strategies developed with whole numbers can also be used with rational numbers.

Example 21

Use rational number properties to mentally compute the following products.

a. $(12 \cdot 25)\frac{1}{4}$ **b.** $\left(5\frac{1}{6}\right)12$ **c.** $\frac{4}{5}(20)$

Solution Each computation shown is a possible approach.

a. $(12 \cdot 25)\frac{1}{4} = 25\left(12 \cdot \frac{1}{4}\right) = 25 \cdot 3 = 75$

b. $\left(5\frac{1}{6}\right)12 = \left(5 + \frac{1}{6}\right)12 = 5 \cdot 12 + \frac{1}{6} \cdot 12 = 60 + 2 = 62$

c. $\frac{4}{5}(20) = 4\left(\frac{1}{5} \cdot 20\right) = 4 \cdot 4 = 16$

Similarly, estimation strategies developed with whole number can be used with rational numbers.

Example 22

Estimate each of the following.

a. $3\frac{1}{4} \cdot 7\frac{8}{9}$ **b.** $24\frac{5}{7} \div 4\frac{1}{8}$

Solution

a. Using rounding, the product will be close to $3 \cdot 8 = 24$. If we use the range strategy, we can say the product must be between $3 \cdot 7 = 21$ and $4 \cdot 8 = 32$.

b. We can use compatible numbers and think of the estimate as $24 \div 4 = 6$ or $25 \div 5 = 5$.

Extending the Notion of Exponents

Recall that a^m was defined for any whole number a and any natural number m as the product of m a's. We define a^m for any rational number a in a similar way as follows.

Definition of *a* to the *m*th Power

$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}}$, where a is any rational number and m is any natural number.

From the definition, $a^3 \cdot a^2 = (a \cdot a \cdot a)(a \cdot a) = a^{3+2} = a^5$. In a similar way, it follows that

Statement 1: If a is a rational number and m and n are any natural numbers, $a^m \cdot a^n = a^{m+n}$.

If Statement 1 is to be true for all whole numbers m and n, then because $a^1 \cdot a^0 = a^{1+0} = a^1$, we must have $a^0 = 1$. Hence, it is useful to give meaning to a^0 when $a \neq 0$ as follows.

Statement 2: For any nonzero rational number a, $a^0 = 1$.

If $a^m \cdot a^n = a^{m+n}$ is extended to all integer powers of a, then how should a^{-3} be defined? If Statement 1 is to be true for all integers m and n, then $a^{-3} \cdot a^3 = a^{-3+3} = a^0 = 1$. Therefore, $a^{-3} = \frac{1}{a^3}$. This is true in general and we have the following.

Statement 3: Definition: For any nonzero rational number a and any natural number n, $a^{-n} = \frac{1}{a^n}$.

In elementary grades the definition of a^{-n} is typically motivated by looking at patterns. Notice that as the following exponents decrease by 1, the numbers on the right are divided by 10. Thus, the pattern might be continued, as shown.

$$\begin{aligned}
10^3 &= 10 \cdot 10 \cdot 10 \\
10^2 &= 10 \cdot 10 \\
10^1 &= 10 \\
10^0 &= 1 \\
10^{-1} &= \frac{1}{10} = \frac{1}{10^1} \\
10^{-2} &= \frac{1}{10} \cdot \frac{1}{10} = \frac{1}{10^2} \\
10^{-3} &= \frac{1}{10^2} \cdot \frac{1}{10} = \frac{1}{10^3}
\end{aligned}$$

If the pattern is extended in this way, then we would predict that $10^{-n} = \frac{1}{10^n}$. Notice that this is inductive reasoning and hence is not a mathematical justification.

Consider whether the property $a^m \cdot a^n = a^{m+n}$ can be extended to include all powers of a, where the exponents are integers. For example, is it true that $2^4 \cdot 2^{-3} = 2^{4+-3} = 2^1$? The definitions of 2^{-3} and the properties of nonnegative exponents ensure this is true, as shown next.

$$2^4 \cdot 2^{-3} = 2^4 \cdot \frac{1}{2^3} = \frac{2^4}{2^3} = \frac{2^1 \cdot 2^3}{2^3} = 2^1$$

Similarly, $2^{-4} \cdot 2^{-3} = 2^{-4+-3} = 2^{-7}$ is true because

$$2^{-4} \cdot 2^{-3} = \frac{1}{2^4} \cdot \frac{1}{2^3} = \frac{1 \cdot 1}{2^4 \cdot 2^3} = \frac{1}{2^{4+3}} = \frac{1}{2^7} = 2^{-7}.$$

In general, with integer exponents, the following theorem holds.

Theorem 6-13

For any nonzero rational number a and any integers m and n, $a^m \cdot a^n = a^{m+n}$.

If $a = 0$ and either m or n is negative, then $a^m \cdot a^n$ is undefined.

Other properties of exponents can be developed by using the properties of rational numbers. For example,

$$\frac{2^5}{2^3} = \frac{2^3 \cdot 2^2}{2^3} = 2^2 = 2^{5-3} \quad \text{and} \quad \frac{2^5}{2^8} = \frac{2^5}{2^5 \cdot 2^3} = \frac{1}{2^3} = 2^{-3} = 2^{5-8}.$$

With integer exponents, the following theorem holds.

Theorem 6-14

For any nonzero rational number a and any integers m and n, $\frac{a^m}{a^n} = a^{m-n}$.

At times, we may encounter an expression like $(2^4)^3$. This expression can be written as a single power of 2 as follows:

$$(2^4)^3 = 2^4 \cdot 2^4 \cdot 2^4 = 2^{4+4+4} = 2^{3 \cdot 4} = 2^{12}$$

In general, if a is any rational number and m and n are positive integers, then

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot \ldots \cdot a^m}_{n \text{ factors}} = \overbrace{a^{m+m+\ldots+m}}^{n \text{ terms}} = a^{nm} = a^{mn}.$$

Does this theorem hold for negative-integer exponents? For example, does $(2^3)^{-4} = 2^{(3)(-4)} = 2^{-12}$? The answer is yes because $(2^3)^{-4} = \dfrac{1}{(2^3)^4} = \dfrac{1}{2^{12}} = 2^{-12}$. Also, $(2^{-3})^4 = \left(\dfrac{1}{2^3}\right)^4 = \dfrac{1}{2^3} \cdot \dfrac{1}{2^3} \cdot \dfrac{1}{2^3} \cdot \dfrac{1}{2^3} = \dfrac{1^4}{(2^3)^4} = \dfrac{1}{2^{12}} = 2^{-12}$.

Theorem 6-15

For any nonzero rational number a and any integers m and n,

$$(a^m)^n = a^{mn}.$$

Using the definitions and theorems developed, we derive additional properties. For example:

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3 \cdot 3} = \frac{2^4}{3^4}$$

This property is generalized as follows.

Theorem 6-16

For any nonzero rational number $\dfrac{a}{b}$ and any integer m,

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

From the definition of negative exponents, the preceding theorem, and division of fractions, we have

$$\left(\frac{a}{b}\right)^{-m} = \frac{1}{\left(\dfrac{a}{b}\right)^m} = \frac{1}{\dfrac{a^m}{b^m}} = \frac{b^m}{a^m} = \left(\frac{b}{a}\right)^m.$$

Theorem 6-17

For any nonzero rational number $\dfrac{a}{b}$ and any integer m, $\left(\dfrac{a}{b}\right)^{-m} = \left(\dfrac{b}{a}\right)^m$.

A property similar to the one in Theorem 6-16 holds for multiplication. For example,

$$(2 \cdot 3)^{-3} = \frac{1}{(2 \cdot 3)^3} = \frac{1}{2^3 \cdot 3^3} = \left(\frac{1}{2^3}\right) \cdot \left(\frac{1}{3^3}\right) = 2^{-3} \cdot 3^{-3}.$$

Theorem 6-18 considers the distributive property of exponentiation over multiplication.

Theorem 6-18

For any nonzero rational numbers a and b and any integer m,

$$(a \cdot b)^m = a^m \cdot b^m.$$

Theorem 6-18 is also true when a or $b = 0$ and $m > 0$.

The properties of exponents are summarized below.

Theorem 6-19: Properties of Exponents

For any nonzero rational numbers a and b and integers m and n, the following are true.

a. $a^0 = 1$

b. $a^{-m} = \frac{1}{a^m}$

c. $a^m \cdot a^n = a^{m+n}$

d. $\frac{a^m}{a^n} = a^{m-n}$

e. $(a^m)^n = a^{mn}$

f. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

g. $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$

h. $(ab)^m = a^m b^m$

Notice that property (h) is for multiplication. Analogous properties do not hold for addition and subtraction. For example, in general, $(a + b)^{-1} \neq a^{-1} + b^{-1}$. To see why, a numerical example is sufficient, but it is instructive to write each side with positive exponents:

$$(a + b)^{-1} = \frac{1}{a + b}$$

$$a^{-1} + b^{-1} = \frac{1}{a} + \frac{1}{b}$$

We know from addition of fractions, $\frac{1}{a + b} \neq \frac{1}{a} + \frac{1}{b}$. Therefore, $(a + b)^{-1} \neq a^{-1} + b^{-1}$.

Example 23

In each of the following statements, show each equality or inequality is true in general for nonzero rational numbers x, a, and b.

a. $(^-x)^{-2} \neq {}^-x^{-2}$

b. $(^-x)^{-3} = {}^-x^{-3}$

c. $ab^{-1} \neq (ab)^{-1}$ unless $a = 1$ or $a = {}^-1$

d. $(a^{-2}b^{-2})^{-1} = a^2b^2$

e. $(a^{-2} + b^{-2})^{-1} \neq a^2 + b^2$

Solution

a. $(^{-}x)^{-2} = \dfrac{1}{(^{-}x)^2} = \dfrac{1}{x^2}$

$^{-}x^{-2} = -(x^{-2}) = {}^{-}\left(\dfrac{1}{x^2}\right) = \dfrac{^{-}1}{x^2}$

Hence, $(^{-}x)^{-2} \neq {}^{-}x^{-2}$.

b. $(^{-}x)^{-3} = \dfrac{1}{(^{-}x)^3} = \dfrac{1}{^{-}x^3} = {}^{-}\left(\dfrac{1}{x^3}\right)$

$^{-}x^{-3} = {}^{-}(x^{-3}) = {}^{-}\left(\dfrac{1}{x^3}\right)$

Hence, $(^{-}x)^{-3} = {}^{-}x^{-3}$.

c. $ab^{-1} = a(b^{-1}) = a \cdot \dfrac{1}{b} = \dfrac{a}{b}$, but $(ab)^{-1} = \dfrac{1}{ab}$. Hence, $ab^{-1} \neq (ab)^{-1}$.

d. $(a^{-2}b^{-2})^{-1} = (a^{-2})^{-1}(b^{-2})^{-1}$
$= a^{(^{-}2)(^{-}1)}b^{(^{-}2)(^{-}1)}$
$= a^2b^2$

e. $(a^{-2} + b^{-2})^{-1} = \left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)^{-1} = \left(\dfrac{a^2 + b^2}{a^2\,b^2}\right)^{-1} = \dfrac{a^2\,b^2}{a^2 + b^2} \neq a^2 + b^2$

Observe that all the properties of exponents refer to powers with either the same base or the same exponent. To evaluate expressions using exponents where different bases and powers are used, perform all the computations or rewrite the expressions in either the same base or the same exponent, if possible. For example, $\dfrac{27^4}{81^3}$ can be rewritten $\dfrac{27^4}{81^3} = \dfrac{(3^3)^4}{(3^4)^3} = \dfrac{3^{12}}{3^{12}} = 1$.

Example 24

Perform the following computations and express your answers without negative exponents.

a. $16^2 \cdot 8^{-3}$
b. $20^2 \div 2^4$
c. $(10^{-1} + 5 \cdot 10^{-2} + 3 \cdot 10^{-3})10^3$
d. $(x^3y^{-2})^{-4}$

Solution

a. $16^2 \cdot 8^{-3} = (2^4)^2 \cdot (2^3)^{-3} = 2^8 \cdot 2^{-9} = 2^{8+^{-}9} = 2^{-1} = \dfrac{1}{2}$

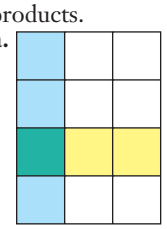
b. $\dfrac{20^2}{2^4} = \dfrac{(2^2 \cdot 5)^2}{2^4} = \dfrac{2^4 \cdot 5^2}{2^4} = 5^2$ or 25

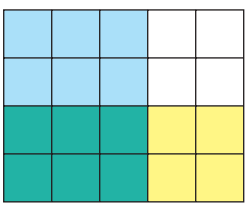
c. $(10^{-1} + 5 \cdot 10^{-2} + 3 \cdot 10^{-3})10^3 = 10^{-1} \cdot 10^3 + 5 \cdot 10^{-2} \cdot 10^3 + 3 \cdot 10^{-3} \cdot 10^3$
$= 10^{-1+3} + 5 \cdot 10^{-2+3} + 3 \cdot 10^{-3+3}$
$= 10^2 + 5 \cdot 10^1 + 3 \cdot 10^0$
$= 153$

d. $(x^3y^{-2})^{-4} = (x^3)^{-4} \cdot (y^{-2})^{-4} = x^{3(^{-}4)} \cdot y^{(^{-}2)(^{-}4)} = x^{-12}y^8 = \left(\dfrac{1}{x^{12}}\right)y^8 = \dfrac{y^8}{x^{12}}$, if $x, y \neq 0$

Assessment 6-3A

1. In the following figures, a unit rectangle is used to illustrate the product of two fractions. Name the fractions and their products.

a. 

b. 

2. Use a rectangular region to illustrate each of the following products.

a. $\frac{1}{3} \cdot \frac{3}{4}$

b. $\frac{2}{3} \cdot \frac{1}{5}$

3. Find each of the following products. Write your answers in simplest form.

a. $\frac{49}{65} \cdot \frac{26}{98}$ **b.** $\frac{a}{b} \cdot \frac{b^2}{a^2}$ **c.** $\frac{xy}{z} \cdot \frac{z^2a}{x^3y^2}$

4. Use the distributive property of multiplication over addition to find each product.

a. $4\frac{1}{2} \cdot 2\frac{1}{3}$

b. $3\frac{1}{3} \cdot 2\frac{1}{2}$

5. Find the multiplicative inverse of each of the following.

a. $\frac{^-1}{3}$ **b.** $3\frac{1}{3}$

c. $\frac{x}{y}$, if $x \neq 0$ **d.** $^-7$

6. Solve for x.

a. $\frac{2}{3}x = \frac{7}{6}$

b. $\frac{3}{4} \div x = \frac{1}{2}$

c. $\frac{5}{6} + \frac{2}{3}x = \frac{3}{4}$

d. $\frac{2x}{3} - \frac{1}{4} = \frac{x}{6} + \frac{1}{2}$

7. Show that the following properties do not hold for the division of rational numbers.

a. Commutative

b. Associative

8. Compute the following mentally. Find the exact answers.

a. $3\frac{1}{4} \cdot 8$

b. $7\frac{1}{4} \cdot 4$

c. $9\frac{1}{5} \cdot 10$

d. $8 \cdot 2\frac{1}{4}$

9. Choose from among the numbers in parentheses the number that best approximates each of the following.

a. $3\frac{11}{12} \cdot 5\frac{3}{100}$ $(8, 20, 15, 16)$

b. $2\frac{1}{10} \cdot 7\frac{7}{8}$ $(16, 14, 4, 3)$

c. $\frac{1}{101} \div \frac{1}{103}$ $\left(0, 1, \frac{1}{2}, \frac{1}{4}\right)$

10. Without actually doing the computations, choose the phrase in parentheses that correctly describes each.

a. $\frac{13}{14} \cdot \frac{17}{19}$ (greater than 1, less than 1)

b. $3\frac{2}{7} \div 5\frac{1}{9}$ (greater than 1, less than 1)

c. $4\frac{1}{3} \div 2\frac{3}{100}$ (greater than 2, less than 2)

11. A sewing project requires $6\frac{1}{8}$ yd of material that sells for \$4 per yard and $3\frac{1}{4}$ yd of material that sells for \$3 per yard. Choose the best estimate for the cost of the project:

a. Between \$30 and \$40

b. Between \$20 and \$34

c. Between \$36 and \$40

d. Between \$33 and \$40

12. Five-eighths of the students at Salem State College live in dormitories. If 6000 students at the college live in dormitories, how many students are there in the college?

13. Alberto owns $\frac{5}{9}$ of the stock in the N.W. Tofu Company. His sister Renatta owns half as much stock as Alberto. What part of the stock is owned by neither Alberto nor Renatta?

14. A suit is on sale for \$180.00. What was the original price of the suit if the discount was $\frac{1}{4}$ of the original price?

15. John took all his money out of his savings account. He spent \$50.00 on a radio and $\frac{3}{5}$ of what remained on presents. Half of what was left he put in his checking account, and the remaining \$35.00 he donated to charity. How much money did John originally have in his savings account?

16. Al gives $\frac{1}{2}$ of his marbles to Bev. Bev gives $\frac{1}{2}$ of these to Carl. Carl gives $\frac{1}{2}$ of these to Dani. If Dani was given four marbles, how many did Al have originally?

17. Write each of the following in simplest form using positive exponents in the final answer.

a. $3^{-7} \cdot 3^{-6}$ **b.** $3^7 \cdot 3^6$

c. $5^{15} \div 5^4$ **d.** $5^{15} \div 5^{-4}$

e. $(-5)^{-2}$ **f.** $\frac{a^2}{a^{-3}}$

18. Write each of the following in simplest form using positive exponents in the final answer.

a. $\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^7$

b. $\left(\frac{1}{2}\right)^9 \div \left(\frac{1}{2}\right)^6$

c. $\left(\frac{2}{3}\right)^5 \cdot \left(\frac{4}{9}\right)^2$

d. $\left(\frac{3}{5}\right)^7 \div \left(\frac{3}{5}\right)^7$

19. If a and b are rational numbers, with $a \neq 0$ and $b \neq 0$, and if m and n are integers, which of the following statements are always true? Justify your answers.

a. $a^m \cdot b^n = (ab)^{m+n}$ **b.** $a^m \cdot b^n = (ab)^{mn}$

c. $a^m \cdot b^m = (ab)^{2m}$ **d.** $(ab)^0 = 1$

e. $(a + b)^m = a^m + b^m$ **f.** $(a + b)^{-m} = \frac{1}{a^m} + \frac{1}{b^m}$

20. Solve for the integer n in each of the following.

a. $2^n = 32$ **b.** $n^2 = 36$

c. $2^n \cdot 2^7 = 2^5$ **d.** $2^n \cdot 2^7 = 8$

21. Solve each of the following inequalities for x, where x is an integer.

a. $3^x \leq 9$ **b.** $25^x < 125$

c. $3^{2x} > 27$ **d.** $4^x > 1$

22. Determine which fraction in each of the following pairs is greater.

a. $\left(\frac{1}{2}\right)^3$ or $\left(\frac{1}{2}\right)^4$

b. $\left(\frac{3}{4}\right)^{10}$ or $\left(\frac{3}{4}\right)^8$

c. $\left(\frac{4}{3}\right)^{10}$ or $\left(\frac{4}{3}\right)^8$

d. $\left(\frac{3}{4}\right)^{10}$ or $\left(\frac{4}{5}\right)^{10}$

e. 32^{50} or 4^{100}

f. $(-27)^{-15}$ or $(-3)^{-75}$

23. Show that the arithmetic mean of two rational numbers is between the two numbers; that is, prove if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}$.

24. In the Corcoran School of Design in 2014, $\frac{17}{25}$ of the students were male.

a. What fraction were female?

b. Does this imply 17 students are male?

25. The reported tax revenue in dollars for Washington DC in 2011 is shown below.

Corporate Income	$36,802,396
Personal Income	32,020,924
Social Security & Other Taxes	620,501
Hotel Tax	212,565,755
Property Taxes	183,005,144
Excise & Fees	21,723,515
Sales Tax	187,656,061
Restaurant	119,487,765
Retail	58,122,531
Airport	7,155,614
Car Rental	2,890,151
Total	**674,394,296**

a. Approximately what is the fractional part of the total tax is the difference in restaurant sales tax and retail sales tax?

b. If the sales tax were abolished, approximately what fraction of the total tax package would be lost?

26. According to the Washington DC City Government, the following lists the number of homicides in the city since 2004.

2013	2012	2011	2010	2009	2008	2007	2006	2005	2004
104	88	108	132	144	186	181	169	196	198

a. What year showed the greatest fractional increase from the previous year?

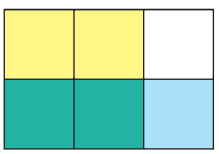
b. What year showed the greatest fractional decrease from the previous year?

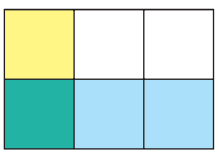
27. If $\frac{1}{33}$ of all deer in the United States are in Mississippi and $\frac{2}{7}$ of the deer are in Montana, what total fractional part of the deer population is in those two states?

28. If $\frac{1}{4}$ of an estate is to be distributed equally to 8 cousins, what fractional part of the entire estate does each cousin receive?

Assessment 6-3B

1. In the following figure a unit rectangle is used to illustrate the product of two fractions. Name the fractions and their products.

a. 

b. 

2. Use a rectangular region to illustrate each of the following products.

a. $\frac{2}{5} \cdot \frac{1}{3}$ **b.** $\frac{2}{3} \cdot \frac{2}{3}$

3. Find each of the following products of rational numbers. Write your answers in simplest form.

a. $2\frac{1}{3} \cdot 3\frac{3}{4}$ **b.** $\frac{22}{7} \cdot 4\frac{2}{3}$

c. $\frac{-5}{2} \cdot 2\frac{1}{2}$ **d.** $2\frac{3}{4} \cdot 2\frac{1}{3}$

e. $\frac{a^2}{b^3} \cdot \frac{b^2}{a^3}$ **f.** $\frac{x^3y^2}{z} \cdot \frac{z}{x^2y}$

4. Use the distributive property to find each product of rational numbers.

a. $2\frac{1}{3} \cdot 4\frac{3}{5}$ **b.** $\left(\frac{x}{y} + 1\right)\left(\frac{y}{x} - 1\right)$

c. $248\frac{2}{5} \cdot 100\frac{1}{8}$

5. Find the multiplicative inverse of each of the following.

a. $\frac{6}{7}$ **b.** 8 **c.** $4\frac{1}{5}$ **d.** $-1\frac{1}{2}$

6. Solve for x.

a. $\frac{2}{3}x = \frac{11}{6}$ **b.** $\frac{3}{4} \div x = \frac{1}{3}$

c. $\frac{5}{6} - \frac{2}{3}x = \frac{3}{4}$ **d.** $\frac{2x}{3} + \frac{1}{4} = \frac{x}{6} - \frac{1}{2}$

7. Find a fraction such that if you add the denominator to the numerator and place the sum over the original denominator, the new fraction has triple the value of the original fraction.

8. Compute the following mentally; find the exact answers.

a. $3\frac{1}{2} \cdot 8$ **b.** $7\frac{3}{4} \cdot 4$

c. $9\frac{1}{5} \cdot 6$ **d.** $8 \cdot 2\frac{1}{3}$

e. $3 \div \frac{1}{2}$ **f.** $3\frac{1}{2} \div \frac{1}{2}$

g. $3 \div \frac{1}{3}$ **h.** $4\frac{1}{2} \div 2$

9. Choose from among the numbers in parentheses the number that best approximates each of the following.

a. $20\frac{2}{3} \div 9\frac{7}{8} \left(2, 180, \frac{1}{2}, 10\right)$

b. $3\frac{1}{20} \cdot 7\frac{77}{100} \left(21, 24, \frac{1}{20}, 32\right)$

c. $\frac{1}{10^3} \div \frac{1}{1001} \left(\frac{1}{10^3}, 1, 1001, 0\right)$

10. Without actually doing the computations, choose the phrase in parentheses that correctly describes each.

a. $4\frac{1}{3} \div 2\frac{13}{100}$ (greater than 2, less than 2)

b. $16 \div 4\frac{3}{18}$ (greater than 4, less than 4)

c. $16 \div 3\frac{8}{9}$ (greater than 4, less than 4)

11. When you multiply a certain number by 3 and then subtract $\frac{7}{18}$, you get the same result as when you multiply the number by 2 and add $\frac{5}{12}$. What is the number?

12. Di Paloma University had a faculty reduction and lost $\frac{1}{5}$ of its faculty. If 320 faculty members were left after the reduction, how many members were there originally?

13. A person has $29\frac{1}{2}$ yd of material available to make doll outfits. Each outfit requires $\frac{3}{4}$ yd of material.

a. How many outfits can be made?

b. How much material will be left over?

14. Every employee's salary at the Sunrise Software Company increases each year by $\frac{1}{10}$ of that person's salary the previous year.

a. If Martha's present annual salary is \$100,000, what will her salary be in 2 yr?

b. If Aaron's present salary is \$99,000, what was his salary 1 yr ago?

c. If Juanita's present salary is \$363,000, what was her salary 2 yr ago?

15. Jasmine is reading a book. She has finished $\frac{3}{4}$ of the book and has 82 pages left to read. How many pages has she read?

16. Peter, Paul, and Mary start at the same time walking around a circular track in the same direction. Peter takes $\frac{1}{2}$ hr to walk around the track. Paul takes $\frac{5}{12}$ hr, and Mary takes $\frac{1}{3}$ hr.

a. How many minutes does it take each person to walk around the track?

b. How many times will each person go around the track before all three meet again at the starting line?

17. Write each of the following rational numbers in simplest form using positive exponents in the final answer.

a. $\left(\frac{1}{3}\right)^{-1}$ **b.** $\frac{a^{-3}}{a}$ **c.** $\frac{(a^{-4})^3}{a^{-4}}$

d. $\frac{a}{a^{-1}}$ **e.** $\frac{a^{-3}}{a^{-2}}$

18. Write each of the following in simplest form using positive exponents in the final answer.

a. $\left(\frac{1}{2}\right)^{10} \div \left(\frac{1}{2}\right)^{2}$

b. $\left(\frac{2}{3}\right)^{5}\left(\frac{4}{9}\right)^{-2}$

c. $\left(\frac{3}{5}\right)^{7} \div \left(\frac{5}{3}\right)^{4}$

d. $\left[\left(\frac{5}{6}\right)^{7}\right]^{3}$

19. If a and b are rational numbers, with $a \neq 0$ and $b \neq 0$, and if m and n are integers, which of the following statements are always true? Justify your answers.

a. $\frac{a^m}{b^n} = \left(\frac{a}{b}\right)^{m-n}$

b. $(ab)^{-m} = \frac{1}{a^m} \cdot \frac{1}{b^m}$

c. $\left(\frac{2}{a^{-1} + b^{-1}}\right)^{-1} = \frac{1}{2} \cdot \frac{1}{a + b}$

d. $2(a^{-1} + b^{-1})^{-1} = \frac{2ab}{a + b}$

e. $a^{mn} = a^m \cdot a^n$

f. $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$

20. Solve, if possible, for n where n is an integer in each of the following.

a. $2^n = -32$ **b.** $n^3 = \frac{-1}{27}$

c. $2^n \cdot 2^7 = 1024$ **d.** $2^n \cdot 2^7 = 64$

e. $(2 + n)^2 = 2^2 + n^2$ **f.** $3^n = 27^5$

21. Solve each of the following inequalities for x, where x is an integer.

a. $3^x \geq 81$ **b.** $4^x \geq 8$

c. $3^{2x} \leq 27$ **d.** $2^x < 1$

22. Determine which fraction in each of the following pairs is greater.

a. $\left(\frac{4}{3}\right)^{10}$ or $\left(\frac{4}{3}\right)^{8}$

b. $\left(\frac{3}{4}\right)^{10}$ or $\left(\frac{4}{5}\right)^{10}$

c. $\left(\frac{4}{3}\right)^{10}$ or $\left(\frac{5}{4}\right)^{10}$

d. $\left(\frac{3}{4}\right)^{100}$ or $\left(\frac{3}{4} \cdot \frac{9}{10}\right)^{100}$

23. In the following, determine which number is greater.

a. 32^{100} or 4^{200}

b. $(-27)^{-15}$ or $(-3)^{-50}$

24. Brandy bought a horse for \$270 and immediately started paying for his keep. She sold the horse for \$540. Considering the cost of his keep she found that she had lost an amount equal to half of what she paid for the horse plus one-fourth of the cost of his keep. How much did Brandy lose on the horse?

25. In 2014 in the Corcoran School of Design, $\frac{8}{25}$ of the students were female. What fraction was male?

26. The reported tax revenue in dollars for Washington DC in 2011 is shown below.

Corporate Income	\$36,802,396
Personal Income	32,020,924
Social Security & Other Taxes	620,501
Hotel Tax	212,565,755
Property Taxes	183,005,144
Excise & Fees	21,723,515
Sales Tax	187,656,061
Restaurant	119,487,765
Retail	58,122,531
Airport	7,155,614
Car Rental	2,890,151
Total	**674,394,296**

a. Approximately what is the fractional part of the total tax is the difference in hotel tax and property tax?

b. If the sales tax was increased $\frac{1}{11}$, approximately what fraction of the total tax package would be gained?

27. According to the Washington DC City Government, the following lists the number of homicides in the city since 2004.

2013	2012	2011	2010	2009	2008	2007	2006	2005	2004
104	88	108	132	144	186	181	169	196	198

a. What is the fractional decrease in homicides from 2005 to 2012?

b. What is the fractional increase in homicides from 2012 to 2013?

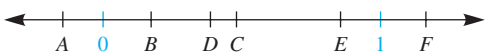
28. If $\frac{1}{33}$ of all deer in the United States are in Mississippi, and $\frac{2}{7}$ of the deer are in Montana, what fractional part of the deer population is not in those two states?

29. If $\frac{1}{4}$ of an estate consists of $\frac{2}{5}$ stocks and $\frac{3}{5}$ bonds, what part of the estate is each of the stocks and bonds?

Mathematical Connections 6-3

Reasoning

1. Suppose you divide a natural number n by a positive rational number less than 1. Will the answer always be less than n, sometimes less than n, or never less than n? Why?
2. If the fractions represented by points C and D on the following number line are multiplied, what point best represents the product? Explain why.



3. If the product of two numbers is 1 and one of the numbers is greater than 1, what do you know about the other number? Explain your answer.

Open-Ended

4. Write a story or model for $1\frac{3}{4} \div \frac{1}{2}$.
5. Complete a survey of your class asking questions such as their favorite color, favorite type of shoe, and favorite math concept. Use fractions to summarize your information and find any differences in male and female preference.
6. Consider the demographics of students in each undergraduate class (freshman, sophomore, junior, and senior) at your university. Describe the fractional differences in males and females in each class.

Cooperative Learning

7. Choose a brick building on your campus. Measure the height of one brick and the thickness of mortar between bricks. Estimate the height of the building and then calculate the height of the building. Were rational numbers used in your computations?
8. Have each student in your group choose a state or city and find the demographics on taxes as in Exercise 26 in Assessment 6-3B. Compare fractions representing sales taxes for each chosen location. Which has the highest? Lowest?
9. In each group of a class, describe your group using fractions in the explanation. Use at least three sets of fractions in your description. Pass the descriptions to the teacher who will then distribute the descriptions to a different group. The goal is to describe your group well enough that others can identify the groups.

Connecting Mathematics to the Classroom

10. Bente says to do the problem $12\frac{1}{4} \div 3\frac{3}{4}$ you just find $12 \div 3 = 4$ and $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3}$ to get $4\frac{1}{3}$. How do you respond?
11. Amy says that dividing a number by $\frac{1}{2}$ is the same as taking half of a number. How do you respond?
12. Dani says that if we have $\frac{3}{4} \cdot \frac{2}{5}$, we could just multiply $\frac{3}{5} \cdot \frac{2}{4} = \frac{3}{5} \cdot \frac{1}{2} = \frac{3}{10}$. Is she correct? Explain why.
13. Noah says that dividing a number by 2 is the same as multiplying it by $\frac{1}{2}$. He wants to know if he is right, and if so, why. How do you respond?
14. Jim is not sure when to use multiplication by a fraction and when to use division. He has the following list of problems. How would you help him solve these problems in a way that would enable him to solve similar problems on his own?
 a. $\frac{3}{4}$ of a package of sugar fills $\frac{1}{2}$ c. How many cups of sugar are in a full package of sugar?
 b. How many packages of sugar are needed to fill 2 c?
 c. If $\frac{1}{3}$ c sugar is required to make two loaves of challah, how many cups of sugar are needed for three loaves?
 d. If $\frac{3}{4}$ c sugar is required for 1 gal of punch, how many gallons can be made with 2 c of sugar?
 e. If you have $22\frac{3}{8}$ in. of ribbon, and need $1\frac{1}{4}$ in. to decorate one doll, how many dolls can be decorated, and how much ribbon will be left over?
15. A student claims that division always makes things smaller so $5 \div \left(\frac{1}{2}\right)$ cannot be 10 because 10 is greater than the number 5 she started with. How do you respond?
16. A student simplified the fraction $\frac{m+n}{p+n}$ to $\frac{m}{p}$. How would you help this student?
17. Jillian says she learned that 17 divided by 5 can be written as $17 \div 5 = 3$ R2, but she thinks that writing $17 \div 5 = \frac{17}{5} = 3\frac{2}{5}$ is much better. How do you respond?

Review Problems

18. Perform each of the following computations. Leave your answers in simplest form or as mixed numbers.
 a. $\frac{-3}{16} + \frac{7}{4}$
 b. $\frac{1}{6} + \frac{-4}{9} + \frac{5}{3}$
 c. $\frac{-5}{2^3 \cdot 3^2} - \frac{-5}{2 \cdot 3^3}$
 d. $3\frac{4}{5} + 4\frac{5}{6}$
 e. $5\frac{1}{6} - 3\frac{5}{8}$
 f. $-4\frac{1}{3} - 5\frac{5}{12}$
19. Each student at Sussex Elementary School takes one foreign language. Two-thirds of the students take Spanish, $\frac{1}{9}$ take French, $\frac{1}{18}$ take German, and the rest take some other foreign language. If there are 720 students in the school, how many do not take Spanish, French, or German?
20. Find each sum or difference; simplify if possible.
 a. $\frac{3x}{xy^2} + \frac{y}{x^2}$
 b. $\frac{a}{xy^2} - \frac{b}{xyz}$
 c. $\frac{a^2}{a^2 - b^2} - \frac{a-b}{a+b}$

21. Determine which of the following is always correct.

a. $\frac{ab + c}{b} = a + c$

b. $\frac{a + b}{a + c} = \frac{b}{c}$

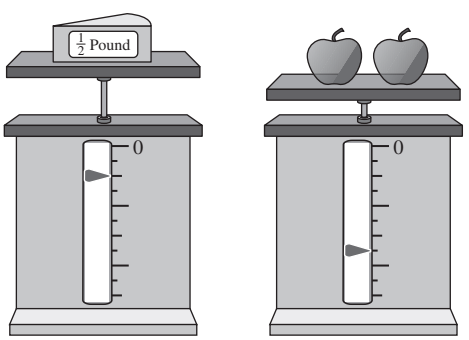
c. $\frac{ab + ac}{ac} = \frac{b + c}{c}$

National Assessments

National Assessment of Educational Progress (NAEP) Questions

Both figures below show the same scale. The marks on the scale have no labels except the zero point.

The weight of the cheese is $\frac{1}{2}$ pound. What is the total weight of the two apples?



NAEP, Grade 8, 2007

Jim has $\frac{3}{4}$ of a yard of string which he wishes to divide into pieces each $\frac{1}{8}$ of a yard long. How many pieces will he have?

A. 3
B. 4
C. 6
D. 8

NAEP, Grade 8, 2003

Nick has a whole pizza.



Nick says he will eat $\frac{1}{2}$ of the pizza.

He says he will give $\frac{3}{8}$ of the pizza to Sam and $\frac{3}{8}$ of the pizza to Joe.

Can Nick do what he says?

◯ Yes ◯ No

NAEP, Grade 4, 2013

6-4 Proportional Reasoning

6-4 Objectives

Students will be able to understand and explain

- Ratios and their relation to rational numbers.
- Proportions and their properties.
- Constant of proportionality.
- Scaling.

Proportional reasoning is an extremely important concept taught in grades K–8. Proportionality has connections to most, if not all, of the other foundational middle-school topics and can provide a context to study these topics.

For grade 7, the *Common Core Standards* state that students should "analyze proportional relationships and use them to solve real-world and mathematical problems." Additionally we find that students should "decide whether two quantities are in a proportional relationship," "identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams and verbal descriptions," represent proportional relationships in equations," and "use proportional relationships to solve multistep ratio and percent problems." (p. 48)

Ratios are encountered in everyday life. For example, there may be a 2-to-3 ratio of Democrats to Republicans on a certain legislative committee, a friend may be given a speeding ticket for driving 69 miles per hour, or eggs may cost $2.40 a dozen. Each of these illustrates a **ratio**.

Definition of Ratio

A **ratio**, denoted as $\frac{a}{b}$, a/b, or $a:b$, where a and b are rational numbers, is a comparison of two quantities.

A ratio of 1:3 for boys to girls in a class means that the number of boys is $\frac{1}{3}$ that of girls; that is, there is 1 boy for every 3 girls. We could also say that the ratio of girls to boys is 3:1, or that there are 3 times as many girls as boys. Ratios can represent **part-to-whole** or **whole-to-part** comparisons. For example, if the ratio of boys to girls in a class is 1:3, then the ratio of boys (part) to children (whole) is 1:4. If there are b boys and g girls, then $\frac{b}{g} = \frac{1}{3}$ and $g = 3b$. Also, the ratio of boys to the entire class is $\frac{b}{b+g} = \frac{b}{b+3b} = \frac{b}{4b} = \frac{1}{4}$. We could also say that the ratio of all children (whole) to boys (part) is 4:1. Some ratios give **part-to-part** comparisons, such as the ratio of the number of boys to girls or the number of students to one teacher. For example, a school might say that the average ratio of students to teachers cannot exceed 24:1.

The ratio of 1:3 for boys to girls in a class does not tell us how many boys and how many girls there are in the class. It only tells us the relative size of the groups. There could be 2 boys and 6 girls, or 3 boys and 9 girls, or 4 boys and 12 girls, or some other numbers that give a ratio equal to $\frac{1}{3}$.

Example 25

There were 7 males and 12 females in the Dew Drop Inn on Monday evening. In the game room next door were 14 males and 24 females.

a. Express the number of males to females at the inn as a ratio (part-to-part).
b. Express the number of males to females at the game room as a ratio (part-to-part).
c. Express the number of males in the game room to the number of people in the game room as a ratio (part-to-whole).

Solution

a. The ratio is $\frac{7}{12}$.

b. The ratio is $\frac{14}{24} = \frac{7}{12}$.

c. The ratio is $\frac{14}{38} = \frac{7}{19}$.

Proportions

In a study, children were shown a picture of a carton of orange juice and were told that the orange juice was made from orange concentrate and water. Then they were shown two glasses—a large glass and a small glass—and they were told that both glasses were filled with orange juice from the carton. They were then asked if the orange juice from each of the two glasses would taste equally "orangey" or if one would taste more "orangey." About half of the students said they were not equally orangey. Of those about half the students said the larger glass would be more "orangey" and about half said the smaller glass would be more "orangey." These students may have been thinking of only one quantity—the water alone or the orange concentrate alone.

Suppose Recipe A for an orange drink calls for 2 cans of orange concentrate for every 3 cans of water. We could say that the ratio of cans of orange concentrate to cans of water is 2:3.

We represent this pictorially in Figure 22(a), where O represents a can of orange concentrate and W represents a can of water. In Figure 22(b) and (c), we continue the process of adding 2 cans of orange concentrate for every 3 cans of water.

Recipe A



Figure 22

From Figure 22 we could develop and continue the **ratio table**, as shown in Table 3.

Table 3

Cans of Orange Concentrate	2	4	6	8	10	12
Cans of Water	3	6	9	?	?	?

In Table 3, the ratios $\frac{2}{3}$ and $\frac{4}{6}$ are equal. The equation $\frac{2}{3} = \frac{4}{6}$ is a proportion. In general, we have the following definition.

Definition of a Proportion

A **proportion** is a statement that two given ratios are equal.

If Recipe B calls for 4 cans of orange concentrate for every 8 cans of water, then the ratio of cans of orange concentrate to cans of water for this recipe is 4:8. We picture this in Figure 23(a).

Recipe B

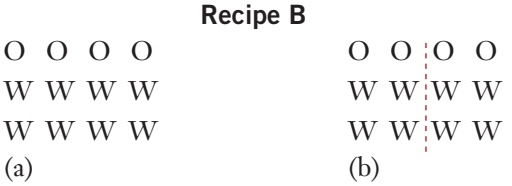


Figure 23

Which of the two recipes produces a drink that tastes more "orangey"? In Figure 22(a), we see that in Recipe A there are 2 cans of orange concentrate for every 3 cans of water. In Figure 23(a), we see that in Recipe B there are 4 cans of orange concentrate for every 8 cans of water.

To compare the two recipes, we need either the same number of cans of orange concentrate or the same number of cans of water. Either is possible. Figure 22(b) shows that for Recipe A there are 4 cans of orange concentrate for every 6 cans of water. In Recipe B, for 4 cans of orange concentrate there are 8 cans of water. Recipe B calls for more water per 4 cans of orange

concentrate, so it is less "orangey." An alternative is to observe that in Figure 23(b), Recipe B shows that there are 2 cans of orange concentrate for every 4 cans of water. We compare this with Figure 22(a), showing 2 cans of orange concentrate for every 3 cans of water, and reach the same conclusion.

From our work in Section 6-1, we know that $\frac{2}{3} = \frac{4}{6}$ because $2 \cdot 6 = 3 \cdot 4$. Hence $\frac{2}{3} = \frac{4}{6}$ is a proportion. Also $\frac{2}{3} \neq \frac{4}{8}$ because $2 \cdot 8 \neq 3 \cdot 4$; this is not a proportion. In general, we have the following theorem that follows from Theorem 6-2 developed in Section 6-1.

Theorem 6-20

If a, b, c, and d are rational numbers and $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} = \frac{c}{d} \text{ is a proportion if, and only if, } ad = bc.$$

The proportion $\frac{a}{b} = \frac{c}{d}$ may be read as "a is to b as c is to d."

Students in the lower grades typically experience problems that are *additive*. Consider the problem below.

Allie and Bente type at the same speed. Allie started typing first. When Allie had typed 8 pages, Bente had typed 4 pages. When Bente has typed 10 pages, how many has Allie typed?

This is an example of an *additive* relationship. Students should reason that since the two people type at the same speed, when Bente has typed an additional 6 pages, Allie should have also typed an additional 6 pages, so she should have typed $8 + 6$, or 14, pages.

Next consider the following problem:

Carl can type 8 pages for every 4 pages that Dan can type. If Dan has typed 12 pages, how many pages has Carl typed?

If students try an *additive* approach, they will conclude that since Dan has typed 8 more pages than in the original relationship, then Carl should have typed an additional 8 pages for a total of 16 pages. However, the correct reasoning is that since Carl types twice as fast as Dan he will type twice as many pages as Dan. Therefore, when Dan has typed 12 pages, Carl has typed 24 pages. The relationship between the ratios is *multiplicative*. Another way to solve this problem is to set up the proportion $\frac{8}{4} = \frac{x}{12}$, where x is the number of pages that Carl will type, and solve for x. Because $\frac{8}{4} = \frac{8 \cdot 3}{4 \cdot 3} = \frac{24}{12}$, then $x = 24$ pages.

In the problem above, one term in the proportion is missing:

$$\frac{8}{4} = \frac{x}{12}$$

One way to solve the equation is to multiply both sides by 12, as follows:

$$\begin{aligned} \frac{8}{4} \cdot 12 &= \frac{x}{12} \cdot 12 \\ 8 \cdot 3 &= x \\ 24 &= x \end{aligned}$$

Another method of solution uses Theorem 6-20. This is often called the *cross-multiplication method*. The equation $\frac{8}{4} = \frac{x}{12}$ is a proportion if, and only if,

$$\begin{aligned} 8 \cdot 12 &= 4x \\ 96 &= 4x \\ 24 &= x. \end{aligned}$$

Example 26

If there are 3 cars for every 8 students at a high school, how many cars are there for 1200 students?

Solution We use the strategy of *setting up a table*, as shown in Table 4.

Table 4

Number of cars	3	x
Number of students	8	1200

The ratio of cars to students is always the same:

$$\begin{aligned} \text{Cars} \quad &\rightarrow \frac{3}{\text{Students} \rightarrow 8} = \frac{x}{1200} \\ 3 \cdot 1200 &= 8x \\ 3600 &= 8x \\ 450 &= x \end{aligned}$$

Thus, there are 450 cars.

Next consider two car rental companies where the rates for 1–4 days are given in Table 5.

Table 5

(a) Ace Car Rental		(b) Better Car Rental	
Days	**Cost**	**Days**	**Cost**
1	\$20	1	\$20
2	\$40	2	\$35
3	\$60	3	\$48
4	\$80	4	\$52

The first two days for Ace Car Rental rates can be used to write a proportion because $\frac{1\ day}{\$20} = \frac{2\ days}{\$40}$. In a proportion, the units of measure must be in the same relative positions. In this case, the numbers of days are in the numerators and the costs are in the denominators.

For the Better Car Rental we see that $\frac{1\ day}{\$20} \neq \frac{2\ days}{\$35}$, so a proportion is not formed.

Consider Table 6, which is a ratio table built from the values for Ace Car Rental.

Table 6

Days (d)	1	2	3	4
Cost (c)	20	40	60	80

The ratios $\frac{d}{c}$ are all equal, that is, $\frac{1}{20} = \frac{2}{40} = \frac{3}{60} = \frac{4}{80}$. Thus, each pair of ratios forms a proportion. In this case, $\frac{d}{c} = \frac{1}{20}$ for all values of c and d. This is expressed by saying that *d is proportional to c* or *d varies proportionally to c* or *d varies directly with c*. In this case, $d = \frac{1}{20}c$ for every c and d. The number $\frac{1}{20}$ is the **constant of proportionality**. We can say that *gas used by a car is proportional to the miles traveled* or *lottery profits vary directly with the number of tickets sold*.

Definition of Constant of Proportionality

If the variables x and y are related by the equality $y = kx \left(\text{or } k = \frac{y}{x}\right)$, then ***y* is proportional to *x*** and ***k*** is the **constant of proportionality** between y and x.

A central idea in proportional reasoning is that a relationship between two quantities is such that the ratio of one quantity to the other remains unchanged as the numerical values of both quantities change.

It is important to remember that in the ratio $a:b$ or $\frac{a}{b}$, a and b do not have to be whole numbers. For example, if in Eugene, Oregon, $\frac{7}{10}$ of the population exercise regularly, then $\frac{3}{10}$ of the population do not exercise regularly, and the ratio of those who do to those who do not is $\frac{7}{10} : \frac{3}{10}$. This ratio can be written 7:3.

It is important to pay special attention to units of measure when working with proportions. For example, if a turtle travels 5 in. every 10 sec, how many feet does it travel in 50 sec? If units of measure are ignored, we might set up the following proportion:

$$\frac{5}{10} = \frac{x}{50}$$

In this proportion the units of measure are not listed. A more informative proportion that often prevents errors is the following:

$$\frac{5 \text{ in.}}{10 \text{ sec}} = \frac{x \text{ in.}}{50 \text{ sec}}$$

This implies that $x = 25$. Consequently, since 12 in. = 1 ft, the turtle travels $\frac{25}{12}$ ft, or $2\frac{1}{12}$ ft, or 2 ft 1 in. in 50 sec.

Another approach for solving proportions uses the **scaling strategy**. Suppose we are asked whether it is better to buy 12 tickets for \$15.00 or 20 tickets for \$23.00. One way to approach the problem is to find the cost of a common number of tickets from each scenario.

Because LCM$(12, 20) = 60$, we could choose to find the cost of 60 tickets under each plan.

In the first plan, since 12 tickets cost \$15.00, then 60 tickets cost \$75.00.

In the second plan, since 20 tickets cost \$23.00, then 60 tickets cost \$69.00.

Therefore, the second plan is a better buy.

The **unit-rate strategy** for solving this problem involves finding the cost of one ticket under each plan and then comparing unit costs.

In the first plan, since 12 tickets cost \$15.00, then 1 ticket costs \$1.25.

In the second plan, since 20 tickets cost \$23.00, then 1 ticket costs \$1.15.



The grade 6 *Common Core Standards* state the following:

Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate how many lawns could be mowed in 35 hours? (p. 42)

NOW TRY THIS 6

Work the problem posed in the grade 6 *Common Core Standards*.

Example 27

Kai, Paulus, and Judy made \$2520 for painting a house. Kai worked 30 hr, Paulus worked 50 hr, and Judy worked 60 hr. They divided the money in proportion to the number of hours worked. If they all earn the same rate of pay, how much did each earn?

Solution Let x be the unit rate or the rate of pay per hour. Then $30x$ denotes the amount of money that Kai received; Paulus received $50x$ and Judy received $60x$. Because the total amount of money received is $30x + 50x + 60x$, we have

$$\begin{aligned} 30x + 50x + 60x &= 2520 \\ 140x &= 2520 \\ x &= 18 \text{ (dollars per hour)}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{Kai received } 30x &= 30 \cdot 18, \text{ or } \$540. \\ \text{Paulus received } 50x &= 50 \cdot 18, \text{ or } \$900. \\ \text{Judy received } 60x &= 60 \cdot 18, \text{ or } \$1080. \end{aligned}$$

Dividing each of the amounts by 18 shows that the proportion is as required.

Consider the proportion $\frac{15}{30} = \frac{3}{6}$. Because the ratios in the proportion are equal and because equal nonzero fractions have equal reciprocals, it follows that $\frac{30}{15} = \frac{6}{3}$. Also notice that the proportions are true because each results in $15 \cdot 6 = 30 \cdot 3$. In general, we have the following theorem.

Theorem 6-21

For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, with $a \neq 0$ and $c \neq 0$, $\frac{a}{b} = \frac{c}{d}$ if, and only if, $\frac{b}{a} = \frac{d}{c}$.

Consider $\frac{15}{30} = \frac{3}{6}$ again. Notice that $\frac{15}{3} = \frac{30}{6}$; that is, the ratio of the numerators is equal to the ratio of the corresponding denominators. In general, we have the following theorem.

Theorem 6-22

For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, with $c \neq 0$, $\frac{a}{b} = \frac{c}{d}$ if, and only if, $\frac{a}{c} = \frac{b}{d}$.

Scale Drawings

Ratios and proportions are used in scale drawings. For example, if the scale is 1:300, then the length of 1 cm in such a drawing represents 300 cm, or 3 m in true size. The **scale** is the ratio of the size of the drawing to the size of the object. The following example shows the use of scale drawings.

Example 28

The floor plan of the main floor of a house in Figure 24 is drawn in the scale of 1:300. Find the dimensions in meters of the living room.

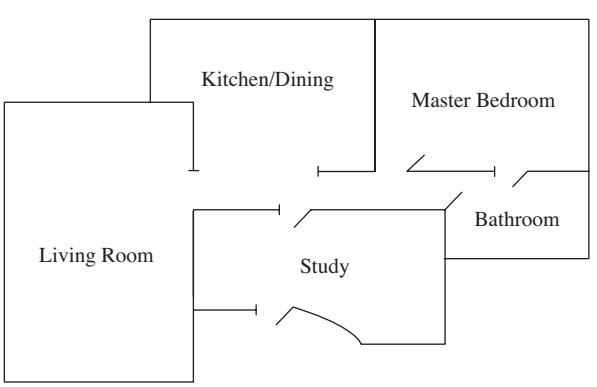


Figure 24

Solution If the dimensions of the living room are approximately 3 cm by 2 cm. Because the scale is 1:300, 1 cm in the drawing represents 300 cm, or 3 m in true size. Hence, 3 cm represents $3 \cdot 3 = 9$ m, and 2 cm represents $2 \cdot 3 = 6$ m. Hence, the dimensions of the living room are approximately 9 m by 6 m.

Example 29

The ratio of men to women at a party was 5:2 before 14 more women appeared. At that point, the ratio was 4:3. How many men and how many women were at the party?

Solution We find the solution using a *bar method.* In Figure 25 each unit of men and women contain the same number of people giving the ratio 5:2.

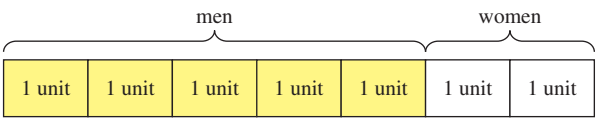


Figure 25

When 14 women joined the group, we have the situation in Figure 26(a). Additionally the ratio is now 4:3, and the number of people in each original unit has changed. The number of men has not changed so 5 of the original units equal 4 of the new ones.

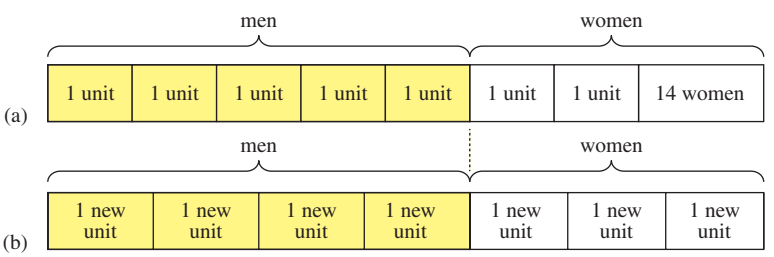


Figure 26

Thus, we have the following equations.

For the men: 5 original units $=$ 4 new units.

For the women: 2 original units $+$ 14 $=$ 3 new units.

Adding the two together we get

7 original units $+$ 14 $=$ 7 new units, or

1 original unit $+$ 2 $=$ 1 new unit.

Equivalently, 4 original units $+$ 8 $=$ 4 new units.

Substituting we have, 4 old units $+$ 8 $=$ 5 old units, or 8 $=$ 1 original unit.

In the Figure 25, there are 5 units of men at 8 men per unit $=$ 40 men, and 2 units of women at 8 women per unit for 16 women. Thus, there were 56 people at the party originally. When 14 more women joined, there were 70 total.

Assessment 6-4A

1. Answer the following regarding the English alphabet.
- **a.** Determine the ratio of vowels to consonants.
- **b.** What is the ratio of consonants to vowels?
- **c.** What is the ratio of consonants to letters?
- **d.** Write a word that has a ratio of 2:3 of vowels to consonants.

2. Solve for x in each of the following proportions.

a. $\frac{12}{x} = \frac{18}{45}$

b. $\frac{x}{7} = \frac{-10}{21}$

c. $\frac{5}{7} = \frac{3x}{98}$

d. $3\frac{1}{2}$ is to 5 as x is to 15.

3. **a.** If the ratio of boys to girls in a class is 2:3, what is the ratio of boys to all the students in the class? Why?
b. If the ratio of boys to girls in a class is m:n, what is the ratio of boys to all the students in the class?
c. If $\frac{3}{5}$ of the class are girls, what is the ratio of girls to boys?

4. There are approximately 2 lb of muscle for every 5 lb of body weight. For a 90-lb person, approximately how much of the weight is muscle?

5. Which is a better buy—4 grapefruits for 80¢ or 12 grapefruits for $2?

6. On a map, $\frac{1}{3}$ in. represents 5 mi. If New York and Aluossim are 18 in. apart on the map, what is the actual distance between them?

7. David reads 40 pages of a book in 50 min. How many pages should he be able to read in 80 min if he reads at a constant rate?

8. Two numbers are in the ratio 3:4. Find the numbers if
a. their sum is 98.
b. their product is 768.

9. Gary, Bill, and Carmella invested in a corporation in the ratio of 2:3:5, respectively. If they divide the profit of $82,000 proportionally to their investment, approximately how much will each receive?

10. Sheila and Dora worked $3\frac{1}{2}$ hr and $4\frac{1}{2}$ hr, respectively, on a programming project. They were paid $176 for the project. How much did each earn if they are both paid at the same rate?

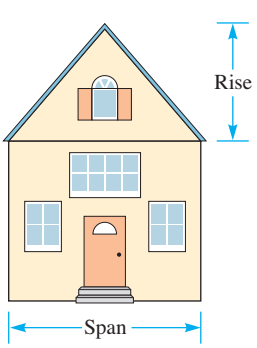
11. Vonna scored 75 goals in her soccer kicking practice. If her success-to-failure rate is 5:4, how many times did she attempt a goal?

12. Express each of the following ratios in the form $\frac{a}{b}$, where a and b are natural numbers.
a. $\frac{1}{6}:1$ **b.** $\frac{1}{3}:\frac{1}{3}$ **c.** $\frac{1}{6}:\frac{2}{7}$

13. Use Theorems 6-21 and 6-22 to write three other proportions that follow from the following proportion.

$$\frac{12¢}{36 \text{ oz}} = \frac{16¢}{48 \text{ oz}}$$

14. The *rise* and *span* for a house roof are identified as shown on the drawing. The *pitch* of a roof is the ratio of the rise to the half-span.
a. If the rise is 10 ft and the span is 28 ft, what is the pitch?



b. If the span is 16 ft and the pitch is $\frac{3}{4}$, what is the rise?

15. Gear ratios are used in industry. A gear ratio is the comparison of the number of teeth on two gears. When two gears are meshed, the revolutions per minute (rpm) are inversely proportional to the number of teeth; that is,

$$\frac{\text{rpm of large gear}}{\text{rpm of small gear}} = \frac{\text{Number of teeth on small gear}}{\text{Number of teeth on large gear}}.$$

a. The rpm ratio of the large gear to the small gear is 4:6. If the small gear has 18 teeth, how many teeth does the large gear have?
b. The large gear revolves at 200 rpm and has 60 teeth. How many teeth are there on the small gear, which has an rpm of 600?

16. A Boeing 747 jet is approximately 230 ft long and has a wingspan of 195 ft. If a scale model of the plane is about 40 cm long, approximately what is the model's wingspan?

17. A recipe calls for 1 tsp of mustard seeds, 3 c of tomato sauce, $1\frac{1}{2}$ c of chopped scallions, and $3\frac{1}{4}$ c of beans. If one ingredient is altered as specified, how must the other ingredients be changed to keep the proportions the same?
a. 2 c of tomato sauce
b. 1 c of chopped scallions
c. $1\frac{3}{4}$ c of beans

18. The electrical resistance of a wire, measured in ohms (Ω), is proportional to the length of the wire. If the electrical resistance of a 5-ft wire is 4 Ω, what is the resistance of 20 ft of the same wire?

19. In a photograph of a father and his daughter, the daughter's height is 2 cm and the father's height is 6 cm. If the father is actually 183 cm tall, how tall is the daughter?

20. The amount of gold in jewelry and other products is measured in karats (K), where 24K represents pure gold. The mark 14K on a chain indicates that the ratio between the mass of the gold in the chain and the mass of the chain is 14:24. If a gold ring is marked 18K and it weighs 4 oz, what is the value of the gold in the ring if pure gold is valued at $1800 per oz?

21. If Amber is paid $8 per hour for typing, the table shows how much she earns.

Hours (h)	1	2	3	4	5
Wages (w)	$8	$16	$24	$32	$40

a. How much would Amber make for a 40-hr work week?
b. What is the constant of proportionality?

22. **a.** In Room A there are 1 man and 2 women; in Room B there are 2 men and 4 women; and in Room C there are 5 men and 10 women. If all the people in Rooms B and C go to Room A, what will be the ratio of men to women in Room A?
b. Prove the following generalization of the proportions used in part (a).

$$\text{If } \frac{a}{b} = \frac{c}{d} = \frac{e}{f}, \text{ then } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a+c+e}{b+d+f}.$$

23. Use the bar method of Example 29 to solve the following exercise. One-half of the length of stick A is $\frac{2}{3}$ of the length of stick B. Stick B is 18 cm shorter than stick A. What is the length of both sticks?

24. A car travels about 26 miles on 1 gallon of gas while a truck travels about 250 miles on 14 gallons of gas. Which gets the better gas mileage?

25. Susan bikes 20 miles in 2 hours while Nick bikes 32 miles in 3 hours. Who travels faster?

26. There are 40 students in a classroom, and the desired ratio of students to computers is 3:1. How many computers are needed for the classroom to achieve the desired ratio?

27. The ratio of oblong tables to round tables at a conference is 5:1. The total number of tables at the conference is 102. How many of each type are there?

28. A recipe uses $1\frac{1}{2}$ c flour to make 2 dozen cookies. Is 4 c flour enough to make 6 dozen cookies with this recipe? Explain your answer.

Assessment 6-4B

1. Answer the following regarding the letters in the word *Mississippi*.
- **a.** Determine the ratio of vowels to consonants.
- **b.** What is the ratio of consonants to vowels?
- **c.** What is the ratio of consonants to letters?

2. Solve for x in each of the following proportions.

a. $\frac{5}{x} = \frac{30}{42}$ **b.** $\frac{x}{8} = \frac{-12}{32}$

c. $\frac{7}{8} = \frac{3x}{48}$ **d.** $3\frac{1}{2}$ is to 8 as x is to 24

3. There are 5 adult drivers to each teenage driver in Aluossim. If there are 12,345 adult drivers in Aluossim, how many teenage drivers are there?

4. A candle is 30 in. long. After burning for 12 min, the candle is 25 in. long. If it continues to burn at the same rate, how long will it take for the whole candle to burn?

5. A rectangular yard has a width-to-length ratio of 5:9. If the distance around the yard is 2800 ft, what are the dimensions of the yard?

6. A grasshopper can jump 20 times its length. If jumping ability in humans (height) were proportional to a grasshopper's (length), how far could a 6-ft-tall person jump?

7. Jim found out that after working for 9 months he had earned 6 days of vacation time. How many days per year does he earn at this rate?

8. At Rattlesnake School the teacher–student ratio is 1:30. If the school has 1200 students, how many additional teachers must be hired to change the ratio to 1:20?

9. At a particular time, the ratio of the height of an object that is perpendicular to the ground to the length of its shadow is the same for all objects. If a 30-ft tree casts a shadow of 12 ft, how tall is a tree that casts a shadow of 14 ft?

10. The following table shows several possible widths W and corresponding lengths L of a rectangle whose area is 10 ft^2.

Width (W) (Feet)	Length (L) (Feet)	Area (Square Feet)
$\frac{1}{2}$	20	$\frac{1}{2} \cdot 20 = 10$
1	10	$1 \cdot 10 = 10$
2	5	$2 \cdot 5 = 10$
$2\frac{1}{2}$	4	$2\frac{1}{2} \cdot 4 = 10$
4	$2\frac{1}{2}$	$4 \cdot 2\frac{1}{2} = 10$
5	2	$5 \cdot 2 = 10$
10	1	$10 \cdot 1 = 10$
20	$\frac{1}{2}$	$20 \cdot \frac{1}{2} = 10$

L

Area $= 10$ ft^2 W

- **a.** Use the values in the table and some additional values to graph the length L on the vertical axis versus the width W on the horizontal axis.
- **b.** What is the algebraic relationship between L and W?

11. Find three sets of x- and y-values for the following proportions.

$$\frac{4 \text{ tickets}}{\$20} = \frac{x \text{ tickets}}{\$y}$$

12. If rent is \$850 for each 2 weeks, how much is the rent for 7 weeks?

13. Leonardo da Vinci in his drawing *Vitruvian Man* showed that the man's armspan was equal to the man's height. Some other ratios are listed below.

$$\frac{\text{Length of hand}}{\text{Length of foot}} = \frac{7}{9}$$

$$\frac{\text{Distance from elbow to end of hand}}{\text{Distance from shoulder to elbow}} = \frac{8}{5}$$

$$\frac{\text{Length of hand}}{\text{Length of big toe}} = \frac{14}{3}$$

Using the ratios above, answer the following questions.
- **a.** If the length of a big toe is 6 cm, how long should the hand be?
- **b.** If a hand is 21 cm, how long is the foot?

14. On a city map, a rectangular park has a length of 4 in. If the actual length and width of the park are 300 ft and 200 ft, respectively, how wide is the park on the map?

15. Jim's car will travel 240 mi on 15 gal of gas. How far can he expect to go on 3 gal of gas?

16. Some model railroads use an O scale in replicas of actual trains. The O scale uses the ratio 1 in./48 in. How many feet long is the actual locomotive if an O scale replica is 18 in. long?

17.
- **a.** On an American flag, what is the ratio of stars to stripes?
- **b.** What is the ratio of stripes to stars?

18. On an American flag, the ratio of the length of the flag to its width must be 19:10.
- **a.** If a flag is to be $9\frac{1}{2}$ ft long, how wide should it be?
- **b.** The flag that was placed on the Moon measured 5 ft by 3 ft. Does this ratio form a proportion with the official length-to-width ratio? Why?

19. If $\frac{x}{y} = \frac{a}{b}, a \neq 0, x \neq 0$, is true, what other proportions do you know are true?

20. If a certain recipe takes $1\frac{1}{2}$ c flour and 4 c milk, how much milk should be used if the cook only has 1 c flour?

21. To estimate the number of fish in a lake, scientists use a tagging and recapturing technique. A number of fish are captured, tagged, and then released back into the lake. After a while, some fish are captured and the number of tagged fish is counted.

Let T be the total number of fish captured, tagged, and released into the lake, n the number of fish in a recaptured sample, and t the number of fish found tagged in that sample. Finally, let x be the number of fish in the lake. The assumption is that the ratio between tagged fish and the total number of fish in any sample is approximately the same, and hence scientists assume $\frac{t}{n} = \frac{T}{x}$. Suppose 173 fish were captured, tagged, and released. Then 68 fish were recaptured and among them 21 were found to be tagged. Estimate the number of fish in the lake.

22. A car travels about 36 miles on 1 gallon of gas while a truck travels about 200 miles on 14 gallons of gas. Which gets the better gas mileage?

23. Susan bikes 18 miles in 2 hours while Nick bikes 30 miles in 3 hours. Who travels faster?

24. There are 32 students in a classroom, and the desired ratio of students to computers is 3:1. How many computers are needed for the classroom to achieve the desired ratio?

25. The ratio of oblong tables to round tables at a conference is 6:1. The total number of tables at the conference is 112. How many of each type are there?

26. A recipe uses $1\frac{1}{2}$ c flour to make 3 dozen cookies. Is 4 c flour enough to make 6 dozen cookies with this recipe? Explain your answer.

27. Jennifer weighs 160 lb on Earth and 416 lb on Jupiter. Find Amy's weight on Jupiter if she weights 120 lb on Earth.

Mathematical Connections 6-4

Reasoning

1. Iris has found some dinosaur bones and a fossil footprint. The length of the footprint is 40 cm, the length of the thigh bone is 100 cm, and the length of the body is 700 cm.

- **a.** Iris found a new track that she believes was made by the same species of dinosaur. If the footprint was 30 cm long and if the same ratio of foot length to body length holds, how long is the dinosaur?
- **b.** In the same area, Iris also found a 50-cm thigh bone. Do you think this thigh bone belonged to the same dinosaur that made the 30-cm footprint that Iris found? Why or why not?

2. Suppose a 10-in. circular pizza costs \$4.00. To find the price, x, of a 14-in. circular pizza, is it correct to set up the proportion $\frac{x}{4} = \frac{14}{10}$? Why or why not?

3. Prove that if $\frac{a}{b} = \frac{c}{d}$ and $a \neq {}^{-}b$, then the following are true.

- **a.** $\frac{a+b}{b} = \frac{c+d}{d}\left(\textit{Hint: } \frac{a}{b} + 1 = \frac{c}{d} + 1\right)$
- **b.** $\frac{a}{a+b} = \frac{c}{c+d}$
- **c.** $\frac{a-b}{a+b} = \frac{c-d}{c+d}$
- **d.** $\frac{d}{c-d} = \frac{b}{a-b}$.

4. Nell said she can tell just by looking at the ratios 15:7 and 15:8 that these do not form a proportion. Is she correct? Why?

5. Sol had photographs that were 4 in. by 6 in., 5 in. by 7 in., and 8 in. by 10 in. Do the dimensions vary proportionately? Explain why.

6. Can $\frac{a}{b}$ and $\frac{a+b}{b}$ ever form a proportion? Why?

7. In a condo complex, $\frac{2}{3}$ of the men were married to $\frac{3}{4}$ of the women. What is the ratio of married people to the total adult population of the condo complex? Explain how you can obtain this ratio without knowing the actual number of men or women.

Open-Ended

8. List three real-world situations that involve ratio and proportion.

9. Find the ratio of Democrats to Republicans in the U.S. House of Representatives and the U.S. Senate. Determine how many each party would need to hold a majority and how many would be needed to stop a filibuster under the existing rules.

10. Most fertilizers used for gardening and crops are listed with a ratio of 1:2:1. Research the meaning of this ratio and find an acceptable mixture for flowers and foliage. If a fertilizer package showed a ratio of 10:20:10, what would be the meaning of this?

11. Research the golden ratio that the Greeks may have used in the design of the Parthenon. Write a report on this ratio and include a drawing of a golden rectangle.

Cooperative Learning

12. In *Gulliver's Travels* by Jonathan Swift we find the following:

The seamstresses took my measure as I lay on the ground, one standing at my neck and another at mid-leg, with a strong cord extended, that each held by the end, while the third measured the length of the cord with a rule of an inch long. Then they measured my right thumb and desired no more; for by a mathematical computation, that twice around the thumb is once around the wrist, and so on to the neck and the waist; and with the help of my old shirt, which I displayed on the ground before them for a pattern, they fitted me exactly.

- **a.** Explore the measurements of those in your group to see if you believe the ratios mentioned for Gulliver.
- **b.** Suppose the distance around a person's thumb is 9 cm. What is the distance around the person's neck?

c. What ratio could be used to compare a person's height to armspan?
d. Do you think there is a ratio between foot length and height? If so, what might it be?
e. Estimate other body ratios and then see how close you are to actual measurements.

Connecting Mathematics to the Classroom

13. Mary is working with measurements and writes the following proportion.

$$\frac{12 \text{ in.}}{1 \text{ ft}} = \frac{5 \text{ ft}}{60 \text{ in.}}$$

How would you help her?

14. Nora said she can use division to decide whether two ratios form a proportion; for example, 32:8 and 40:10 form a proportion because $32 \div 8 = 4$ and $40 \div 10 = 4$. Is she correct? Why?

15. Al is 5 ft tall and has a shadow that is 18 in. long. At the same time, a tree has a shadow that is 15 ft long. Al sets up and solves the proportion as follows:

$$\frac{5 \text{ ft}}{15 \text{ ft}} = \frac{18 \text{ in.}}{x}, \quad \text{so } x = 54 \text{ in.}$$

How would you help him?

16. Mandy read that the arm of the Statue of Liberty is 42 ft long. She would like to know how long the Statue of Liberty's nose is. How would you advise her to proceed?

17. One student in the class says that her sister is in a school using the Singapore math materials and much of the work with fractions is done with bars. The student continues, saying, "I don't understand how bars can help me understand it with an exercise like that found in exercise 13 above." Can you help her?

Review Problems

18. If the numerator of a rational number is 6 times the denominator and the numerator is also 5 more than the denominator, what are the numerator and denominator?

19. Explain whether $\frac{3}{4}$ is a proper fraction or an improper fraction and why.

20. Explain why any integer is a rational number.

21. A student says that $\frac{20}{30}$ can be simplified by crossing out the 0s and that in general, this procedure works for simplifying fractions. Explain whether or not the statement is true using the rational number $\frac{25}{35}$.

22. In an old Sam Loyd puzzle, a watch is described as having stopped when the minute and hour hands formed a straight line and the second hand was not on 12. At what time can this happen?

23. If $\frac{2}{3}$ of a class was absent due to illness and $\frac{4}{5}$ of the school was absent due to illness, is the fractional portion of the absent class equivalent to the fractional portion of the school absenteeism?

24. Prove that $\frac{99}{98}$ is less than $\frac{97}{96}$.

25. **a.** Amal can finish a job in $\frac{1}{2}$ of a day working by herself. Her son Sharif can finish the same job in $\frac{1}{4}$ of a day working alone. How long will it take to finish the job if they work together?
b. If Amal can finish a job in a hours and Sharif in b hours, then how long will it take to finish the job if they work together?

National Assessments

National Assessment of Educational Progress (NAEP) Questions

Sarah has a part-time job at Better Burgers restaurant and is paid $5.50 for each hour she works. She has made the chart below to reflect her earnings but needs your help to complete it.

A. Fill in the missing entries in the chart.

Hours Worked	Money Earned (in dollars)
1	$5.50
4	
	$38.50
$7\frac{3}{4}$	$42.63

B. If Sarah works h hours, then in terms of h, how much will she earn?

NAEP, Grade 8, 2007

The length of a photograph is 5 inches and its width is 3 inches. The photograph is enlarged proportionally. The length of the enlarged photograph is 10 inches. What is the width of the enlarged photograph?

A. 6 inches
B. 7 inches
C. 9 inches
D. 15 inches
E. $16\frac{2}{3}$ inches

NAEP, Grade 8, 2013

Hint for Solving the Preliminary Problem

Find the fraction of the whole washer that the two holes represent and then find the portion not represented by the holes. This should aid in finding the area of the washer.

Chapter 6 Summary

KEY CONCEPTS	DEFINITIONS, DESCRIPTIONS, AND THEOREMS
Section 6-1	
Rational number (p. 258)	A number in the form $\frac{a}{b}$, or a/b, where a and b are integers and $b \neq 0$. In the rational number $\frac{a}{b}$, a is the *numerator* and b is the *denominator*.
Uses of rational numbers (p. 258)	• Division problem • Portion, or part, of a whole • Ratio • Probability
Types of fractions (p. 259)	*Proper fraction*—any fraction $\frac{a}{b}$, where $0 \leq a < b$; a proper fraction is always less than 1. *Improper fraction*—any fraction $\frac{a}{b}$, where $a \geq b > 0$; an improper fraction is always greater than or equal to 1. *Equivalent, or equal, fractions*—numbers that represent the same point on a number line.
Fundamental Law of Fractions (p. 263)	*Theorem:* If $\frac{a}{b}$ is a fraction and n is a nonzero rational number, then $\frac{a}{b} = \frac{an}{bn}$.
Simplest form (p. 263)	A rational number $\frac{a}{b}$ is in *simplest form*, or *lowest terms* if, and only if, $\text{GCD}(a, b) = 1$; that is, if a and b have no common factor greater than 1.
Equality of fractions (p. 266)	Two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $b \neq 0$ and $d \neq 0$, are *equal* (or *equivalent*) if, and only if, $ad = bc$.
Ordering of rational numbers (p. 267)	*Like denominators*—If a, b, and c are any integers and $b > 0$, then $\frac{a}{b} > \frac{c}{b}$ if, and only if, $a > c$. *Unlike denominators*—If a, b, c, and d are any integers such that $b > 0$ and $d > 0$, then $\frac{a}{b} > \frac{c}{d}$ if, and only if $ad > bc$.
Denseness property for rational numbers (p. 268)	Given any two different rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, there is another rational numbers between these two numbers. *Theorem:* Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers with positive denominators, where $\frac{a}{b} < \frac{c}{d}$. Then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.
Section 6-2	
Addition of rational numbers (p. 275)	*Like denominators*—If $\frac{a}{b}$ and $\frac{c}{b}$ are rational numbers, then $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. *Unlike denominators*—If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Section 6-2

Addition properties for rational numbers (p. 279)

- *Additive inverse*—for any rational number $\frac{a}{b}$, there exists a unique rational number $-\frac{a}{b}$ such that $\frac{a}{b} + \left(-\frac{a}{b}\right) = 0 = \left(-\frac{a}{b}\right) + \frac{a}{b}$.
- *Property of equality*—if $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers such that $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f}$ is any rational number, then $\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$.

Subtraction of rational numbers (p. 281)

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d}$ is the unique rational number $\frac{e}{f}$ such that $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$.

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$.

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.

Estimation with rational numbers (p. 283)

Round fractions to *convenient*, or *benchmark* numbers, such as $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{3}, \frac{3}{4}$, or 1.

Section 6-3

Multiplication of rational numbers (p. 292)

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Multiplication properties for rational numbers (p. 293)

- *Multiplicative identity*—the rational number 1 is the unique number such that for every rational number $\frac{a}{b}$, $1 \cdot \frac{a}{b} = \frac{a}{b} = \frac{a}{b} \cdot 1$.
- *Multiplicative inverse*—for any nonzero rational number $\frac{a}{b}$, the multiplicative inverse (reciprocal) is the unique rational number $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1 = \frac{b}{a} \cdot \frac{a}{b}$.
- *Property of 0*—let $\frac{a}{b}$ be any rational number. Then $\frac{a}{b} \cdot 0 = 0 = 0 \cdot \frac{a}{b}$.
- *Property of equality*—let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers such that $\frac{a}{b} = \frac{c}{d}$. Then $\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}$.
- *Properties of inequality*—let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers:

 If $\frac{a}{b} > \frac{c}{d}$ and $\frac{e}{f} > 0$, then $\frac{a}{b} \cdot \frac{e}{f} > \frac{c}{d} \cdot \frac{e}{f}$.

 If $\frac{a}{b} > \frac{c}{d}$ and $\frac{e}{f} < 0$, then $\frac{a}{b} \cdot \frac{e}{f} < \frac{c}{d} \cdot \frac{e}{f}$.

Distributive property of multiplication over addition or subtraction (p. 294)

Let $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ be any rational numbers. Then

$$\frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f} \quad \text{and} \quad \frac{a}{b}\left(\frac{c}{d} - \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{c}{d} - \frac{a}{b} \cdot \frac{e}{f}.$$

Division of rational numbers (p. 297)

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, with $\frac{c}{d} \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{e}{f}$ if, and only if, $\frac{e}{f}$ is the unique rational number such that $\frac{c}{d} \cdot \frac{e}{f} = \frac{a}{b}$.

Algorithm for Division of Fractions (p. 299)

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, with $\frac{c}{d} \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

Section 6-3

a^m (p. 301)	If a is any rational number and m is any natural number, then $a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors}}$.
a^{-n} (p. 302)	For any nonzero rational number a and any natural number n, $a^{-n} = \frac{1}{a^n}$.
Exponents of rational numbers theorems (p. 302)	*Theorem:* For any nonzero rational number a and any integers m and n, $a^m \cdot a^n = a^{m+n}$. *Theorem:* For any nonzero rational number a and any integers m and n, $\frac{a^m}{a^n} = a^{m-n}$. *Theorem:* For any nonzero rational number a and any integers m and n, $(a^m)^n = a^{mn}$. *Theorem:* For any nonzero rational number $\frac{a}{b}$ and any integer m, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$. *Theorem:* For any nonzero rational number $\frac{a}{b}$ and any integer m, $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$. *Theorem:* For any nonzero rational numbers a and b and any integer m, $(a \cdot b)^m = a^m \cdot b^m$.

Section 6-4

Ratio (p. 311)	A comparison of two quantities a and b, where a and b are rational numbers, denoted as $\frac{a}{b}$, a/b, or $a{:}b$ and read as "*a to b*".
Proportion (p. 313)	A statement that two given ratios are equal. *Theorem:* If a, b, c, and d are rational numbers and $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if, and only if, $ad = bc$.
Constant of Proportionality (p. 316)	If the variables x and y are related by the equality $y = kx$, or $k = \frac{y}{x}$, then y *is proportional to* x and k is the *constant of proportionality* between y and x.
Proportionality theorems (p. 318)	*Theorem:* For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, with $a \neq 0$ and $c \neq 0$, $\frac{a}{b} = \frac{c}{d}$ if, and only if, $\frac{b}{a} = \frac{d}{c}$. *Theorem:* For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, with $c \neq 0$, $\frac{a}{b} = \frac{c}{d}$ if, and only if, $\frac{a}{c} = \frac{b}{d}$.

Chapter 6 Review

1. For each of the following, draw a diagram illustrating the fraction.

a. $\frac{3}{4}$ **b.** $\frac{2}{3}$ **c.** $\frac{3}{4} \cdot \frac{2}{3}$

2. Write three rational numbers equal to $\frac{5}{6}$.

3. Write each of the following rational numbers in simplest form.

a. $\frac{24}{28}$ **b.** $\frac{ax^2}{bx}$ **c.** $\frac{0}{17}$

d. $\frac{45}{81}$ **e.** $\frac{b^2 + bx}{b + x}$ **f.** $\frac{16}{216}$

g. $\frac{x + a}{x - a}$ **h.** $\frac{xa}{x + a}$

4. In each of the following pairs, replace the comma with $>$, $<$, or $=$ to make a true statement.

a. $\frac{6}{10}, \frac{120}{200}$ **b.** $\frac{-3}{4}, \frac{-5}{6}$

c. $\left(\frac{4}{5}\right)^{10}, \left(\frac{4}{5}\right)^{20}$ **d.** $\left(1+\frac{1}{3}\right)^{2}, \left(1+\frac{1}{3}\right)^{3}$

5. Find the additive and multiplicative inverses for each of the following.

a. 3 **b.** $3\frac{1}{7}$

c. $\frac{5}{6}$ **d.** $-\frac{3}{4}$

6. Order the following numbers from least to greatest.

$$-1\frac{7}{8}, 0, -2\frac{1}{3}, \frac{69}{140}, \frac{71}{140}, \left(\frac{71}{140}\right)^{300}, \frac{1}{2}, \left(\frac{74}{73}\right)^{300}$$

7. Can $\frac{4}{5}\cdot\frac{7}{8}\cdot\frac{5}{14}$ be written as $\frac{4}{8}\cdot\frac{7}{14}\cdot\frac{5}{5}$ to obtain the same answer? Why or why not?

8. Use mental math to compute the following. Explain your method.

a. $\frac{1}{3}\cdot(8\cdot 9)$ **b.** $36\cdot 1\frac{5}{6}$

9. John has $54\frac{1}{4}$ yd of material.

a. If he needs to cut the cloth into pieces that are $3\frac{1}{12}$ yd long, how many pieces can he cut?

b. How much material will be left over?

10. Without actually performing the given operations, choose the most appropriate estimate (among the numbers in parentheses) for the following expressions.

a. $\dfrac{30\frac{3}{8}}{4\frac{1}{9}}\cdot\dfrac{8\frac{1}{3}}{3\frac{8}{9}}$ $\quad(15, 20, 8)$

b. $\left(\frac{3}{800}+\frac{4}{5000}+\frac{15}{6}\right)6$ $\quad(15, 0, 132)$

c. $\frac{1}{407}\div\frac{1}{1609}$ $\quad\left(\frac{1}{4}, 4, 0\right)$

11. Write a story problem that models $4\frac{5}{8}\div\frac{1}{2}$. Solve the problem by drawing appropriate diagrams.

12. Find two rational numbers between $\frac{3}{4}$ and $\frac{4}{5}$.

13. Suppose the $\boxed{\div}$ button on your calculator is broken, but the $\boxed{1/x}$ button works. Explain how you could compute $504792/23$.

14. Jim is starting a diet. When he arrived home, he ate $\frac{1}{3}$ of the half of a pizza that was left from the previous night. The whole pizza contains approximately 2000 calories. How many calories did Jim consume?

15. If a person got heads on a flip of a fair coin one-half the time and obtained 376 heads, how many times was the coin flipped?

16. If a person obtained 240 heads when flipping a coin 1000 times, what fraction of the time did the person obtain heads? Put the answer in simplest form.

17. If the University of New Mexico won $\frac{3}{4}$ of its women's basketball games and $\frac{5}{8}$ of its men's basketball games, explain whether it is reasonable to say that the university won $\frac{3}{4}+\frac{5}{8}$ of its basketball games.

18. The carvings of the faces at Mount Rushmore in South Dakota measure 60 ft from chin to forehead. If the distance from chin to forehead is typically 9 in., and the distance between the pupils of the eyes is typically $2\frac{1}{2}$ in., what is the approximate distance between the pupils on the carving of George Washington's head?

19. A student argues that the following fraction is not a rational number because it is not the quotient of two integers:

$$\frac{\frac{2}{3}}{\frac{3}{4}}$$

How would you respond?

20. Molly wants to fertilize 12 acres of park land. If it takes $9\frac{1}{3}$ bags for each acre, how many bags does she need?

21. If $\frac{2}{3}$ of all students in the academy are female and $\frac{2}{5}$ of those are blondes, what fraction describes the number of blond females in the academy?

22. Explain which is greater: $\frac{-11}{9}$ or $\frac{-12}{10}$.

23. Solve for x in each of the following.

a. $7^{x}=343$ **b.** $2^{-3x}=\frac{1}{512}$

c. $2x-\frac{5}{3}=\frac{5}{6}$ **d.** $x+2\frac{1}{2}=5\frac{2}{3}$

e. $\frac{20+x}{x}=\frac{4}{5}$ **f.** $2x+4=3x-\frac{1}{3}$

24. Write each of the following in simplest form. Leave all answers with positive exponents.

a. $\dfrac{(x^{3}a^{-1})^{-2}}{xa^{-1}}$ **b.** $\left(\dfrac{x^{2}y^{-2}}{x^{-3}y^{2}}\right)^{-2}$

25. Find each sum or difference.

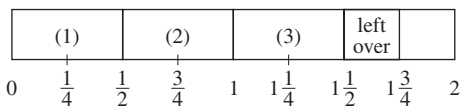
a. $\frac{3a}{xy^{2}}+\frac{b}{x^{2}y^{2}}$

b. $\frac{5}{xy^{2}}-\frac{2}{3x}$

c. $\frac{a}{x^{3}y^{2}z}-\frac{b}{xyz}$

d. $\frac{7}{2^{3}3^{2}}+\frac{5}{2^{2}3^{3}}$

26. Mike drew the following picture to find out how many pieces of ribbon $\frac{1}{2}$ yd long could be cut from a strip of ribbon $1\frac{3}{4}$ yd long.



From the picture he concluded that $1\frac{3}{4} \div \frac{1}{2}$ is 3 pieces with $\frac{1}{4}$ yd left over, so the answer is $3\frac{1}{4}$ pieces. He checked this using the algorithm $\frac{7}{4} \cdot \frac{2}{1} = \frac{14}{4} = 3\frac{1}{2}$ and is confused why he has two different answers. How would you help him?

27. Tom tossed a coin 30 times and got 17 heads.
- **a.** What is the ratio of heads to coin tosses?
- **b.** What is the ratio of heads to tails?
- **c.** What is the ratio of tails to heads?

28. Which bottle of juice is a better buy (cost per fluid ounce): 48 fl oz for \$3 or 64 fl oz for \$4?

29. Eighteen-karat gold contains 18 parts (grams) gold and 6 parts (grams) other metals. Amy's new ring contains 12 parts gold and 3 parts other metals. Is the ring 18-karat gold? Why?

30. A recipe for fruit salad serves 4 people. It calls for 3 oranges and 16 grapes. How many oranges and grapes do you need to serve 11 people?

31. If the scale on a drawing of a house is 1 cm to $2\frac{1}{2}$ m, what is the length of the house if it measures 3 cm on the scale drawing?

32. In water (H_2O), the ratio of the weight of oxygen to the weight of hydrogen is approximately 8:1. How many ounces of hydrogen are in 1 lb of water?

33. A manufacturer produces the same kind of computer chip in two plants. In the first plant, the ratio of defective chips to good chips is 15:100 and in the second plant, that ratio is 12:100. A buyer of a large number of chips is aware that some come from the first plant and some from the second. However, she is not aware of how many come from each. The buyer would like to know the ratio of defective chips to good chips in any given order. Can she determine that ratio? If so, explain how. If not, explain why not.

34. Suppose the ratio of the lengths of the sides in two squares is $1:r$. What is the ratio of their areas? ($A = s^2$)

35. The Grizzlies won 18 games and lost 7.
- **a.** What is the ratio of games won to games lost?
- **b.** What is the ratio of games won to games played?

36. Express each of the following as a ratio $\frac{a}{b}$ where a and b are whole numbers.

a. $\frac{1}{5} : 1$ **b.** $\frac{2}{5} : \frac{3}{4}$

37. The ratio of boys to girls in Mr. Good's class is 3 to 5, the ratio of boys to girls in Ms. Garcia's is the same, and you know that there are 15 girls in Ms. Garcia's class. How many boys are in Ms. Garcia's class?

38. If the ratio of the number of states of the United States using the *Common Core Standards* to those not using it is 9:10, how many states are not using these *Standards*?

39. From about 1978 to 1985, there was a trend towards conservatism among university professors with about $\frac{1}{20}$ of professors identifying themselves as strongly left-wing, about a third identifying themselves as liberals, about $\frac{1}{4}$ identifying themselves as moderates, $\frac{1}{4}$ as conservative, and $\frac{1}{20}$ as strongly conservative. What fraction identified themselves with none of the labels?

40. Since 1985, the fraction of liberal professors has grown steadily, with research finding somewhere between 7 to 9 liberals for each professor of another political persuasion. What statement with ratios could you make about professors of other political persuasions?

41. In an equilateral triangle, all sides have the same length and the perimeter is the sum of those lengths. What is the ratio of the perimeters of two equilateral (all sides of equal length) triangles whose sides each have lengths 6 cm and 10 cm respectively?

42. If a cup of coffee with 1 oz of cream is compared to another cup with $\frac{9}{10}$ oz cream, which has more cream?

43. In a package of tulip bulbs, the seller guarantees that at least $\frac{45}{99}$ of the bulbs will bloom. If you planted a package of 121 bulbs, at least how many would be expected not to bloom?

44. A woman's will decreed that her cats be shared among her three daughters as follows: $\frac{1}{2}$ of the cats to the eldest daughter, $\frac{1}{3}$ of the cats to the middle daughter, and $\frac{1}{9}$ of the cats to the youngest daughter. Since the woman had 17 cats, the daughters decided that they could not carry out their mother's wishes. The judge who held the will agreed to lend the daughters a cat so that they could share the cats as their mother wished. Now, $\frac{1}{2}$ of 18 is 9; $\frac{1}{3}$ of 18 is 6; and $\frac{1}{9}$ of 18 is 2. Since $9 + 6 + 2 = 17$, the daughters were able to divide the 17 cats and return the borrowed cat. They obviously did not need the extra cat to carry out their mother's bequest, but they could not divide 17 into halves, thirds, and ninths. Has the woman's will really been followed?

45. Prince Juan was allowed to take a number of bags of gold as he went into exile. However, a guard at the first bridge he crossed demanded half the bags of gold plus one more bag. Juan met this demand and proceeded to the next bridge. Guards at the second, third, and fourth bridges made identical demands, all of which the prince met. When Juan finally crossed all the bridges, a single bag of gold was left. With how many bags did Juan start?

Chapter 7

Rational Numbers as Decimals and Percent

7-1 Introduction to Finite Decimals

7-2 Operations on Decimals

7-3 Repeating Decimals

7-4 Percents and Interest



Preliminary Problem

On a photocopy machine, you can alter an original by entering a percentage for the copy. If you enter 200%, the machine will double all the dimensions of the original. Debbie set the machine to 150% and copied a 3 cm by 6 cm photo. She decided the copy was not large enough, so she placed the copy into the machine and left the setting on 150%. (a) What are the dimensions of the new copy? (b) If she wanted to reproduce another photo to the same size of the new copy and do it using only one step, what setting should she use? (c) If she had a copy that had been enlarged using a setting of 150%, what setting should she use to reduce the copy to its original size?

If needed, see Hint on page 383.

Although the Hindu-Arabic numeration system discussed in Chapter 3 was used in many regions around the sixth century, the extension of the system to decimals by Dutchman Simon Stevin did not take place until approximately a thousand years later. The only significant improvement in the system since Stevin's time has been in notation. Even today, there is no universally accepted form of writing a decimal. For example, in the United States, we write 6.75; in England, this number is written $6 \cdot 75$; and in Germany and France, it is written 6,75.

CCSS **In the grade 4 *Common Core Standards*, we find that students should:**

Use decimal notation for fractions with denominators 10 or 100. For example, rewrite 0.62 as $\frac{62}{100}$; describe a length as 0.62 meters; locate 0.62 on a number line diagram. (p. 31)

In Section 7-1, we explore relationships between fractions and decimals and show how decimals are an extension of the base-ten system. Later in the chapter, we consider operations on decimals, properties of decimals, and percents.

7-1 Introduction to Finite Decimals

7-1 Objectives

Students will be able to understand and explain

- Decimal notation.
- Connections between fractions and decimals using various models and strategies.
- Why terminating decimals occur and how to tell if a decimal will terminate.
- Ordering of terminating decimals.

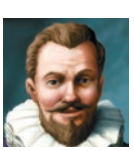
The word *decimal* comes from the Latin *decem*, meaning "ten." Thus, integers written in base ten are technically decimals, although most people do not think of them as such.

CCSS **In the grade 5 *Common Core Standards*, we find that students should:**

Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., $347.392 = 3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times \left(\frac{1}{10}\right) + 9 \times \left(\frac{1}{100}\right) + 2 \times \left(\frac{1}{1000}\right)$. (p. 35)

We encounter decimals when dealing with money. For example, if a bike costs \$128.95, the dot in \$128.95 is the **decimal point**. Because \$0.95 is $\frac{95}{100}$ of a dollar, $\$128.95 = 128 + \frac{95}{100}$.

Historical Note



In 1584, Simon Stevin (1548–1620), a Dutch quartermaster general, wrote *La Thiende (The Tenth)*, giving rules for computing with decimals. He not only stated these rules but also suggested practical applications for decimals and recommended that his government adopt them. To show place value, Stevin used circled numerals between digits. For example, he wrote 0.4789 as 4 ① 7 ② 8 ③ 9 ④. ●



THIENDE. 13

HET ANDER DEEL DER THIENDE VANDE WERCKINCHE.

I VOORSTEL VANDE VERGADERINGHE.

Wesende ghegeven Thiendetalen te vergaderen: hare Somme te vinden.

TGHEGHEVEN. Het sijn drie oirdens van Thiendetalen, welcker eerste 27⓪8①4③7③, de tweede, 37⓪6①7②5③, de derde, 875⓪7①8②2③, TBEGHEERDE. Wy moeten haer Somme vinden. WERCKING. Men sal de ghegheven ghetalen in oirden stellen als hier neven, die vergaderende naer de ghemeene manire der vergaderinghe van heelegetalen aldus:

⓪	①	②	③		
2	7	8	4	7	
	3	7	6	7	5
8	7	5	7	8	2
9	4	1	3	0	4

dollars. Also because \$0.95 is 9 dimes and 5 cents, 1 dime is $\frac{1}{10}$ of a dollar, and 1 cent is $\frac{1}{100}$ of a dollar, we see that \$0.95 is $9 \cdot \frac{1}{10} + 5 \cdot \frac{1}{100}$ dollars. Consequently,

$$128.95 = 1 \cdot 10^2 + 2 \cdot 10 + 8 \cdot 1 + 9 \cdot \frac{1}{10} + 5 \cdot \frac{1}{10^2}.$$

The digits in 128.95 correspond, respectively, to the following place values: 10^2, 10, 1, $\frac{1}{10^1}$, and $\frac{1}{10^2}$. Similarly, 12.61843 represents

$$1 \cdot 10 + 2 \cdot 1 + \frac{6}{10^1} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{4}{10^4} + \frac{3}{10^5}.$$

Since $1 \cdot 10 + 2 \cdot 1 = 12$ and

$$\begin{aligned} \frac{6}{10^1} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{4}{10^4} + \frac{3}{10^5} &= \frac{6 \cdot 10^4}{10^1 \cdot 10^4} + \frac{1 \cdot 10^3}{10^2 \cdot 10^3} + \frac{8 \cdot 10^2}{10^3 \cdot 10^2} + \frac{4 \cdot 10^1}{10^4 \cdot 10^1} + \frac{3}{10^5} \\ &= \frac{6 \cdot 10^4 + 1 \cdot 10^3 + 8 \cdot 10^2 + 4 \cdot 10^1 + 3}{10^5} \\ &= \frac{61{,}843}{100{,}000}, \end{aligned}$$

we can write $12.61843 = 1 \cdot 10 + 2 \cdot 1 + \frac{6}{10^1} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{4}{10^4} + \frac{3}{10^5}$ as $12\frac{61{,}843}{100{,}000}$.

Technically the decimal 12.61843 is read "twelve and sixty-one thousand eight hundred forty-three hundred-thousandths" although most people will say "twelve point six, one, eight, four, three." The word "and" is used only to represent the decimal point. Each place of a decimal may be named by its power of 10. For example, the places of 12.61843 can be named as shown in Table 1.

Table 1

1	2	·	6	1	8	4	3
Tens	Units	*and*	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths

In most U.S. school mathematics programs, fractions are introduced before decimals. Children see $2\frac{1}{4}$ before they see 2.25, although they both represent the same number. With the introduction of the metric system, decimals become even more important.

The three main ideas in this section are:

1. Decimal numerals are another way of writing selected fractions. Connections between fractions and decimals and how they are represented are important concepts in elementary school.
2. The decimal point indicates a separating point between the *units* digit and parts of a unit.
3. The base-ten place-value system extends two directions from the decimal point. In one direction, the values become greater; in the other direction, the values become lesser.

Decimals can be introduced with concrete materials. For example, suppose that a long in the base-ten block set represents 1 unit instead of letting the cube represent 1 unit. Then the cube represents $\frac{1}{10}$ and 5.4 could be represented as in Figure 1(a).

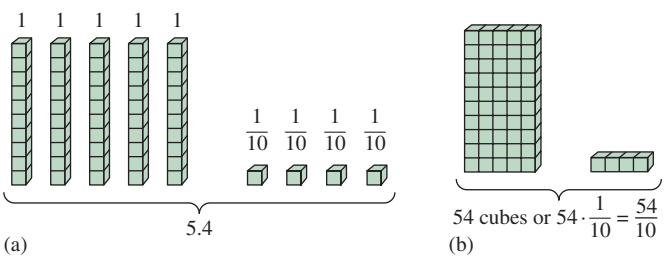


Figure 1

We could use Figure 1(b) to obtain a different interpretation of 5.4 and see that 5.4 is also equivalent to 54 tenths, or $\frac{54}{10}$. This equivalence can be stated symbolically as

$$5.4 = 5 + 0.4 = 5 + \frac{4}{10} = \frac{50}{10} + \frac{4}{10} = \frac{54}{10}.$$

This approach gives students a concrete connection between fractions and decimals.

NOW TRY THIS 1

In a set of base-ten blocks, let 1 flat represent 1 unit.

a. What does 1 long represent?
b. What does 1 cube represent?
c. Represent 1.23 using the blocks with these representations.

To represent a decimal such as 2.235, think of a block shown in Figure 2(a) as a unit. Then a flat represents $\frac{1}{10}$, a long represents $\frac{1}{100}$, and a cube represents $\frac{1}{1000}$. Using these objects, we show a representation of 2.235 in Figure 2(b).

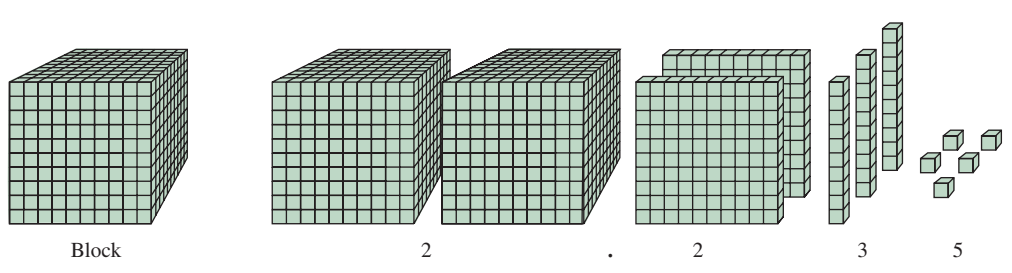


Figure 2

Table 2 shows other examples of decimals, their fractional meanings, and their common fraction notations.

Table 2

Decimal	Expanded Fractional Meaning	Common Fraction Notation
5.3	$5 + \frac{3}{10}$	$5\frac{3}{10} = \frac{53}{10}$
0.02	$0 + \frac{0}{10} + \frac{2}{100}$	$\frac{2}{100}$
2.0103	$2 + \frac{0}{10} + \frac{1}{100} + \frac{0}{1000} + \frac{3}{10{,}000}$	$2\frac{103}{10{,}000} = \frac{20{,}103}{10{,}000}$
$^{-}3.6$	$^{-}\left(3 + \frac{6}{10}\right)$	$^{-}3\frac{6}{10} = -\frac{36}{10}$

Decimals in standard form can be written in expanded form using place value and negative exponents. For example,

$$12.61843 = 1 \cdot 10^{1} + 2 \cdot 10^{0} + 6 \cdot 10^{-1} + 1 \cdot 10^{-2} + 8 \cdot 10^{-3} + 4 \cdot 10^{-4} + 3 \cdot 10^{-5}.$$



From the student page "Decimal Place Value" shown on page 334, we see that in grade 5, students learn equivalent forms of decimals.

Example 1 shows how to convert rational numbers whose denominators are powers of 10 to decimals.

Example 1

Convert each of the following to decimals.

a. $\frac{25}{10}$ **b.** $\frac{56}{100}$ **c.** $\frac{205}{10{,}000}$

Solution

a. $\frac{25}{10} = \frac{2 \cdot 10 + 5}{10} = \frac{2 \cdot 10}{10} + \frac{5}{10} = 2 + \frac{5}{10} = 2.5$

b. $\frac{56}{100} = \frac{5 \cdot 10 + 6}{10^2} = \frac{5 \cdot 10}{10^2} + \frac{6}{10^2} = 0 + \frac{5}{10^1} + \frac{6}{10^2} = 0.56$

c. $\frac{205}{10{,}000} = \frac{2 \cdot 10^2 + 0 \cdot 10^1 + 5}{10^4} = \frac{2 \cdot 10^2}{10^4} + \frac{0 \cdot 10^1}{10^4} + \frac{5}{10^4}$

$= \frac{2}{10^2} + \frac{0}{10^3} + \frac{5}{10^4} = 0 + \frac{0}{10^1} + \frac{2}{10^2} + \frac{0}{10^3} + \frac{5}{10^4} = 0.0205$

The conversions in Example 1 can also be performed using negative exponents. For example,

$$\frac{25}{10} = (2 \cdot 10 + 5) \cdot 10^{-1} = (2 \cdot 10) \cdot 10^{-1} + 5 \cdot 10^{-1} = 2 \cdot (10 \cdot 10^{-1}) + 5 \cdot 10^{-1}$$
$$= 2 \cdot 10^{0} + 5 \cdot 10^{-1} = 2 \cdot 1 + 5 \cdot 10^{-1} = 2.5.$$

School Book Page

Decimal Place Value

Lesson 1-4

Common Core

5.NBT.3.a Read and write decimals to thousandths using base-ten numerals, number names, and expanded form.

Decimal Place Value

How can you represent decimals?

Jo picked a seed from her flower. The seed measured 0.245 gram. What are some different ways you can represent 0.245?



ones	tenths	hundredths	thousandths
0 .	2	4	5

Standard Form: 0.245

The 5 is in the thousandths place. Its value is 0.005.

Expanded Form: 0.2 + 0.04 + 0.005

Word Form: two hundred forty-five thousandths

Another Example What are equivalent decimals?

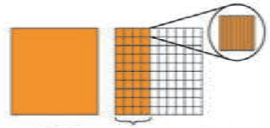
Equivalent decimals name the same amount.

Name two other decimals equivalent to 1.4.

One and four tenths is the same as one and forty hundredths.
So, 1.4 = 1.40.

One and four tenths is the same as one and four hundred thousandths.
So, 1.4 = 1.400.

So, 1.4 = 1.40 = 1.400.



Guided Practice*

MATHEMATICAL PRACTICES

Do you know HOW?

Write the word form for each number and give the value of the underlined digit.

1. 4.737
2. 9.806

Write each number in standard form.

3. 6 + 0.6 + 0.03 + 0.007
4. four and sixty-eight hundredths

Write two decimals that are equivalent to the given decimal.

5. 3.700
6. 5.60

Do you UNDERSTAND?

7. **Reason** The number 3.453 has two 3s. Why does each 3 have a different value?
8. How do you read the decimal point in word form?
9. **Generalize** José finished a race in 2.6 hours and Pavel finished the same race in 2.60 hours. Which runner finished the race first?

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14 *For another example, see Set D on page 23.

Independent Practice

Write the word form for each number and give the value of the underlined digit.

10. 2.300
11. 9.207
12. 1.982
13. 6.178

Write each number in standard form.

14. two and six thousandths
15. five and one hundred four thousandths
16. 3 + 0.3 + 0.009
17. 7 + 0.6 + 0.05 + 0.007

Write two decimals that are equivalent to the given decimal.

18. 2.200
19. 8.1
20. 9.50
21. 4.200

Problem Solving

MATHEMATICAL PRACTICES

22. **Be Precise** Kay is buying juice at the market. She has $9 and each bottle of juice costs $2. Does she have enough money to buy 5 bottles of juice? Explain.

23. **Model** Which point on the number line below best represents 0.368?



A W B X C Y D Z

24. The Borneo stick insect has a total length, including legs, of 21.5 inches. Write 21.5 in word form.
25. Worker leafcutter ants can measure 0.5 inches. Name two decimals that are equivalent to 0.5.
26. **Generalize** Why are 7.630 and 7.63 equivalent?

Lesson 1-4 15

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The ideas in Example 1 are reinforced with a calculator. In Example 1(b), press 5 6 ÷ 1 0 0 = and watch the display. Next divide by 10 again and look at the new placement of the decimal point. Once more, divide by 10 (which amounts to dividing the original number, 56, by 10,000) and note the placement of the decimal point. This leads to the general rule for dividing or multiplying a decimal by a positive integer power of 10:

To divide a decimal by 10^n where n is a positive integer, start at the decimal point, count n place values to the left, adjoining zeros if necessary, and insert the decimal point to the left of the nth place value counted.

To multiply a decimal by 10^n where n is a positive integer, move the decimal point n places to the right adjoining zeros, if needed.

The fractions in Example 1 are easy to convert to decimals because the denominators are powers of 10. If the denominator of a fraction is not a power of 10, as in $\frac{3}{5}$, we use the problem-solving strategy of *converting the problem to one that we already know how to do*. First, we rewrite $\frac{3}{5}$ as a fraction in which the denominator is a power of 10, and then we convert the fraction to a decimal.

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10} = 0.6$$

The reason for multiplying the numerator and the denominator by 2 is apparent when we observe that $10 = 5 \cdot 2$ or $2 \cdot 5$.

When the denominator of a fraction is a power of 10, converting the fraction to a decimal is a simple use of division of a whole number by a power of 10. Any power of 10 is of the form 10^n, and in general, because $10^n = (2 \cdot 5)^n = 2^n \cdot 5^n$, the prime factorization of the denominator that is a power of 10^n must be $2^n \cdot 5^n$. We use this idea in Example 2.

Example 2

Express each of the following as decimals.

a. $\frac{7}{2^6}$ **b.** $\frac{1}{2^3 \cdot 5^4}$ **c.** $\frac{1}{125}$ **d.** $\frac{7}{250}$

Solution

a. $\frac{7}{2^6} = \frac{7 \cdot 5^6}{2^6 \cdot 5^6} = \frac{7 \cdot 15{,}625}{(2 \cdot 5)^6} = \frac{109{,}375}{10^6} = 0.109375$

b. $\frac{1}{2^3 \cdot 5^4} = \frac{1 \cdot 2^1}{2^3 \cdot 5^4 \cdot 2^1} = \frac{2}{2^4 \cdot 5^4} = \frac{2}{(2 \cdot 5)^4} = \frac{2}{10^4} = 0.0002$

c. $\frac{1}{125} = \frac{1}{5^3} = \frac{1 \cdot 2^3}{5^3 \cdot 2^3} = \frac{8}{(5 \cdot 2)^3} = \frac{8}{10^3} = 0.008$

d. $\frac{7}{250} = \frac{7}{2 \cdot 5^3} = \frac{7 \cdot 2^2}{2 \cdot 5^3 \cdot 2^2} = \frac{28}{(2 \cdot 5)^3} = \frac{28}{10^3} = 0.028$



A calculator quickly converts fractions to decimals. For example, to find $\frac{7}{2^6}$, press 7 ÷ 2 y^x 6 =. To convert $\frac{1}{125}$ to a decimal, press 1 ÷ 1 2 5 = or press 1 2 5 1/x =.

The given calculator answers, as well as the answers in Example 2, are illustrations of **terminating decimals**—*decimals that can be written with a finite number of places to the right of the decimal point.* Not every rational number can be written as a terminating decimal. For example, if we attempt to rewrite $\frac{2}{11}$ as a terminating decimal using the method just developed, we try to find a natural number b such that the following holds:

$$\frac{2}{11} = \frac{2b}{11b}, \quad \text{where } 11b \text{ is a power of 10}$$

The only prime factors of a power of 10 are 2 and 5. Because $11b$ has 11 as a factor, by the Fundamental Theorem of Arithmetic, we cannot write $11b$ as a power of 10. Therefore, it seems that $\frac{2}{11}$ cannot be written as a terminating decimal. A similar argument using the Fundamental Theorem of Arithmetic holds in general, so we have the following theorem.

Theorem 7-1

A rational number $\frac{a}{b}$ in simplest form can be written as a terminating decimal if, and only if, the prime factorization of the denominator contains no primes other than 2 or 5.

Example 3

Which of the following fractions can be written as terminating decimals?

a. $\frac{7}{8}$ **b.** $\frac{11}{250}$ **c.** $\frac{21}{28}$ **d.** $\frac{37}{768}$

Solution

a. Because the denominator, 8, is 2^3, and the fraction is in simplest form, $\frac{7}{8}$ can be written as a terminating decimal.

b. Because the denominator, 250, is $2 \cdot 5^3$, and the fraction is in simplest form, $\frac{11}{250}$ can be written as a terminating decimal.

c. $\frac{21}{28}$ can be written in simplest form as $\frac{21}{28} = \frac{3 \cdot 7}{2^2 \cdot 7} = \frac{3}{2^2}$. The denominator is now 2^2, so $\frac{21}{28}$ can be written as a terminating decimal.

d. In simplest form, the fraction $\frac{37}{768} = \frac{37}{2^8 \cdot 3}$ has a denominator with a factor of 3, so $\frac{37}{768}$ cannot be written as a terminating decimal.

REMARK As Example 3(c) shows, to determine whether a rational number $\frac{a}{b}$ can be represented as a terminating decimal, we consider the prime factorization of the denominator *only* if the fraction is in simplest form.

Ordering Terminating Decimals

In the grade 4 *Common Core Standards*, we find that students should be able to:

compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model. (p. 31)

Given two numbers, the one located farther to the right on a standard horizontal number line is the greater. A terminating decimal is easily located on a number line because it can be represented as a rational number $\frac{a}{b}$ and b is a power of 10. For example, consider 0.56, or $\frac{56}{100}$. One way to think about $\frac{56}{100}$ is as the rightmost endpoint of the 56th part of a unit segment divided into 100 equal parts, as in Figure 3.

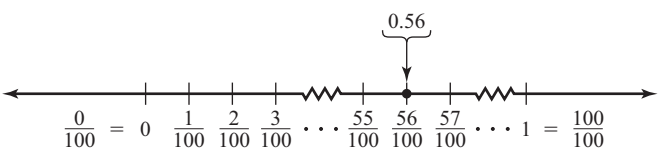


Figure 3

Two terminating decimals can be ordered by converting each to rational numbers in the form $\frac{a}{b}$ and determining which is greater. For example, because $0.36 = \frac{36}{100}$, $0.9 = 0.90 = \frac{90}{100}$, and $\frac{90}{100} > \frac{36}{100}$, it follows that $0.9 > 0.36$. One could also tell that $0.9 > 0.36$ because \$0.90 is 90¢ and \$0.36 is 36¢ and $90¢ > 36¢$.

IMAP Video

Watch Madison compare decimals and see a common misconception about ordering decimals.

Comparing 0.36 and 0.9 as above suggests a method for comparing any two decimals. For example, consider the decimals 0.345 and 0.1474. Note that 0.345 is a decimal with thousandths as the least place value, and 0.1474 has ten-thousandths as its least place value. To compare the decimals, we could write both with ten-thousandths as the least place value. Note that 0.345 could be written as 0.3450. Now, we have 0.3450 and 0.1474 both with ten-thousandths as the least place value. These decimals could be written as follows:

$$\frac{3450}{10{,}000} \quad \text{and} \quad \frac{1474}{10{,}000}$$

These two fractions can be compared by looking only at the numerators. Because 3450 is greater than 1474, $\frac{3450}{10{,}000} > \frac{1474}{10{,}000}$, and hence $0.345 > 0.1474$.

This process is frequently shortened by looking only at the decimals lined up as follows:

$$0.3450 = 0.345$$
$$0.1474 = 0.1474$$

With the decimals lined up in this fashion, 3 tenths is greater than 1 tenth. Thus, $0.345 > 0.1474$. This suggests a way to order terminating decimals without conversion to fractions. The steps used to compare terminating decimals are similar to those for comparing whole numbers.

1. *Line up the numbers by place value and append zeros, if necessary.*
2. *Start at the left and find the first place where the face values are different.*
3. *Compare these digits. The digit with the greater face value in this place represents the greater of the two numbers.*

Assessment 7-1A

1. Write each of the following as a sum in expanded place value form.
 a. 0.023 b. 206.06
 c. 312.0103 d. 0.000132
2. Rewrite each of the following as a decimal.
 a. $4 \cdot 10^3 + 3 \cdot 10^2 + 5 \cdot 10 + 6 + 7 \cdot 10^{-1} + 8 \cdot 10^{-2}$
 b. $4 \cdot 10^3 + 6 \cdot 10^{-1} + 8 \cdot 10^{-3}$
 c. $4 \cdot 10^4 + 3 \cdot 10^{-2}$
 d. $2 \cdot 10^{-1} + 4 \cdot 10^{-4} + 7 \cdot 10^{-7}$
3. Write each of the following as a decimal.
 a. Five hundred thirty-six and seventy-six ten-thousandths
 b. Three and eight thousandths
 c. Four hundred thirty-six millionths
 d. Five million and two tenths
4. Write each of the following numerals in words.
 a. 0.34
 b. 20.34
 c. 2.034
 d. 0.000034
5. Write each of the following terminating decimals as $\frac{a}{b}$ in simplest form where $a, b \in I$ and $b \neq 0$.
 a. 0.436 b. 25.16
 c. $^-316.027$ d. 28.1902
 e. $^-4.3$ f. $^-62.01$
6. Mentally determine which of the following can be represented as a terminating decimal.
 a. $\frac{4}{5}$ b. $\frac{61}{2^2 \cdot 5}$
 c. $\frac{3}{6}$ d. $\frac{1}{2^5}$
 e. $\frac{4}{34}$ f. $\frac{133}{625}$
7. Where possible, write each of the numbers in exercise 6 as a terminating decimal.
8. Seven minutes is part of an hour. If 7 min were to be expressed as a decimal part of an hour, explain whether it would be a terminating decimal.
9. Given the U.S. monetary system, what reason can you think of for having coins only for a penny, nickel, dime, quarter, and half-dollar as coins less than $1.00?
10. In each of the following, order the decimals from greatest to least:
 a. 13.4919, 13.492, 13.49183, 13.49199
 b. $^-1.453, ^-1.45, ^-1.4053, ^-1.493$
11. Write the numbers in each of the following sentences as decimals.
 a. A mite has body length of about fourteen thousandths of an inch.
 b. The Earth orbits the Sun once every three hundred sixty-five and twenty-four hundredths days.
12. Use a grid with 100 squares and represent 0.32. Explain your representation.
13. If the decimals 0.804, 0.84, and 0.8399 are arranged on a typical horizontal number line, which is furthest to the right?
14. Write a decimal that has a ten-thousandths place and is between 8.34 and 8.341.
15. a. Show that between any two terminating decimals, there is another terminating decimal.
 b. Argue that part (a) can be used to show that there are infinitely many terminating decimals between any two terminating decimals.
16. If "decimals" in other number bases work the same as in base ten, explain the meaning of 3.145_{six}.
17. The five top swimmers in an event had the following times:

Emily	64.54 sec	Kathy	64.02 sec
Molly	64.46 sec	Rhonda	63.54 sec
Martha	63.59 sec		

 List them in the order they placed with the "best" time first (least time is best here).
18. Suppose a carpenter's ruler was marked in thirty-seconds. What would be the fraction and its decimal equivalent for the mark on the ruler between $\frac{1}{16}$ and $\frac{2}{16}$?

Assessment 7-1B

1. Write each of the following as a sum in expanded place value form.
 a. 0.045 b. 103.03
 c. 245.6701 d. 0.00034
2. Rewrite each of the following as a decimal.
 a. $5 \cdot 10^3 + 2 \cdot 10^2 + 4 \cdot 10^{-1}$
 b. $4 \cdot 10^{-3} + 2 \cdot 10^4$
 c. $2 \cdot 10^2 + 3 \cdot 10^4$
 d. $10^{-3} + 10^{-5}$
3. Write each of the following as a decimal.
 a. Two thousand twenty-seven thousandths
 b. Two thousand and twenty-seven thousandths
 c. Two thousand twenty and seven thousandths
 d. Four hundred-thousandths
4. Write each of the following numerals in words.
 a. 0.45
 b. 2.035
 c. 45.0006
 d. 0.0000445

5. Write each of the following terminating decimals as $\frac{a}{b}$ in simplest form where $a, b \in I$ and $b \neq 0$.
 a. 28.32 **b.** 34.1736 **c.** $^{-}27.32$
6. Mentally determine which of the following represent terminating decimals.
 a. $\frac{4}{8}$ **b.** $\frac{1}{2^6}$
 c. $\frac{137}{625}$ **d.** $\frac{1}{17}$
 e. $\frac{3}{25}$ **f.** $\frac{14}{35}$
7. Where possible, write each of the numbers in exercise 6 as a terminating decimal.
8. What whole numbers of minutes (less than 60) could be expressed as terminating decimal parts of an hour?
9. If in a set of base-ten-blocks, one block represented $\frac{1}{10}$, what is the value of each of the following?
 a. 1 cube
 b. 1 flat
 c. 1 long
 d. 3 blocks, 1 long, and 4 cubes
10. In each of the following, order the decimals from least to greatest:
 a. 24.9419, 24.942, 24.94189, 24.94199
 b. $^{-}34.25$, $^{-}34.251$, $^{-}34.205$, $^{-}34.2519$
11. Write the numbers in each of the following sentences as decimals.
 a. A flea has a body length about one sixteenth of an inch.
 b. Venus orbits the sun about once every 224 and $\frac{7006}{10,000}$ days.
12. Use a grid with 100 squares and represent 0.23.
13. If the decimals 0.8114, 0.8119, 0.82 are arranged on a typical number line, which is furthest to the right?
14. Write a decimal that is between 8.345 and 8.3456.
15. Argue that there are infinitely many terminating decimals between any two specific terminating decimals such as 0.0625 and 0.125.
16. If "decimals" in other number bases work the same as in base ten, explain the meaning of 0.00334_{seven}.
17. The five top swimmers in an event had the following times:

Eddie	63.51 sec	Karl	62.99 sec
Marius	63.43 sec	Ricky	62.51 sec
Michael	62.56 sec		

 List them in the order they placed with the "best" time first (least time is best here).
18. A normal carpenter's ruler marked off in sixteenths. What would be the values at each of the marks in decimal form?

Mathematical Connections 7-1

Reasoning

1. Using Simon Stevin's notation (see Historical Note), how would you write the following decimals?
 a. 0.3256
 b. 0.0032
2. If 1 mL is 0.001 L, how should 18 mL be expressed as a terminating decimal number of liters (L stands for liter)?
3. Explain whether 1 day can be expressed as a terminating decimal part of a 365-day year.
4. Using the number line model to depict terminating decimals, explain whether or not you think there is a greatest terminating decimal less than 1. Explain.
5. Explain why you think a sign on a copy machine reading ".05¢ a copy" is put up by mistake.
6. Explain how you could use base-ten blocks to represent two and three hundred forty-five thousandths.
7. Explain why in Theorem 7-1 the rational number must be in simplest form before examining the denominator.
8. Explain why appending any number of zeros to the right of a finite decimal does not change its value. For example, $0.34 = 0.340 = 0.3400$.
9. How can you tell each of the following is incorrect just by looking at them?
 a. $\frac{1}{5} = 2.0$
 b. $\frac{2}{3} = 0.6$

Open-Ended

10. Determine how decimal notation is symbolized in different countries.
11. Examine three elementary school textbooks and report how the introductions of the topics of exponents and decimals differ, if they do. Report the grade level that decimals are introduced.

Cooperative Learning

12. Using base-five blocks similar to base-ten blocks, in your group discuss how you might introduce a "decimal" notation in base five.
13. In small groups, research the history of decimals to find contributions of the Arabs, the Chinese, and the people of Renaissance Europe. Explain whether you believe Stevin "invented" the decimal system.
14. With your group, play the *Where's the Best Place* game below. The object of the game is to build the greatest number possible.
 - Each player builds a number chart like the one shown. 0.☐☐☐☐☐
 - Number ten cards 0 through 9, shuffle and place them face down in a pile.
 - One player draws a card and each player must decide where to place the card in his or her number chart. After a

digit is written, it cannot be erased, and the digit cannot be used again in that game.

- The game is over when each place in the number chart is filled. The player with the greatest number wins the game.
 - **a.** Suppose a 9 is drawn first. Where should it be placed? Why?
 - **b.** Suppose a 0 is drawn first. Where should it be placed? Why?
 - **c.** The number charts for four players are shown below. (i) Can Ann win the game? (ii) Can Ben win the game?
 - **d.** Suppose 2 is drawn next. Who will win the game?

Ann	0	.	6	7		5	Ben	0	.	6	5		7
Carl	0	.	6	7	5		Dani	0	.	6	5	7	

Connecting Mathematics to the Classroom

15. A student claims that 0.86 is greater than 0.9 because 86 is greater than 9. How do you respond?

16. A student argues that fractions should no longer be taught once students learn how to work with decimal numbers. How do you respond?

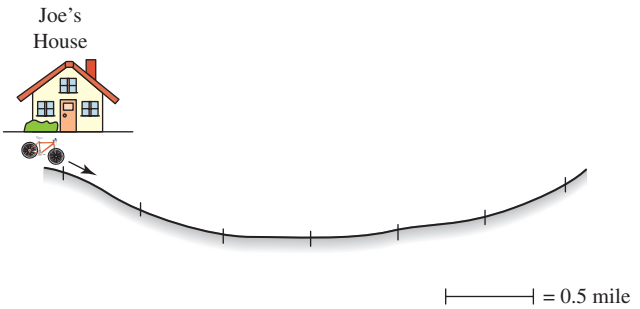
17. A student claims that $0.304 = 0.34$ because 0 is nothing. How do you respond?

18. A student claims that because 0.1, 0.01, 0.001, . . . is a geometric sequence so is 1.1, 1.01, 1.001, ... How do you respond?

National Assessments

National Assessment of Educational Progress (NAEP) Questions

Joe rode his bicycle from his house to his friend's house.
He rode 1.7 miles along the path below.
The path is marked every 0.5 mile.
Put an X on the path to show how far Joe rode to his friend's house.



NAEP, Grade 4, 2011

50-YARD RACE

Girl	Time (seconds)
Christa	8.28
Kelly	7.82
Lorinda	7.9
Sonja	8.31
Tanya	8.2

The table shows the times for five girls who ran a 50-yard race. The goal was for each girl to run the race in less than 8.25 seconds. Which girls met this goal?

A. Christa and Sonja
B. Christa and Tanya
C. Kelly, Lorinda, and Sonja
D. Kelly, Lorinda, and Tanya

NAEP, Grade 4, 2013

The diameter of a red blood cell, in inches, is 3×10^{-4}. This expression is the same as which of the following numbers?

A. 0.00003
B. 0.0003
C. 0.003
D. 3,000
E. 30,000

Did you use the calculator on this question?

NAEP, Grade 8, 2003

7-2 Operations on Decimals

7-2 Objectives

Students will be able to understand and explain

- How concrete models, drawings, and strategies can be used to develop efficient algorithms for decimal operations.
- Exponential and scientific notation for decimals.
- Strategies for decimal mental computations and estimations.

CCSS **Grade 5 *Common Core Standards* expects students to:**

add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. (p. 35)

Adding Decimals

To develop an algorithm for addition of terminating decimals, consider the sum 2.16 + 1.73. In elementary school, base-ten blocks are recommended to demonstrate such an addition problem. Figure 4 shows how the addition can be performed.

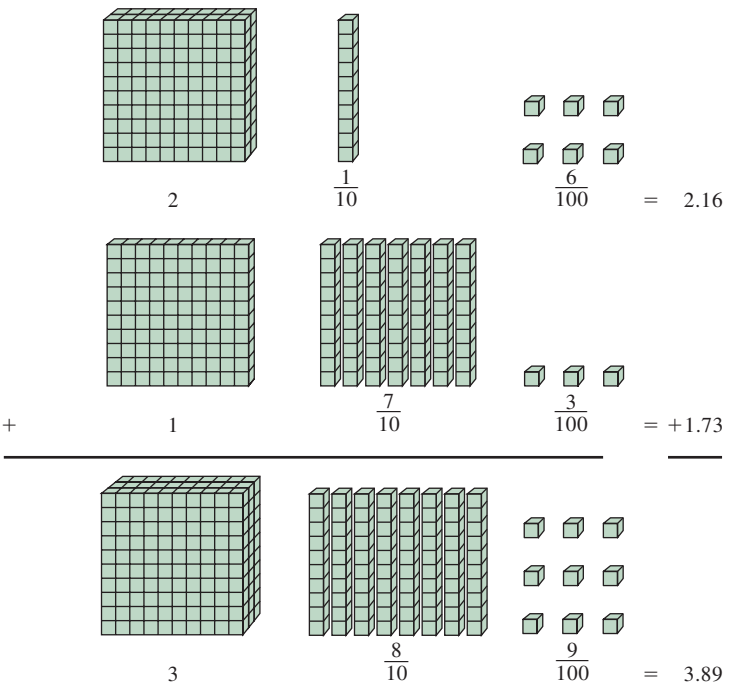


Figure 4

The computation in Figure 4 can be explained by *changing it to a problem we already know how to solve*; that is, to a sum involving fractions. We then use the commutative and associative properties of addition to aid in the computation as follows:

$$\begin{aligned} 2.16 + 1.73 &= \left(2 + \frac{1}{10} + \frac{6}{100}\right) + \left(1 + \frac{7}{10} + \frac{3}{100}\right) \\ &= (2 + 1) + \left(\frac{1}{10} + \frac{7}{10}\right) + \left(\frac{6}{100} + \frac{3}{100}\right) \\ &= 3 + \frac{8}{10} + \frac{9}{100} \\ &= 3.89 \end{aligned}$$

A different way of looking at the addition $2.16 + 1.73$ uses a different representation of the decimals as fractions with denominators that are powers of 10 as shown in the following:

$$2.16 = \frac{216}{100} \quad \text{and} \quad 1.73 = \frac{173}{100} \quad \text{so } 2.16 + 1.73 = \frac{216}{100} + \frac{173}{100}$$
$$= \frac{216 + 173}{100} = \frac{389}{100} = 3.89$$

Similarly, $2.1 + 1.73 = \frac{21}{10} + \frac{173}{100} = \frac{210}{100} + \frac{173}{100} = \frac{383}{100} = 3.83$.

In each case, the decimals are converted to fractions with the same denominator that is a power of 10. Once this is done, the numerators can be added as whole numbers to form the numerator of the answer fraction, which can then be converted back to a decimal. If these decimals are written as shown below, we "line up the decimal points" and add as we do with whole numbers, placing the decimal point in the appropriate place in the answer.

$$\begin{array}{r} 2.16 \\ +1.73 \\ \hline 3.89 \end{array} \quad \text{and} \quad \begin{array}{r} 2.1 \\ +1.73 \\ \hline 3.83 \end{array} \quad \text{or} \quad \begin{array}{r} 2.10 \\ +1.73 \\ \hline 3.83 \end{array}$$

In the preceding computations, we add units to units, tenths to tenths, and hundredths to hundredths. The lining-up-the-decimal-points technique works for both addition and subtraction.

Multiplying Decimals

Just as in the presented algorithms for adding terminating decimals by representing them as fractions, we develop and explain an algorithm for multiplication of decimals. Consider the product $4.62 \cdot 2.4$ shown below:

$$(4.62)(2.4) = \frac{462}{100} \cdot \frac{24}{10} = \frac{462}{10^2} \cdot \frac{24}{10^1} = \frac{462 \cdot 24}{10^2 \cdot 10^1} = \frac{11{,}088}{10^3} = 11.088$$

The answer to this computation was obtained by multiplying the whole numbers 462 and 24 and then dividing the result by 10^3.



On the student page shown on page 343, observe how multiplication of a whole number times a decimal can be accomplished by first shading 0.36 four times. The product is the sum of all the shaded portions: 1.44. In this manner, multiplication of a whole number times a decimal is interpreted almost exactly as the multiplication of a whole number by a whole number. Consider the shading method in the *Another Example* of the student page to show why $0.5 \times 0.3 = 0.15$.

An algorithm for multiplying decimals can be stated as follows:

> *If there are n digits to the right of the decimal point in one number and m digits to the right of the decimal point in a second number, multiply the two numbers, ignoring the decimals, and then place the decimal point so that there are n + m digits to the right of the decimal point in the product.*

Example 4

Compute each of the following:

a. $(6.2)(1.43)$ **b.** $(0.02)(0.013)$ **c.** $(1000)(3.6)$

Solution

a.
$$\begin{array}{r} 1.43 \quad \text{(2 digits after the decimal point)} \\ \times\ 6.2 \quad \text{(1 digit after the decimal point)} \\ \hline 286 \\ 858 \\ \hline 8.866 \quad (2+1, \text{or 3 digits after the decimal point}) \end{array}$$

b.
$$\begin{array}{r} 0.013 \\ \times\ 0.02 \\ \hline 0.00026 \end{array}$$

c.
$$\begin{array}{r} 3.6 \\ \times 1000 \\ \hline 3600.0 \end{array}$$

Some people feel that teaching pencil-and-paper computation with decimal multiplication and division cannot be justified in light of readily available calculators and computers. Instead they suggest that more emphasis be placed on estimation skills. Estimation would then be used to place the decimal point in the multiplication or division. With this in mind the following algorithm is suggested.

> *Ignore the decimal points and do the multiplication or division as if all numbers were whole numbers. When finished, place the decimal point in the result by estimation.*

For example, we would multiply $8.2 \cdot 2.63$ by multiplying $82 \cdot 263 = 21566$. Since $8 \cdot 2 = 16$ and $8 \cdot 3 = 24$, the decimal point is placed as 21.566.

Lesson 6-4

Common Core

5.NBT.7 ...Multiply... decimals to hundredths, using concrete models or drawings... properties of operations...; relate the strategy to a written method and explain the reasoning used.

Models for Multiplying Decimals

Number Talk grid paper

How can you multiply whole numbers and decimals?

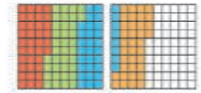
Bari displayed four paintings side-by-side in one row. Each painting has the same width. What is the total width of the 4 paintings?

Choose an Operation Multiply to find the total width of the four paintings.



What You Think

Find 0.36 × 4. Multiplying 0.36 × 4 is like adding 0.36 four times on a decimal model.



The product is the total area shaded.

0.36 × 4 = 1.44

What You Write

Multiply. Add the number of decimal places to place the decimals.

1 2
0.36 ← 2 decimal places
× 4 ← + 0 decimal places
1.44 ← 2 decimal places

The total width is 1.44 meters. You can also multiply using a calculator.

Press: 0.36 × 4 ENTER =

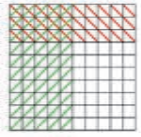
Display: 1.44

Another Example How can you multiply a decimal by a decimal?

Find 0.5 × 0.3. Use what you know about multiplying whole numbers to multiply decimals.

What You Think

Think of 0.5 as shading the first five columns of a decimal model. Think of 0.3 as shading the first 3 rows of the decimal model.



The product is the area where the shading overlaps.

0.5 × 0.3 = 0.15

What You Write

Multiply. Count the number of decimal places in each factor to place the decimal in the product.

1
0.5 ← 1 decimal place
× 0.3 ← + 1 decimal place
0.15 ← 2 decimal places

You can multiply using a calculator.

Press: 0.5 × 0.3 ENTER =

Display: 0.15

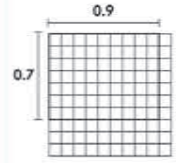
Explain It

1. How can you place the decimal in the product when you multiply with decimals?

Other Examples

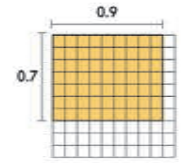
Use area models to multiply a decimal by a decimal.

Find 0.9 × 0.7



Use each factor as a side of a rectangle on a hundredths grid.

Shade the area of the rectangle bound by side lengths 0.9 and 0.7. Count the number of hundredths cells in the shaded area to find the product.



The shaded area contains 63 hundredths cells, so **0.7 × 0.9 = 0.63**.

Guided Practice*

MATHEMATICAL PRACTICES

Do you know HOW?

In **1** through **4**, place the decimal point in the product. You may use grids to help.

1. 2 × 0.32 = 64
2. 3 × 0.92 = 276
3. 4.2 × 5.4 = 2268
4. 5.7 × 0.03 = 171

In **5** and **6**, find the product.

5. 0.8 × 4
6. 0.7 × 21

Do you UNDERSTAND?

7. Explain when you need to add zeros to the left of a product.
8. **Use Tools** In the example at the top, which method is easier to use to find the product, paper and pencil or a calculator?

DIGITAL eTools www.pearsonsuccessnet.com

*For another example, see Set B on page 162.

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NOW TRY THIS 2

a. Try the above technique using estimation to place the decimal point with the parts of Example 4.

b. Example 4(c) suggests that multiplication by 1000, or 10^3, results in moving the decimal point in the product three places to the right. **(i)** Explain why this is true using expanded notation and the distributive property of multiplication over addition for rational numbers. **(ii)** In general, how does multiplication by 10^n, where n is a positive integer, affect the product? Why?

Scientific Notation



Many calculators display the decimals for the fractions $\frac{3}{45{,}689}$ and $\frac{5}{76{,}146}$ as $\boxed{6.5661319\quad -05}$ and $\boxed{6.566333\quad -05}$, respectively. The displays are in **scientific notation**. The first display is a notation for $6.5661319 \cdot 10^{-5}$ and the second for $6.566333 \cdot 10^{-5}$.

Scientists use scientific notation to express either very small or very large numbers. For example, "the Sun is 93,000,000 mi from Earth" is expressed as "the Sun is $9.3 \cdot 10^7$ mi from Earth." A micron, a metric unit of measure that is 0.000001 m, is written $1 \cdot 10^{-6}$ m.

Definition of Scientific Notation

In **scientific notation**, a positive number is written as the product of a number greater than or equal to 1 and less than 10 and an integer power of 10. To write a negative number in scientific notation, treat the number as a positive number and adjoin the negative sign in front of the result.

The following numerals are in scientific notation:

$$8.3 \cdot 10^8, \quad 1.2 \cdot 10^{10}, \quad {}^{-}7.32 \cdot 10^8, \quad \text{and} \quad 7.84 \cdot 10^{-6}$$

The numbers $0.43 \cdot 10^9$ and $12.3 \cdot 10^{-6}$ are not in scientific notation because 0.43 and 12.3 are not greater than or equal to 1 and less than 10. To write a number like 934.5 in scientific notation, we divide by 10^2 to get 9.345 and then multiply by 10^2 to retain the value of the original number:

$$934.5 = \left(\frac{934.5}{10^2}\right)10^2 = 9.345 \cdot 10^2$$

This results in moving the decimal point two places to the left (dividing by 10^2) and then multiplying by 10^2. Similarly, to write 0.000078 in scientific notation, we first multiply by 10^5 to obtain 7.8 and then divide by 10^5 (or, equivalently, multiply by 10^{-5}) to keep the original value:

$$0.000078 = (0.000078 \cdot 10^5)10^{-5} = 7.8 \cdot 10^{-5}$$

This amounts to moving the decimal point five places to the right and multiplying by 10^{-5}.

Example 5

Write each of the following in scientific notation.

a. 413,682,000 **b.** 0.0000231 **c.** 83.7
d. ${}^{-}10{,}000{,}000$ **e.** $0.34 \cdot 10^{-6}$

Solution

a. $413{,}682{,}000 = \left(\frac{413{,}682{,}000}{10^8}\right)10^8 = 4.13682 \cdot 10^8$

b. $0.0000231 = (0.0000231 \cdot 10^5)10^{-5} = 2.31 \cdot 10^{-5}$

c. $83.7 = 8.37 \cdot 10^1$ (Move the decimal point one place to the left and multiply by 10.)

d. $^-10{,}000{,}000 = \frac{^-10{,}000{,}000}{10^7} \cdot 10^7 = {}^-1 \cdot 10^7$

e. $0.34 \cdot 10^{-6} = (3.4 \cdot 10^{-1}) \cdot 10^{-6} = 3.4 \cdot (10^{-1} \cdot 10^{-6}) = 3.4 \cdot 10^{-7}$

Example 6

Convert the following to standard numerals.

a. $6.84 \cdot 10^{^-5}$ **b.** $3.12 \cdot 10^7$ **c.** $^-4.08 \cdot 10^4$

Solution

a. $6.84 \cdot 10^{^-5} = 6.84\left(\frac{1}{10^5}\right) = 0.0000684$

b. $3.12 \cdot 10^7 = 31{,}200{,}000$

c. $^-4.08 \cdot 10^4 = {}^-40{,}800$

Computations involving scientific notation make use of the laws of exponents. For example, $(5.6 \cdot 10^5)(6 \cdot 10^4)$ can be rewritten $(5.6 \cdot 6)(10^5 \cdot 10^4) = 33.6 \cdot 10^9$, which is $3.36 \cdot 10^{10}$ in scientific notation. Also,

$$(2.35 \cdot 10^{^-15})(2 \cdot 10^8) = (2.35 \cdot 2)(10^{^-15} \cdot 10^8) = 4.7 \cdot 10^{^-7}.$$



Calculators with an EE key can be used to represent numbers in scientific notation. For example, to find $(5.2 \cdot 10^{16})(9.37 \cdot 10^4)$, press

5 · 2 EE 1 6 × 9 · 3 7 EE 4 =.

Dividing Decimals

The set of nonzero whole numbers and the set of nonzero integers are not closed under division. However, the set of nonzero rational numbers is closed under division (why?).

The following examples deal with terminating decimals. Consider $75.45 \div 3$.

$$75.45 \div 3 = \frac{7545}{100} \div \frac{3}{1} = \frac{7545}{100} \cdot \frac{1}{3} = \frac{7545}{3} \cdot \frac{1}{100} = 2515 \cdot \frac{1}{100} = 25.15$$

By writing the dividend as a fraction whose denominator is a power of 10 and using the commutative property of multiplication, the division is changed into a division of whole numbers $(7545 \div 3)$ times a power of 10, namely, $\frac{1}{100}$, or $10^{^-2}$. A different method of accomplishing the

change of the division into whole number division is to multiply the dividend and the divisor by the same power of 10, as follows:

$$75.45 \div 3 = \frac{75.45}{3} = \frac{75.45 \cdot 10^2}{3 \cdot 10^2} = \frac{7545}{300} = 25\frac{45}{300} = 25\frac{15}{100} = 25.15$$

When the divisor is a whole number as shown above, the division can be handled as with whole number division and the decimal point placed directly over the decimal point in the dividend. When the divisor is not a whole number, as in $1.2032 \div 0.32$, we can obtain a whole-number divisor by expressing the quotient as a fraction and then multiplying the numerator and denominator of the fraction by a power of 10 as seen on the student page shown on page 347. This corresponds to rewriting the division problem in form (a) as an equivalent problem in form (b), as follows:



(a) $0.32\overline{)1.2032}$ (b) $32\overline{)120.32}$

In elementary school texts, this process is usually described as "moving" the decimal point two places to the right in both the dividend and the divisor. It is usually indicated with arrows, as shown in the following.

$$\begin{array}{r} 3.76 \\ 0.32\overline{)1.2032} \\ \underline{96} \\ 243 \\ \underline{224} \\ 192 \\ \underline{192} \\ 0 \end{array}$$ Multiply divisor and dividend by 100.

As seen above, in some cases of division of decimals, just as in division of whole numbers, the remainder for the division is 0.

Example 7

Compute each of the following.

a. $13.169 \div 0.13$ **b.** $9 \div 0.75$

Solution

a.
$$\begin{array}{r} 101.3 \\ 0.13\overline{)13.169} \\ \underline{13} \\ 16 \\ \underline{13} \\ 39 \\ \underline{39} \\ 0 \end{array}$$

b.
$$\begin{array}{r} 12 \\ 0.75\overline{)9.00} \\ \underline{75} \\ 150 \\ \underline{150} \\ 0 \end{array}$$

In Example 7(b), we appended two zeros in the dividend because $\frac{9}{0.75} = \frac{9 \cdot 100}{0.75 \cdot 100} = \frac{900}{75}$.

Lesson 7-6

Common Core

5.NBT.7 ... divide decimals to hundredths, using concrete models or drawings and strategies based on place value, ...; relate the strategy to a written method and explain the reasoning used.

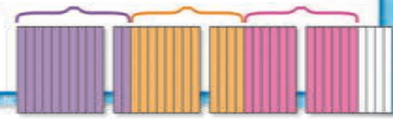
Dividing a Decimal by a Decimal

How can you divide a decimal using a decimal divisor?

Michelle purchases several bottles of water. Before tax is added, the total cost is $3.60 and the cost of each bottle is $1.20. How many bottles did she buy?

Choose an Operation

Divide 3.60 by 1.20.



Step 1

Estimate.

$4 \div 1 = 4$

Think of a power of 10 that will make the divisor a whole number.

$1.20\overline{)3.60}$

Multiply 1.20 by 10^2 or 100.

Step 2

Multiply the divisor and dividend by the same power of 10 and place the decimal point in the quotient.

$1.20\overline{)3.60}$

Find: $120\overline{)360}$.

Step 3

Divide.

$$\begin{array}{r} 3 \\ 120\overline{)360} \\ -\,360 \\ \hline 0 \end{array}$$

3 is close to 4. The answer is reasonable.

Michelle purchased 3 bottles of water.

Another Example How can you divide a decimal by a decimal?

Find $0.021 \div 0.35$.

Step 1

Use multiplication to estimate.

$\square \times 0.35 = 0.021$.

You know:

$1 \times 0.35 = 0.35$ and

$0.1 \times 0.35 = 0.035$.

So, the quotient is < 0.1.

Step 2

Multiply the divisor and dividend by the same power of 10 to make the divisor a whole number, and place the decimal point in the quotient.

$0.35\overline{)0.021} = 35\overline{)2.1}$

Step 3

Divide. Annex zeros as needed.

$$\begin{array}{r} 0.06 \\ 35\overline{)2.10} \\ -\,210 \\ \hline 0 \end{array}$$

Since $0.06 < 0.1$, the answer is reasonable.

Guided Practice*

MATHEMATICAL PRACTICES

Do you know HOW?

Find each quotient.

Tip *Use estimation to check answers for reasonableness.*

1. $2 \div 0.5$
2. $1.25 \div 0.25$
3. $2.1 \div 0.7$
4. $6.6 \div 0.3$

Do you UNDERSTAND?

5. **Look for Patterns** When dividing by a decimal, why can you multiply the divisor and dividend by the same power of 10?
6. **Communicate** Why is a zero added to the dividend in Step 3 of the example above?

Independent Practice

Leveled Practice In **7** through **10**, estimate each quotient.

7. $3.6 \div 0.7$
8. $9.8 \div 0.2$
9. $17.8 \div 3.1$
10. $89.05 \div 4.8$

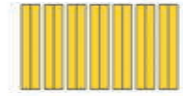
In **11** through **22**, find each quotient.

11. $62 \div 0.25$
12. $48.4 \div 0.02$
13. $0.02 \div 0.05$
14. $182.88 \div 0.08$
15. $107.25 \div 0.03$
16. $5.68 \div 8$
17. $624 \div 0.6$
18. $23.1 \div 0.7$
19. $24.2 \div 55$
20. $0.3567 \div 8.7$
21. $3.6 \div 9$
22. $4.788 \div 0.42$

Problem Solving

MATHEMATICAL PRACTICES

23. **Critique Reasoning/Arguments** Susan solves $1.4 \div 0.2$ using the diagram at the right. Is her reasoning correct? Explain her thinking.



24. **Communicate** Tim estimates that $60 \div 5.7$ is about 10. Will the actual quotient be greater than or less than 10? Explain.
25. **Reason** Dex estimates that $49{,}892 \div 0.89$ is about 5,000. Is his estimate reasonable? Why or why not?
26. **Think About the Structure** What value should you multiply the divisor and dividend by to begin dividing 89 by 0.04?

A 1 B 10 C 100 D 1,000

*For another example, see Set D on page 187.

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Example 8

An owner of a gasoline station must collect a gasoline tax of \$0.11 on each gallon of gasoline sold. One week, the owner paid \$1595.00 in gasoline taxes. The pump price of a gallon of gas that week was \$3.35.

a. How many gallons of gas were sold during the week?
b. What was the revenue after taxes for the week?

Solution

a. To find the number of gallons of gas sold during the week, we must divide the total gas tax bill by the amount of the tax per gallon.

$$\frac{1595}{0.11} = \frac{159500}{11} = 14{,}500$$

Thus, 14,500 gallons were sold.

b. To obtain the revenue after taxes, first determine the revenue before taxes. To do so, multiply the number of gallons sold by the cost per gallon:

$$(14{,}500)(\$3.35) = \$48{,}575$$

Next, subtract the gasoline taxes from the total revenue.

$$\$48{,}575 - \$1{,}595 = \$46{,}980$$

Thus, the revenue after gasoline taxes is \$46,980.

Next consider a division resulting in a nonzero remainder as shown below.

$$\begin{array}{r} 3.64 \quad \text{Quotient} \\ 0.33\overline{)1.2032} \\ \underline{99} \\ 213 \\ \underline{198} \\ 152 \\ \underline{132} \\ 20 \quad \text{Remainder} \end{array}$$

Note that all digits in the dividend are used, and the remainder 20 is shown at the bottom of the division. The quotient is 3.64 and the remainder shown as 20 actually represents 0.0020, as can be seen by the original decimal alignment. This could be verified by checking: $3.64 \cdot 0.33 + 0.0020 = 1.2032$. In the next section, we will show how to write $\frac{12032}{33}$ as an infinite repeating decimal.

Mental Computation



In the grade 5 *Common Core Standards*, we find that students "apply their understanding of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop fluency in these computations and make reasonable estimates of their results." (p. 33)

Some of the tools used for mental computations with whole numbers can be used to perform mental computations with decimals, as seen in the following:

1. *Breaking and bridging*

$$1.5 + 3.7 + 4.48$$
$$= 4.5 + 0.7 + 4.48$$
$$= 5.2 + 4.48$$
$$= 9.2 + 0.48 = 9.68$$

$1.5 + 3 = 4.5$

$4.5 + 0.7 = 5.2$

$5.2 + 4 = 9.2$

$9.2 + 0.48 = 9.68$

2. *Using compatible numbers*

Decimal numbers are compatible when they add to a whole number.

$$\begin{array}{r} 7.91 \\ 3.85 \\ 4.09 \\ +\ 0.15 \\ \hline \end{array} \qquad \begin{array}{r} 12 \\ \\ +\ 4 \\ \hline 16 \end{array}$$

$7.91 + 4.09$

$3.85 + 0.15$

$12 + 4$

3. *Making compatible numbers*

$$\begin{array}{rl} 9.27 = & 9.25 + 0.02 \\ +3.79 = & 3.75 + 0.04 \\ \hline & 13.00 + 0.06 = 13.06 \end{array}$$

4. *Balancing with decimals in subtraction*

$$\begin{array}{r} 4.63 \\ -\ 1.97 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{rl} 4.63 + 0.03 & = 4.66 \\ -(1.97 + 0.03) & = -2.00 \\ \hline & 2.66 \end{array}$$

5. *Balancing with decimals in division*

$$0.25\overline{)8}$$
$$\times 4 \qquad \times 4$$
$$32$$
$$1\overline{)32}$$

REMARK Balancing with decimals in division uses the property $\frac{a}{b} = \frac{ac}{bc}$, if $c \neq 0$.

Rounding Decimals

Frequently, it is not necessary to know the exact numerical answer to a question. For example, if we want to know the distance to the Moon or the population of New York City, the approximate answers of 239,000 mi and 8,300,000 people, respectively, may be adequate.

Often a situation determines how you should round. For example, suppose a purchase came to $38.65, and the cashier used a calculator to determine the 6% sales tax by multiplying $0.06 \cdot 38.65$. The display showed 2.319. Because the display is between 2.31 and 2.32, and it is closer to 2.32, the cashier rounds up the sales tax to $2.32.

Suppose a display of 8.7345649 needs to be reported to the nearest hundredth. The display is between 8.73 and 8.74 but is closer to 8.73, so we round it down to 8.73. Next suppose the

number 6.8675 needs to be rounded to the nearest thousandth. Notice that 6.8675 is exactly halfway between 6.867 and 6.868. *In such cases, it is common practice to round up*; therefore the answer to the nearest thousandth is 6.868. We write this as $6.8675 \approx 6.868$ and say 6.8675 is approximately equal to 6.868. You may encounter different rounding rules when the number is exactly half way between two values.

Example 9

Round each of the following numbers.

a. 7.456 to the nearest hundredth
b. 7.456 to the nearest tenth
c. 7.456 to the nearest unit
d. 7456 to the nearest thousand
e. 745 to the nearest ten
f. 74.56 to the nearest ten

Solution

a. $7.456 \approx 7.46$
b. $7.456 \approx 7.5$
c. $7.456 \approx 7$
d. $7456 \approx 7000$
e. $745 \approx 750$
f. $74.56 \approx 70$



Rounding can also be done on some calculators using the FIX key. If we want the number 2.3669 to be rounded to thousandths, we enter FIX 3. The display will show 0.000. If we then enter 2.3669 and press the = key, the display will show 2.367.

Estimating Decimal Computations Using Rounding

Rounded numbers can be useful for estimating answers to computations. For example, consider each of the following:

1. Karly goes to the grocery store to buy items that cost the following amounts. She estimates the total cost by rounding each amount to the nearest dollar and adding the rounded numbers.

$$\begin{array}{rcr} \$2.39 & \rightarrow & \$2 \\ 0.89 & \rightarrow & 1 \\ 6.13 & \rightarrow & 6 \\ 4.75 & \rightarrow & 5 \\ \underline{+\ 5.05} & \rightarrow & \underline{5} \\ & & \$19 \end{array}$$

 Thus, Karly's estimate for her grocery bill is $19.

2. Karly's bill for car repairs was $72.80, and she has a coupon for $17.50 off. She can estimate her total cost by rounding each amount to the nearest 10 dollars and subtracting.

$$\begin{array}{rcr} \$72.80 & \rightarrow & \$70 \\ \underline{-\ 17.50} & \rightarrow & \underline{-\ 20} \\ & & \$50 \end{array}$$

 Thus, an estimate for the repair bill is $50. Notice that this is not a very good estimate. A better estimate is obtained by finding $73 - 18$, which is $55.

3. Karly sees a flash of lightning and hears the thunder 3.2 sec later. She knows that sound travels at 0.33 km/sec. She may estimate the distance she is from the lightning by rounding the time to the nearest unit and the speed to the nearest tenth and multiplying.

$$\begin{array}{rcr} 0.33 & \rightarrow & 0.3 \\ \times\ 3.2 & \rightarrow & \times\ 3 \\ \hline & & 0.9 \end{array}$$

Thus, Karly estimates that she is approximately 0.9 km from the lightning.

An alternative approach is to recognize that $0.33 \approx \frac{1}{3}$ and 3.2 is close to 3.3, so an approximation using compatible numbers is $\left(\frac{1}{3}\right)3.3$, or 1.1 km.

4. Karly wants to estimate the number of miles she gets per gallon of gas. She started with a full tank. If she had driven 298 mi and it took 12.4 gal to refill the gas tank, she rounds and divides as follows:

$$12.4\overline{)298} \rightarrow 12\overline{)\begin{array}{r}25\\300\end{array}}$$

Thus, she estimates about 25 mi/gal.

5. Karly wanted to place the decimal point in a product resulting from the multiplication estimation when she bought 21.45 lb of sesame seeds to grind for tahini at $3.40 per pound. The multiplication (without a decimal) resulted in 7293. She knew that very rough estimates could be obtained by rounding 21.45 to 20 and $3.40 to $3. The product had to be in the neighborhood of 20×3, or 60. So her placement of the decimal point was $72.93.

NOW TRY THIS 3

Other estimation strategies, such as front-end, clustering, and grouping to convenient numbers, that we investigated with whole numbers also work with decimals. Take Karly's grocery store bill in exercise 1 on the preceding page and use a front-end-with-adjustment strategy to estimate the bill.

Do you prefer rounding or front-end estimation? Why?

Round-off Errors

Round-off errors are typically compounded when computations are involved. For example, if two distances 42.6 mi and 22.4 mi are rounded to the nearest tenth, then the sum of the distances appears to be $42.6 + 22.4$, or 65.0 mi. To the nearest hundredth, the distances might have been more accurately reported as 42.55 and 22.35 mi, respectively. The sum of these distances is 64.9 mi. Alternatively, the original distances may have been rounded from 42.64 and 22.44 mi. The sum now is 65.08, or 65.1 rounded to the nearest tenth. The original sum of 65.0 mi is between 64.9 and 65.1 mi, but the exact answer need not be 65.0 mi to the nearest tenth.

The *greatest possible error* in measuring is defined as one half of the measuring unit. For example, if we measure a length to be 4.7 cm, then because the measurement was made to the nearest tenth, the greatest possible error is $\frac{1}{2}$ of 0.1 cm or 0.05 cm.

Similar errors arise in other arithmetic operations.

When computations are done with approximate numbers, the final result should not be reported using more significant digits than the number used with the fewest significant digits.

Nonzero digits are always significant. Zeros before other digits are non significant. Zeros between other nonzero digits are significant. Zeros to the right of a decimal point are significant.

Assessment 7-2A

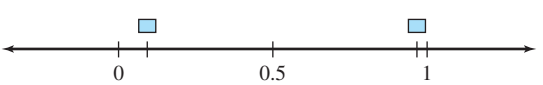
1. Use base-ten blocks to model 3.2 + 2.32.
2. Write the decimals as fractions and then add. Write your answer in decimal form.
 a. 8.6 + 23.1 + 0.92 **b.** 2.32 + 21.008
3. Write the decimals as fractions and then multiply. Write your answer in decimal form. Show all work.
 a. 0.5 · 0.6 **b.** 203 · 0.03 **c.** 0.003 · 0.006
4. Assume the multiplication shown below is correct except for the placement of the decimal point in the product.

$$934.23 \cdot 3.2 = 2989536$$

 a. Insert the decimal point in the correct position without actually performing the multiplication. Describe the process you used to place the decimal point.
 b. Describe a second way to determine where to place the decimal point.
5. John's calculator calculates correctly but does not display decimal points. Use estimation to place the decimal point in the correct position for each of the following.
 a. $6 \cdot 854.14 = 512484$
 b. $81.6 \cdot 212.34 = 17326944$
 c. $137.025/1.75 = 783$
6. Multiply each number below by 0.01 using mental computation.
 a. 2586 **b.** 34.79
 c. 0.24 **d.** 0.0037
7. When you multiply a natural number by a decimal,
 a. when do you get a result less that the natural number?
 b. when do you get a result greater than the natural number?
8. If Maura went to the store and bought a chair for $17.95, a lawn rake for $13.59, a spade for $14.86, a lawn mower for $179.98, and two six-packs of mineral water for $2.43 each, what was the bill?
9. **a.** Complete the following magic square; that is, make the sum of every row, column, and diagonal the same.

8.2		
3.7	5.5	
	9.1	2.8

 b. If each cell of the magic square has 0.85 added to it, is the square still magic?
 c. If the answer to part (b) is "yes," what is the sum of each row?
10. Karin bought 25 lb of peaches at $4.00/lb and 15 lb of apples at $2.00/lb. If she wanted to buy 10 more pounds of another kind of fruit to make the average price per pound equal to $3.50, what price should she pay for the additional 10 lb?
11. Automobile engines were once measured in cubic inches ($in.^3$) but are now usually measured in cubic centimeters (cm^3). Susan's 1963 Thunderbird has a 390 $in.^3$ engine. If 2.54 cm is equivalent to 1 in., approximately how many cubic centimeters is this?
12. A stock rose $0.24 in the market on Thursday. If the resulting price was $73.245, what was the price of the stock before the rise?
13. A U.S. $1 bill was valued at 1.046 Canadian dollars in November, 2013. What was the value of 27.32 American dollars in Canadian dollars that day?
14. A kilowatt hour means 1000 watts of electricity are being used continuously for 1 hr. The electric utility company in Laura's town charges $0.06715 for each kilowatt hour used. Laura heats her house with three electric wall heaters that use 1200 watts each.
 a. How much does it cost to heat her house for 1 day?
 b. How many hours would a 75-watt lightbulb have to stay on to result in $1 for electricity charges?
15. If one liter is 4.224 cups, how many liters is 36.5 cups?
16. If each of the following sequences is arithmetic, continue the decimal patterns for 3 more terms.
 a. 0.9, 1.8, 2.7, 3.6, 4.5,
 b. 0.3, 0.5, 0.7, 0.9, 1.1,
17. If the first term of a geometric sequence is 0.9 and its ratio is 0.2, what is the sum of the first five terms?
18. Interpret 0.2222 as a sum of a finite geometric sequence whose first term is 0.2. (*Hint:* Write 0.2222 as the sum of fractions whose denominators are powers of 10.)
19. Estimate the placement of each of the following values on the given number line by placing the letter for each computation in the appropriate box.
 a. 0.3 ÷ 0.31
 b. 0.3 · 0.31



20. A bank statement from a local bank shows that a checking account has a balance of $83.62. The balance recorded in the checkbook shows only $21.69. After checking the canceled checks against the record of these checks, the customer finds that the bank has not yet recorded six checks in the amounts of $3.21, $14.56, $12.44, $6.98, $9.51, and $7.49. Is the bank record correct? (Assume the person's checkbook records *are* correct.)
21. Mike's class is building a model of the new World Trade Center (WTC) in New York City. They used a scale of 0.005 for their model; that is, each measure of the actual building is multiplied by 0.005 to find the measure of the corresponding part on the model. If the height of the WTC is 1776 ft, what is the height of the model?

22. Suppose that to predict how tall a four-year-old will be at age 12, you multiply his or her height by 1.5. If Allie is 102.3 cm when she turns 4, predict how tall she will be when she turns 12.

23. The value of a yen in Japan in November 2013 was given by the equation \$1 = 99.925 yen.
- **a.** What was the value of \$150?
- **b.** Pauli saw a shirt marked at 3500 yen in November 2013. What was the price in U.S. dollars?

24. There are 2.54 cm in 1 in.
- **a.** Estimate the number of inches in 35 cm.
- **b.** Compute the number of inches in 35 cm to the nearest hundredth.
- **c.** Jamie is 154.5 cm tall. What is her height in inches?

25. A kilometer is approximately 0.62 miles.
- **a.** Molly ran a 30-km race. How many miles is this?
- **b.** In Canada, the speed limit for a certain stretch of road was marked 90 km/hr. What is the corresponding speed limit in mi/hr?
- **c.** If the speed limit is 75 mi/hr, what is it in km/hr?

26. Convert each of the following to standard form.
- **a.** $3.2 \cdot 10^{-9}$
- **b.** $3.2 \cdot 10^{9}$
- **c.** $4.2 \cdot 10^{-1}$
- **d.** $6.2 \cdot 10^{5}$

27. Write the numerals in each of the following sentences in scientific notation.
- **a.** The diameter of Earth is about 12,700,000 m.
- **b.** The distance from Pluto to the Sun is about 4,486,000,000 km.
- **c.** Each year, about 50,000,000 cans are discarded in the United States.

28. Solve the following for x where x is a decimal.
- **a.** $8.56 = 3 - 2x$
- **b.** $2.3x - 2 = x + 2.55$

29. Write the numerals in each of the following sentences in standard form.
- **a.** The mass of a dust particle is $7.53 \cdot 10^{-10}$ g.
- **b.** The speed of light is approximately $2.98 \cdot 10^{5}$ km/sec.
- **c.** Jupiter is approximately $7.7857 \cdot 10^{8}$ km from the Sun.

30. Write the results of each of the following in scientific notation.
- **a.** $(8 \cdot 10^{12})(6 \cdot 10^{15})$
- **b.** $(16 \cdot 10^{12}) \div (4 \cdot 10^{5})$
- **c.** $(5 \cdot 10^{8})(6 \cdot 10^{9}) \div (15 \cdot 10^{15})$

31. Round each of the following numbers as specified.
- **a.** 203.651 to the nearest hundred
- **b.** 203.651 to the nearest ten
- **c.** 203.651 to the nearest unit
- **d.** 203.651 to the nearest tenth
- **e.** 203.651 to the nearest hundredth

32. Jane's car travels 243 mi on 12 gal of gas. How many miles to the gallon does her car get?

33. Audrey wants to buy some camera equipment to take pictures on her daughter's birthday. To estimate the total cost, she rounds each price to the nearest dollar and adds the rounded prices. What is her estimate for the items listed?

Camera	\$54.56
Film	\$4.50
Case	\$17.85

34. Estimate the sum or difference in each of the following by using (i) rounding and (ii) front-end estimation. Then perform the computations to see how close your estimates are to the actual answers.

a.
$$\begin{array}{r} 65.84 \\ 24.29 \\ 12.18 \\ +\ 19.75 \end{array}$$

b.
$$\begin{array}{r} 89.47 \\ -\ 32.16 \end{array}$$

c.
$$\begin{array}{r} 5.85 \\ 6.13 \\ 9.10 \\ +\ 4.32 \end{array}$$

d.
$$\begin{array}{r} 223.75 \\ -\ 87.60 \end{array}$$

35. Find the least and the greatest possible products for the expression using the digits 1 through 9. Each digit may be used only once in each case.

$$\square.\square \times \square$$

36. Some digits in the following number are covered by squares.

$$4\square\square 3\square.\square\square 8\square$$

If each of the digits 1 through 9 is used exactly once in the number, determine the greatest possible number.

37. Iris worked a 40-hour week at \$8.25/hr. Mentally compute her salary for the week and explain how you did it.

38. Mentally compute the number to fill in the blank in each of the following equations.
- **a.** $8.4 \cdot 6 = 4.2 \cdot$ ____
- **b.** $10.2 \div 0.3 = 20.4 \div$ _____
- **c.** $ab = (a/2) \cdot$ ____
- **d.** $a \div b = (2a) \div$ ____

39. Which of the following result in equal quotients?
- **i.** $7 \div 0.25$
- **ii.** $70 \div 2.5$
- **iii.** $0.7 \div 0.25$
- **iv.** $700 \div 25$

40. **a.** Fill in the parentheses in each of the following to write a true equation.

$$2 \cdot 1 + 0.25 = (\)^2$$
$$3 \cdot 2 + 0.25 = (\)^2$$

Conjecture what the next two equations in this pattern will be.

b. Generalize your answer in part (a) by filling in an appropriate expression in the equation

$$n(n - 1) + 0.25 = (\)^2,$$

where $n = 2, 3, 4, \ldots$ and prove it is true.

Assessment 7-2B

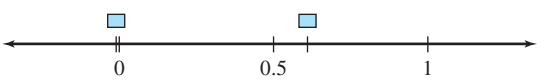
1. Use base-ten blocks to model 3.7 + 1.22.
2. Write the decimals as fractions and then add. Write your answer in decimal form.
 a. 5.3 + 13.2 + 0.86 **b.** 5.42 + 32.005
3. Write the decimals as fractions and then multiply. Write your answer in decimal form. Show all work.
 a. 0.3 · 0.8 **b.** 502 · 0.04 **c.** 0.004 · 0.06
4. Assume the multiplication shown below is correct except for the placement of the decimal point in the product.

$$564.23 \cdot 6.7 = 3780341$$

 a. Insert the decimal point in the correct position without actually performing the multiplication. Describe the process you used to place the decimal point.
 b. Describe a second way to determine where to place the decimal point?
5. John's calculator calculates correctly but does not display decimal points. Use estimation to place the decimal point in the correct position for each of the following.
 a. $5 \cdot 254.15 = 127075$
 b. $31.6 \cdot 122.34 = 3865944$
 c. $813.45 \div 4.25 = 1914$
6. Multiply each number below by 0.001 using mental computation.
 a. 5280 **b.** 54.56
 c. 0.275 **d.** 0.78
7. When you multiply a positive decimal less than 1 by a decimal greater than 1, will the result always be greater than 1? Explain.
8. At a store, Samuel bought a bouquet for $14.99, a candy bar for 79¢, a memory stick for $49.99, and a bottle of water for $1.49. What was this total?
9. **a.** Complete the following magic square; that is, make the sum of every row, column, and diagonal the same.

7.5		
3.0	4.8	
	8.4	2.1

 b. If each cell of the original magic square is multiplied by 0.5, is the square still magic? If "yes," what is the sum of each row?
10. Keith bought 30 lb of nuts at $3.00/lb and 20 lb of nuts at $5.00/lb. If he wanted to buy 10 more pounds of a different kind of nut to make the average price per pound equal to $4.50, what price should he pay for the additional 10 lb?
11. Automobile engines were once measured in cubic inches ($in.^3$) but are now usually measured in cubic centimeters (cm^3). Dan's 1991 Taurus has a 3000 cm^3 engine. If 2.54 cm is equivalent to 1 in., approximately how many cubic inches is this?
12. A stock's price dropped from $63.28 per share to $27.45. What was the loss on a single share of the stock?
13. A U.S. $1 bill was valued at 1.046 Canadian dollars on November 10, 2013. What was the value of 28.43 American dollars in Canadian dollars that day?
14. If one quart is 4 cups, how many quarts is 18.5 cups?
15. If each of the following sequences is either arithmetic or geometric, continue the decimal patterns for 3 more terms.
 a. 1, 0.5, 0.25, 0.125,
 b. 0.2, 1.5, 2.8, 4.1, 5.4,
16. If the first term of a finite geometric sequence is 0.4 and its ratio is 0.3, what is the sum of the first five terms?
17. Interpret the decimal 0.3333333 as a sum of a finite geometric sequence whose first term is 0.3. (*Hint:* Write 0.3333333 as the sum of fractions whose denominators are powers of 10.)
18. Estimate the placement of each of the following values on the given number line by placing the letter for each computation in the appropriate box.
 a. 0.3 + 0.31
 b. 0.3 − 0.31



19. Mary Kim invested $964 in 18 shares of stock. A month later, she sold the 18 shares at $61.48 per share. She also invested in 350 shares of another stock for $27,422.50. She sold this stock for $85.35 a share and paid $495 in commissions. What was Mary Kim's profit or loss on the transactions to the nearest dollar?
20. Mike's class is building a model of the Willis Tower (formerly Sears Tower). They used a scale of 0.005 for their model; that is, each measure of the actual building is multiplied by 0.005 to find the measure of the corresponding part on the model. If the height of the Willis Tower is 1451 ft, what is the height of the model?
21. Suppose that to predict how tall a four-year-old will be at age 12, you multiply his or her height by 1.5. If Allie is 105.2 cm when she turns 4, predict how tall she will be when she turns 12.
22. The value of a yen in Japan in November 2013 was given by the equation $1 = 99.925 yen.
 a. What is the value of $200?
 b. Pauli saw a shirt marked at 7000 yen in November 2013. What was the price in U.S. dollars?
23. There are 2.54 cm in 1 in.
 a. Estimate the number of inches in 160 cm.
 b. Compute the number of inches in 160 cm to the nearest hundredth.
 c. Jamie is 182 cm tall. What is her height in inches?
24. One kilometer is approximately 0.62 miles.
 a. Molly ran a 50-km race. How many miles is this?
 b. In Canada, the speed limit for a certain stretch of road was marked 120 km/hr. What is the corresponding speed limit in mi/hr?
 c. If the speed limit is 50 mi/hr, what is it in km/hr?

25. Convert each of the following to standard numerals:
- **a.** $3.5 \cdot 10^{7}$
- **b.** $3.5 \cdot 10^{-7}$
- **c.** $^{-}2.4 \cdot 10^{-3}$

26. Write the numerals in each of the following sentences in scientific notation.
- **a.** The population of Montana is 998,199 people.
- **b.** The area of North America is 24,490,000 mi^2.

27. Solve the following for x where x is a decimal.
- **a.** $2x + 1.3 = 4.1$
- **b.** $4.2 - 3x = 10.2$

28. Write the numerals in each of the following sentences in standard form.
- **a.** A computer requires $4.4 \cdot 10^{-6}$ sec to do an addition problem.
- **b.** There are about $1.99 \cdot 10^{4}$ km of coastline in the United States.
- **c.** Earth has existed for approximately $3 \cdot 10^{9}$ yr.

29. Write the results of each of the following in scientific notation.
- **a.** $(5 \cdot 10^{7})(7 \cdot 10^{12})$
- **b.** $(^{-}13 \cdot 10^{4}) \div 65$
- **c.** $(3 \cdot 10^{7})(4 \cdot 10^{5}) \div (6 \cdot 10^{-7})$

30. Round each of the following numbers as specified.
- **a.** 715.04 to the nearest hundred
- **b.** 715.04 to the nearest tenth
- **c.** 715.04 to the nearest unit
- **d.** 715.04 to the nearest ten
- **e.** 715.04 to the nearest thousand

31. Jane drives at a constant speed of 55.5 mph. How far should she expect to drive in $\frac{3}{4}$ hr?

32. Use estimation to choose a decimal to multiply by 9 in order to get within 1 of 93. Explain how you made your choice and check your estimate.

33. Estimate the sum or difference in each of the following by using (i) rounding and (ii) front-end estimation. Then perform the computation to see how close your estimates are to the actual answers.

a.
$$\begin{array}{r} 47.62 \\ 27.99 \\ 13.14 \\ +\ 7.61 \\ \hline \end{array}$$

b.
$$\begin{array}{r} 79.86 \\ -\ 27.37 \\ \hline \end{array}$$

c.
$$\begin{array}{r} 5.85 \\ 6.17 \\ 9.1 \\ +\ 4.23 \\ \hline \end{array}$$

d.
$$\begin{array}{r} 232.65 \\ -\ 78.92 \\ \hline \end{array}$$

34. Find the least and the greatest possible products for the expression using the digits 1 through 9. Each digit may be used only once in each case.

$$\square.\square \times \square.\square$$

35. Some digits in the following number are covered by squares:

$$4\square\square 3\square.\square\square 8\square$$

If each of the digits 1 through 9 is used exactly once in the number, determine the least possible number.

36. Iris worked a 40-hour week at \$6.25/hr. Mentally compute her salary for the week and explain how you did it.

37. Mentally compute the number to fill in the blank in each of the following equations.
- **a.** $12.4 \cdot 7 = 6.2 \cdot$ ____
- **b.** $12.4 \div 0.2 =$ ____ $\div 0.1$
- **c.** $ab = (a \cdot 10^{-1}) \cdot$ ____
- **d.** $12.3 = 10^{-2} \cdot$ ____

38. Which of the following result in equal quotients?
- (i) $9 \div 0.35$
- (ii) $90 \div 3.5$
- (iii) $900 \div 35$
- (iv) $0.9 \div 0.035$

39. **a.** Fill in the parentheses in each of the following to write a true equation.

$$1 \cdot 2 + 0.25 = (\)^2$$
$$2 \cdot 3 + 0.25 = (\)^2$$

Conjecture what the next two equations in this pattern will be.

b. Do the computations to determine if your next two equations are correct.

c. Generalize your answer in part (a) by filling in an appropriate expression in the equation

$$n(n + 1) + 0.25 = (\)^2$$

where $n = 1, 2, 3, \ldots$.

Mathematical Connections 7-2

Reasoning

1. How much greater is $0.76 \cdot 3$ than $0.75 \cdot 3$? How can you tell without actually multiplying each product?

2. How is multiplication of decimals like multiplication of whole numbers? How is it different?

3. Why are estimation skills important in dividing decimals?

4. In the text, multiplication and division were done using both fractional and decimal forms. Discuss the advantages and disadvantages of each.

5. Explain why subtraction of terminating decimals can be accomplished by lining up the decimal points, subtracting as if the numbers were whole numbers, and then placing the decimal point in the difference.

6. Is a product of two positive decimals each less than 1 always less than each of the decimals? Justify your answer.

7. What is the significance of the 1 above the 6 in the addition shown below?

$$\begin{array}{r} {\scriptstyle 1\ \ \ } \\ 21.67 \\ +\ 6.28 \\ \hline 27.95 \end{array}$$

Open-Ended

8. Find several examples of the use of decimals in the newspaper. Tell whether you think the numbers are exact or estimates. Also tell why you think decimals were used instead of fractions.

9. How could a calculator be used to develop or reinforce the understanding of multiplication of decimals?

10. Do you think that decimals should always be converted to fractions to explain computational procedures?

Cooperative Learning

11. In your group, decide on all the prerequisite skills that students need before learning to perform arithmetic operations on decimals.

12. In your group, each person will write a test of 5 questions that examines students' ability to estimate products of decimals. Next, compare the tests to choose the best questions, and then cooperate to devise a single test.

Connecting Mathematics to the Classroom

13. In multiplying $0.125 \cdot 0.08$, Roger knew the rule that says he should multiply $125 \cdot 8$ and then move the decimal point $3 + 2 = 5$ places. When he did the calculation on the calculator he saw the answer was 0.01. He was confused on whether the rule really works. How do you help him?

14. Jamal wants to know why 0.3 and 0.30 are equal. How do you respond?

15. Fred says that to multiply $32.7 \cdot 2.6$, he just multiplies $327 \cdot 26 = 8502$, and then because $33 \cdot 3 = 99$, he records the answer as 85.02. What is your response?

16. A student multiplies $(6.5)(8.5)$ to obtain the following:

$$\begin{array}{r} 8.5 \\ \times\, 6.5 \\ \hline 4\,2\,5 \\ 5\,1\,0 \\ \hline 5\,5.2\,5 \end{array}$$

However, when the student multiplies $8\frac{1}{2} \cdot 6\frac{1}{2}$, she obtains the following:

$$\begin{array}{rl} 8\frac{1}{2} & \\ \times\, 6\frac{1}{2} & \\ \hline 4\frac{1}{4} & \left(\frac{1}{2} \cdot 8\frac{1}{2}\right) \\ 48 & (6 \cdot 8) \\ \hline 52\frac{1}{4} & \end{array}$$

How is this possible?

17. A student tries to calculate $0.999^{10{,}000}$ on a calculator and finds the answer to be $4.5173346 \cdot 10^{-5}$. The student wonders how it could be that when a number so close to 1 is raised to the 10,000 power the result is a number close to 0. How do you respond?

18. How would you respond to the following situations?

a. A student claims that $\frac{9443}{9444}$ and $\frac{9444}{9445}$ are equal because each fraction equals approximately 0.9998941.

b. Another student claims that the fractions are not equal and wants to know if there is any way the same calculator can determine which is greater.

Review Problems

19. Write 14.0479 in expanded form.

20. Without dividing, determine which of the following represent terminating decimal.

a. $\frac{24}{36}$ **b.** $\frac{49}{56}$

21. If the denominator of a fraction is 26, is it possible that the fraction could be written as a terminating decimal? Why or why not?

22. $\frac{35}{56}$ can be written as a terminating decimal. Explain why.

National Assessments

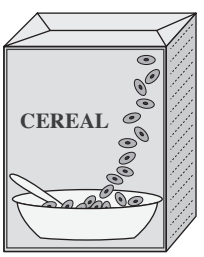
National Assessment of Educational Progress (NAEP) Questions

It costs \$0.25 to operate a clothes dryer for 10 minutes at a laundromat. What is the total cost to operate one clothes dryer for 30 minutes, a second for 40 minutes, and a third for 50 minutes?

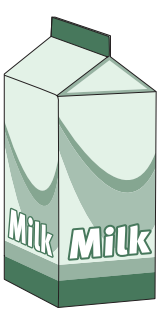
A. \$3.25 B. \$3.00 C. \$2.75 D. \$2.00 E. \$1.20

Add the numbers $\frac{7}{10}$, $\frac{7}{100}$, and $\frac{7}{1{,}000}$. Write this sum as a decimal.

NAEP, Grade 8, 2007



\$4.95



\$2.45

Sales Tax Table	
Amount of Sales	Amount of Tax
\$6.00	\$0.36
6.20	0.37
6.40	0.38
6.60	0.40
6.80	0.41
7.00	0.42
7.20	0.43
7.40	0.44
7.60	0.46
7.80	0.47
8.00	0.48

Carlos bought the cereal and milk shown. Use the table to find out the total amount Carlos spent, including tax.
Total amount spent: _____

Show how you found your answer.

NAEP, Grade 4, 2007

Which of these would be easiest to solve by using mental math?

A. \$65.12 − \$28.19
B. 358 × 2
C. 1,625 ÷ 3
D. \$100.00 + \$10.00

NAEP, Grade 4, 2007

7-3 Repeating Decimals

7-3 Objectives

Students will be able to understand and explain

- Why repeating decimals occur and how to tell if a decimal will repeat.
- How a fraction can be converted to a repeating decimal and vice versa.
- Ordering repeating decimals efficiently.
- That $0.\overline{9} = 1$ and investigate various ways to show this is true.

CCSS In the grade 7 *Common Core Standards*, we find that students should be able to "convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats." (p. 49)

Earlier in the chapter, we developed procedures for converting some rational numbers to decimals. For example, $\frac{7}{8}$ can be written as a terminating decimal as follows:

$$\frac{7}{8} = \frac{7}{2^3} = \frac{7 \cdot 5^3}{2^3 \cdot 5^3} = \frac{875}{1000} = 0.875$$

The decimal for $\frac{7}{8}$ can also be found by division:

$$\begin{array}{r} 0.875 \\ 8\overline{)7.000} \\ \underline{64} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

However, we showed that $\frac{2}{11}$ was not a terminating decimal. In the following discussion, we investigate nonterminating repeating decimals.

Repeating Decimals

If we use a calculator to find a decimal representation for $\frac{2}{11}$, the calculator may display 0.1818181. It seems that the block of numbers 18 repeats. To examine what digits, if any, the calculator did not display, consider the following division:

$$\begin{array}{r} 0.18 \\ 11\overline{)2.00} \\ \underline{11} \\ 90 \\ \underline{88} \\ 2 \end{array}$$

At this point, if the division is continued, the division pattern repeats, since the remainder 2 divided by 11 repeats the division. Thus, the quotient is 0.181818 A decimal of this type is a **repeating decimal**, and the repeating block of digits is the **repetend**. The repeating decimal is written $0.\overline{18}$, where the bar indicates that the block of digits underneath is repeated continuously. The **period** of the repetend is the number of digits that repeat. In $0.\overline{18}$ the period is 2. We use the least number of digits possible in the repetend to represent the decimal. For example, the numeral 3.21515151515 . . . is written $3.2\overline{15}$ rather than $3.2\overline{1515}$ or $3.21\overline{51}$. A repeating decimal such as $0.\overline{18}$ is the infinite sum $0.18 + 0.0018 + 0.000018 + \ldots$. Because we can't add infinitely many numbers, we need to define the meaning of such a sum. We interpret it by looking at partial finite sums: 0.18, 0.18 + 0.0018, 0.18 + 0.0018 + 0.000018, ... The more terms we

add, the closer the partial sums get to a specific number; the partial sums given above get closer and closer to $\frac{2}{11}$. The value of the infinite sum is defined as that number.

Example 10

Use a calculator to convert the following to decimals.

a. $\frac{1}{7}$ **b.** $\frac{2}{13}$ **c.** $\frac{5}{18}$

Solution When we use a calculator to divide, it seems that the division pattern repeats.

a. $\frac{1}{7} = 0.\overline{142857}$ **b.** $\frac{2}{13} = 0.\overline{153846}$ **c.** $\frac{5}{18} = 0.2\overline{7}$

To see why in Example 10 the division pattern repeats as predicted, consider the following divisions:

a.
$$\begin{array}{r} 0.142857 \\ 7\overline{)1.000000} \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 1 \end{array}$$

b.
$$\begin{array}{r} 0.153846 \\ 13\overline{)2.000000} \\ \underline{13} \\ 70 \\ \underline{65} \\ 50 \\ \underline{39} \\ 110 \\ \underline{104} \\ 60 \\ \underline{52} \\ 80 \\ \underline{78} \\ 2 \end{array}$$

c.
$$\begin{array}{r} 0.27 \\ 18\overline{)5.0000} \\ \underline{36} \\ 140 \\ \underline{126} \\ 14 \end{array}$$

In $\frac{1}{7}$, the remainders obtained in the division are 3, 2, 6, 4, 5, and 1. These are all the possible nonzero remainders that can be obtained when dividing by 7. If we had obtained a remainder of 0, the decimal would terminate. Consequently, the seventh division cannot produce a new remainder. Whenever a remainder recurs, the process repeats itself. Using similar reasoning, we could predict that the repetend for $\frac{2}{13}$ could not have more than 12 digits, because there are only 12 possible nonzero remainders. However, one of the remainders could repeat sooner than that, which was actually the case in part (b). In part (c), we could not have more than 17 nonzero remainders. The remainder 14 begins repeating after one step, and the repeating part is not directly after the decimal point. In general, if $\frac{a}{b}$ is any rational number in simplest form with $b \neq 0$ and $b > a$, and it does not represent a terminating decimal, the repetend has at most $b - 1$ digits. Therefore, a *rational number may always be represented either as a terminating decimal or as a repeating decimal.*

Students frequently forget to simplify a given rational number before determining the possible length of the repetend. For example, consider $\frac{6}{21}$. Even though the denominator is 21,

the maximum possible length of the repetend is not 20. Because $\frac{6}{21} = \frac{2}{7}$, the maximum possible length of the repetend is 6.

NOW TRY THIS 4

a. Write $\frac{1}{9}$ as a decimal.

b. Based on your answer in part (a), mentally compute the decimal representation for each of the following.

i. $\frac{2}{9}$ **ii.** $\frac{3}{9}$ **iii.** $\frac{5}{9}$ **iv.** $\frac{8}{9}$

Example 11

Use a calculator to convert $\frac{1}{17}$ to a repeating decimal.



Solution In using a calculator, if we press [1] [÷] [1] [7] [=], we obtain the following, shown as part of a division problem:

$$\begin{array}{r} 0.0588235 \\ 17\overline{)1.} \end{array}$$

Without knowing whether the calculator has an internal round-off feature and with the calculator's having an eight-digit display, we find the greatest number of digits to be trusted in the quotient is six following the decimal point. (Why?) If we use those six places and multiply 0.058823 times 17, we may continue the operation as follows:

[·] [0] [5] [8] [8] [2] [3] [×] [1] [7] [=]

We then obtain 0.999991, which we may place in the preceding division:

$$\begin{array}{r} 0.0\,5\,8\,8\,2\,3 \\ 1\,7\overline{)1.0\,0\,0\,0\,0\,0} \\ \underline{9\,9\,9\,9\,9\,1} \\ 9 \end{array}$$

What is left to divide is $1 - 0.999991$. Next, we divide 9 by 17 to obtain 0.5294118. Again ignoring the rightmost digit, we continue as before, completing the division as follows, where the repeating pattern is apparent:

$$\begin{array}{r} 0.05882352941176470588235 \\ 17\overline{)1.00000000000000000000000} \\ \underline{999991} \\ 9000000 \\ \underline{8999987} \\ 13000000 \\ \underline{12999985} \\ 15 \end{array}$$

Thus, $\frac{1}{17} = 0.\overline{0588235294117647}$, and the repetend is 16 digits long.

Example 11 illustrates how a calculator with only a finite display of digits can be used to do division beyond what the calculator was designed to do. A common suggestion for elementary students is that they only use the division algorithm selectively to learn the process. The calculator example given and similar ones are used to see if we truly understand the division process and the place values involved along the way.

Writing a Repeating Decimal in the Form $\frac{a}{b}$, where $a, b \in I, b \neq 0$

We have already considered how to write terminating decimals in the form $\frac{a}{b}$, where a, b are integers and $b \neq 0$. For example,

$$0.55 = \frac{55}{10^2} = \frac{55}{100}.$$

To write $0.\overline{5}$ in a similar way, we see that because the repeating decimal has infinitely many digits, the denominator cannot be written as a single power of 10. To overcome this difficulty, we must somehow eliminate the infinitely repeating part of the decimal. If we let $n = 0.\overline{5}$, then the *subgoal* is to write an equation for n without a repeating decimal. It can be shown that $10(0.555\ldots) = 5.555\ldots = 5.\overline{5}$. Hence, $10n = 5.\overline{5}$. Using this information, we subtract the corresponding sides of the equations to obtain an equation whose solution can be written without a repeating decimal.

$$\begin{aligned} 10n &= 5.\overline{5} \\ -\quad n &= 0.\overline{5} \\ \hline 9n &= 5 \\ n &= \frac{5}{9} \end{aligned}$$

Performing the subtraction gives an equation that contains only integers. The repeating blocks "cancel" each other. Thus, $0.\overline{5} = \frac{5}{9}$. This result can be checked by performing the division $5 \div 9$.

Suppose a decimal has a repetend of more than one digit such as $0.\overline{235}$. One approach is to multiply the decimal $0.\overline{235}$ by 10^3, since there is a three-digit repetend. Let $n = 0.\overline{235}$. Our *subgoal* is to write an equation for n without the repeating decimal:

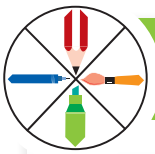
$$\begin{aligned} 1000n &= 235.\overline{235} \\ -\quad n &= \quad 0.\overline{235} \\ \hline 999n &= 235 \\ n &= \frac{235}{999} \end{aligned}$$

Thus, $0.\overline{235} = \frac{235}{999}$.

We generalize the above method by first noticing that because $0.\overline{5}$ repeats in one-digit blocks to write it in the form $\frac{a}{b}$, we first multiply by 10^1. Because $0.\overline{235}$ repeats in three-digit blocks, we first multiply by 10^3. In general, *if the repetend is immediately to the right of the decimal point, first multiply by 10^m, where m is the number of digits in the repetend, and then continue as in the preceding cases.*



The above approach to finding the fraction equivalent of a repeating decimal is seen on the partial student page shown on page 361.



School Book Page — Repeating Decimals

1-1b Activity Lab

Repeating Decimals

In Lesson 1-1, you learned how to write a terminating decimal as a fraction. You use algebra to write a repeating decimal as a fraction.

EXAMPLE **Writing a Repeating Decimal as a Fraction**

In a recent survey, $0.\overline{45}$ of those asked chose blue as their favorite color. Write $0.\overline{45}$ as a fraction in simplest form.

Step 1 Represent the given decimal with a variable.

$n = 0.\overline{45}$

Step 2 Multiply by 10^d, where d = the number of digits that repeat. In this case, multiply by 10^2 or 100. Since 2 digits repeat in $0.\overline{45}$.

$100n = 45.\overline{45}$

Step 3 Subtract to eliminate the repeating part.

$100n = 45.454545\ldots$

$-\ n = -\ 0.454545\ldots$ ← Use the Subtraction Property of Equality.

$99n = 45.000000\ldots$ ← Simplify.

$99n = 45$

Step 4 Solve the new equation.

$\frac{99n}{99} = \frac{45}{99}$ ← Divide each side by 99.

$n = \frac{45}{99} = \frac{5}{11}$ ← Simplify using the GCF, 9.

The repeating decimal $0.\overline{45}$ equals $\frac{5}{11}$.

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Now, suppose the repeating block does *not* occur immediately after the decimal point. For example, let $n = 2.3\overline{45}$. A strategy for solving this problem is to *change it to a related problem* we know how to solve; that is, change it to a problem where the repeating block immediately follows the decimal point. This becomes a *subgoal*. To accomplish this, we multiply both sides by 10:

$$n = 2.3\overline{45}$$
$$10n = 23.\overline{45}$$

We now proceed as with previous problems. Because $10n = 23.\overline{45}$ and the number of digits in the repetend is 2, we multiply by 10^2 as follows:

$$100(10n) = 2345.\overline{45}$$

Thus,

$$\begin{aligned} 1000n &= 2345.\overline{45} \\ -\ 10n &= \quad 23.\overline{45} \\ \hline 990n &= 2322 \end{aligned}$$
$$n = \frac{2322}{990}, \text{ or } \frac{129}{55}.$$

Hence, $2.3\overline{45} = \frac{2322}{990} = \frac{129}{55} = 2\frac{19}{55}$. We can check that the answer is correct by using a calculator to convert $\frac{129}{59}$ back to decimal form.

A Surprising Result

To find the $\frac{a}{b}$ form of $0.\overline{9}$, we proceed as follows. Let $n = 0.\overline{9}$; then $10n = 9.\overline{9}$. Next we subtract and continue to solve for n:

$$\begin{aligned} 10n &= 9.\overline{9} \\ -\ n &= 0.\overline{9} \\ \hline 9n &= 9 \\ n &= 1 \end{aligned}$$

Hence, $0.\overline{9} = 1$. This approach to the problem may not be convincing. Another approach to show that $0.\overline{9}$ is really another name for 1 is shown next:

$$\begin{aligned} \frac{1}{3} &= 0.3333\ldots \\ +\ \frac{2}{3} &= 0.6666\ldots \\ \hline 1 &= 0.9999\ldots \text{ or } 0.\overline{9} = 1 \end{aligned}$$

This last decimal represents the infinite sum $\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots$.

Still another approach is as follows:

$$\begin{aligned} \frac{1}{9} &= 0.11111111\ldots \\ 1 = 9 \cdot \frac{1}{9} &= 9 \cdot 0.11111111\ldots \\ &= 0.999999\ldots \end{aligned}$$

Some may prefer a visual approach to show that $0.\overline{9} = 1$. Consider the number line in Figure 5(a). Most would agree that $0.\overline{9}$ would be between 0.9 and 1.0, so we start there. Then $0.\overline{9}$ is between 0.99 and 1.0, as in Figure 5(b).

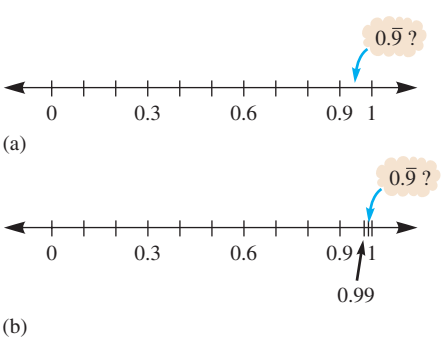


Figure 5

Then we can proceed similarly to argue that $0.\overline{9}$ would be between 0.999 and 1, and so forth. Next, it is reasonable to ask if there is any tiny amount a such that $0.\overline{9} + a = 1$? The answer has to be no. (Why?) If there is no such number a, then $0.\overline{9}$ cannot be less than 1. Because $0.\overline{9}$ cannot be greater than 1, we know that $0.\overline{9} = 1$. In more advanced mathematics courses, sums like $0.\overline{9}$, or $0.9 + 0.09 + 0.009 + \ldots$, are defined as the *limits of finite sums.*

Ordering Repeating Decimals

We now know that any repeating decimal can be written as a rational number in the form $\frac{a}{b}$, where $b \neq 0$. Thus, any repeating decimal can be represented on a number line in a manner

similar to the way that terminating decimals can be placed on the number line. Also, because we have seen that between any two rational numbers in $\frac{a}{b}$ form there are infinitely many more rational numbers of that form (denseness property), it is reasonable that there should be infinitely many repeating decimals between any two other decimals.

To order repeating decimals, we consider where a repeating decimal might lie on a number line, or we compare them using place value in a manner similar to how we ordered terminating decimals. For example, to order repeating decimals such as $1.\overline{3478}$ and $1.34\overline{7821}$ we write the decimals one under the other, in their equivalent forms without the bars, and line up the decimal points (or place values) as follows:

$$1.34783478\ldots$$
$$1.34782178\ldots$$

The digit to the left of the decimal points and the first four digits after the decimal points are the same in each of the numbers. However, since the digit in the hundred-thousandths place of the top number, which is 3, is greater than the digit 2 in the hundred-thousandths place of the bottom number, $1.\overline{3478}$ is greater than $1.34\overline{7821}$.

It is easy to compare two fractions, such as $\frac{21}{43}$ and $\frac{37}{75}$, using a calculator. We convert each to a decimal and then compare the decimals.

$$\boxed{2}\boxed{1}\boxed{\div}\boxed{4}\boxed{3}\boxed{=} \rightarrow 0.4883721$$

$$\boxed{3}\boxed{7}\boxed{\div}\boxed{7}\boxed{5}\boxed{=} \rightarrow 0.4933333$$

Examining the digits in the hundredths place, we see that

$$\frac{37}{75} > \frac{21}{43}.$$

Example 12

Find a rational number in decimal form between $0.\overline{35}$ and $0.\overline{351}$.

Solution First, line up the decimals.

$$0.353535\ldots$$
$$0.351351\ldots$$

Then, to find a decimal between these two, observe that starting from the left, the first place at which the two numbers differ is the thousandths place. Clearly, one decimal between these two is 0.352. Others include 0.3514, $0.35\overline{15}$, and 0.35136. In fact, there are infinitely many others.

Assessment 7-3A

1. Will $\frac{1}{123456789}$ be a terminating decimal? Explain how you know.

2. **a.** Do $0.45\overline{777}$, $0.45\overline{77}$, and $0.45\overline{7}$ represent different numbers? Explain.
b. Which is the preferred form in part (a)?
c. What is the period of the decimal in part (b)?

3. Write the first 10 decimal places for each of the following.
a. $0.\overline{123}$ **b.** $0.1\overline{23}$
c. $0.12\overline{3}$ **d.** $0.12\overline{343}$

4. Which of the following is greater: $0.36\overline{25}$ or $0.3\overline{625}$?

5. Write the following using a bar to indicate the repetend and give the period of each repetend.
a. $0.4444\ldots$ **b.** $0.36454545\ldots$
c. $0.18273273273\ldots$

6. Find the decimal representation for each of the following.
a. $\frac{4}{9}$ **b.** $\frac{2}{7}$ **c.** $\frac{3}{11}$ **d.** $\frac{1}{15}$
e. $\frac{2}{75}$ **f.** $\frac{1}{99}$ **g.** $\frac{5}{6}$ **h.** $\frac{1}{13}$
i. $\frac{1}{21}$ **j.** $\frac{3}{19}$

7. Convert each of the following repeating decimals to $\frac{a}{b}$ form, where a, b are integers and $b \neq 0$.
 a. $0.\overline{4}$ **b.** $0.\overline{61}$ **c.** $1.3\overline{96}$
 d. $0.\overline{55}$ **e.** $^{-}2.3\overline{4}$ **f.** $^{-}0.0\overline{2}$
8. Express 1 min as a repeating decimal part of an hour.
9. Order the following decimals from greatest to least.
 $^{-}1.45\overline{4}, ^{-}1.\overline{45}\overline{4}, ^{-}1.\overline{45}, ^{-}1.45\overline{4}, ^{-}1.454$
10. Find three more terms for the following sequence.
 $0, 0.5, 0.\overline{6}, 0.75, 0.8, 0.8\overline{3}, _____, _____, _____$
11. Write $(0.\overline{5})^2$ as a repeating decimal.
12. Give an argument why $3\frac{1}{7}$ must be a repeating decimal.
13. Suppose $a = 0.\overline{32}$ and $b = 0.\overline{123}$. Find $a + b$ by adding from left to right. How many digits are in the repetend of the sum?
14. Explain whether a terminating decimal could ever be written as a repeating decimal.
15. Find three decimals between each of the following pairs of decimals.
 a. $3.\overline{2}$ and 3.22
 b. $462.2\overline{4}$ and 462.243
16. Find the decimal halfway between the decimals $0.\overline{4}$ and 0.5.
17. **a.** Find three rational numbers between $\frac{3}{4}$ and $0.\overline{75}$.
 b. Find three rational numbers between $\frac{1}{3}$ and $0.\overline{34}$.
18. **a.** What is the 21st digit in the decimal expansion of $\frac{3}{7}$?
 b. What is the 5280th digit in the decimal expansion of $\frac{1}{17}$?
19. **a.** Write each of the following as a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
 i. $0.\overline{1}$ **ii.** $0.\overline{01}$ **iii.** $0.\overline{001}$
 b. What fraction would you expect represents $0.\overline{0001}$?
 c. Mentally compute the decimal equivalent for $\frac{1}{90}$.
20. Use the fact that $0.\overline{9} = 1$ to find the terminating decimal equal to the following.
 a. $0.0\overline{9}$ **b.** $0.3\overline{9}$ **c.** $9.\overline{99}$
21. Use the fact that $0.\overline{1} = \frac{1}{9}$ to convert each of the following into rational numbers.
 a. $0.\overline{2}$ **b.** $0.\overline{3}$ **c.** $9.\overline{9}$
22. Use the fact that $0.\overline{01} = \frac{1}{99}$ and $0.\overline{001} = \frac{1}{999}$ to mentally convert each of the following to rational numbers.
 a. $0.\overline{05}$ **b.** $0.\overline{003}$
23. Find the sum of the finite geometric sequence whose first term is 0.4, whose ratio is 0.5, and which has five terms.
24. Find a rational number in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$, for the following repeating decimals.
 a. $0.2\overline{9}$ **b.** $2.0\overline{29}$
25. Consider the repeating decimals $0.\overline{235}$ and $0.\overline{2356}$. How many places do you expect in the repetend of the sum of the two decimals? Why?
26. Find values of x such that each of the following is true. Write answers both in $\frac{a}{b}$ form, where a and b are integers and $b \neq 0$, and as decimals.
 a. $1 - 3x = 8$ **b.** $1 = 3x + 8$
 c. $1 = 8 - 3x$
27. If a, b, and c are digits, write $0.\overline{abc}$ as a fraction.
28. What is the repetend of $\frac{1}{333{,}333}$?
29. If $\frac{9}{23} = 0.\overline{391304347826086956521 7}$, what is the 100th digit in the decimal expansion?

Assessment 7-3B

1. Will $\frac{3}{1234567}$ be a terminating decimal? Explain how you know.
2. **a.** Do $0.34\overline{666}$, $0.34\overline{66}$, and $0.34\overline{6}$ represent different numbers? Explain.
 b. Which is the preferred form in part (a)?
 c. What is the period of the decimal in part (a)?
3. Write the first 10 decimal places for each of the following.
 a. $0.\overline{246}$ **b.** $0.2\overline{46}$
 c. $0.24\overline{6}$ **d.** $0.24\overline{68}$
4. Which of the following is greater: $0.57\overline{89}$ or $0.5\overline{789}$?
5. Write the following using a bar to indicate the repetend and give the period of each repetend.
 a. 0.9999 . . . **b.** 0.567323232 . . .
 c. 0.1827482748274 . . .
6. Find the decimal representation for each of the following.
 a. $\frac{2}{3}$ **b.** $\frac{7}{9}$ **c.** $\frac{1}{24}$ **d.** $\frac{3}{60}$
 e. $\frac{2}{99}$ **f.** $\frac{7}{6}$ **g.** $\frac{2}{21}$ **h.** $\frac{4}{19}$
7. Convert each of the following repeating decimals to $\frac{a}{b}$ form, where a and b are integers and $b \neq 0$.
 a. $0.\overline{7}$ **b.** $0.\overline{46}$ **c.** $2.\overline{37}$
 d. $2.3\overline{4}$ **e.** $^{-}4.3\overline{4}$ **f.** $^{-}0.0\overline{3}$
8. Express 1 sec as a repeating decimal part of an hour.
9. Order the following decimals from least to greatest.
 $^{-}4.34, ^{-}4.\overline{34}, ^{-}4.3\overline{4}, ^{-}4.3\overline{43}, ^{-}4.4\overline{34}$
10. List the next three terms in the following arithmetic sequence.
 $0, 0.\overline{3}, 0.\overline{6}, 1, 1.\overline{3},$
11. Write $0.4\overline{9} \cdot 0.\overline{62}$ as a repeating decimal.
12. Give an argument why $\frac{2}{26}$ must be a repeating decimal. What can you say about length of the repetend without performing the division.?
13. Find $a + b$ if $a = 1.2\overline{34}$ and $b = 0.\overline{1234}$. Is the answer a rational number? How many digits are in the repetend?

14. Explain whether a repeating decimal could ever be written as a terminating decimal.

15. Find three decimals between each of the following pairs of decimals.
a. $4.\overline{3}$ and 4.3
b. $203.\overline{76}$ and $203.\overline{7}$

16. Find the decimal halfway between $0.\overline{9}$ and 1.1.

17. Find three rational numbers between the following.
a. $\frac{2}{3}$ and 0.67 **b.** $\frac{2}{3}$ and $0.6\overline{7}$

18. What is the 23rd decimal in the expansion of $\frac{1}{17}$?

19. **a.** Write each of the following as a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
i. $0.\overline{2}$ **ii.** $0.0\overline{2}$ **iii.** $0.00\overline{2}$
b. What fraction would you expect represents $0.000\overline{2}$?
c. Mentally compute the decimal equivalent for $\frac{4}{90}$.

20. Use the fact that $0.\overline{9} = 1$, to find the terminating decimal equal to the following.
a. $1.\overline{9}$ **b.** $0.00\overline{9}$ **c.** $0.3\overline{9}$

21. Use the fact that $0.\overline{1} = \frac{1}{9}$, $0.\overline{01} = \frac{1}{99}$, and $0.\overline{001} = \frac{1}{999}$ to convert each of the following mentally to the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
a. $0.\overline{4}$ **b.** $0.\overline{12}$ **c.** $0.\overline{111}$

22. Use the fact that $0.\overline{01} = \frac{1}{99}$ and $0.\overline{001} = \frac{1}{999}$ to convert each of the following to a rational number $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
a. $3.\overline{25}$ **b.** $3.\overline{125}$

23. Find the sum of the finite geometric sequence whose first term is 0.1, whose ratio is 0.3, and which has four terms.

24. Find the $\frac{a}{b}$ form, where a and b are integers and $b \neq 0$, for each of the following.
a. $0.2\overline{9}$ **b.** $0.0002\overline{9}$

25. Consider the repeating decimals $0.\overline{23}$ and $0.\overline{235}$. How many places do you expect in the repetend of the sum of the two decimals? Why?

26. Find values of x such that each of the following is true. Write answers both in $\frac{a}{b}$ form, where a and b are integers and $b \neq 0$, and as decimals.
a. $3x = 8$ **b.** $3x + 1 = 8$ **c.** $3x - 1 = 8$

27. If a and b are digits, write $0.\overline{ab}$ as a fraction.

28. What is the repetend of $\frac{1}{333{,}333{,}333}$?

29. If $\frac{9}{23} = 0.\overline{391304347826086956521 7}$, what is the 900th digit in the decimal expansion?

Mathematical Connections 7-3

Reasoning

1. **a.** If a grocery store advertised three lemons for $2.00, what is the cost of one lemon?
b. If you choose to buy exactly one lemon at the cost given in part (a), what will the grocery store charge?
c. How is the store treating the repeating decimal cost of one lemon?
d. Explain whether or not a grocery store would ever use a repeating decimal as a cost for an item.
e. Explain whether you think cash registers ever work with repeating decimals.

2. A friend claims that every finite decimal is equal to some infinite decimal. Is the claim true? Explain why or why not.

3. Some addition problems are easier to compute with fractions and some are easier to do with decimals. For example, $\frac{1}{7} + \frac{5}{7}$ is easier to compute than $0.\overline{142857} + 0.\overline{714285}$ and $0.4 + 0.25$ is easier to compute than $\frac{2}{5} + \frac{1}{4}$. Describe situations in which you think it would be easier to compute the additions with fractions than with decimals, and vice versa.

4. Find decimal expansions of $\frac{1}{11}$, $\frac{1}{111}$, and $\frac{1}{1111}$, and then guess the decimal expansion of $\frac{1}{11111}$. Explain your reasoning and check your answer with a calculator.

Open-Ended

5. Notice that $\frac{1}{7} = 0.\overline{142857}$, $\frac{2}{7} = 0.\overline{285714}$, $\frac{3}{7} = 0.\overline{428571}$, $\frac{4}{7} = 0.\overline{571428}$, $\frac{5}{7} = 0.\overline{714285}$, and $\frac{6}{7} = 0.\overline{857142}$.
a. Describe a common property that all of these repeating decimals share.
b. Suppose you memorized the decimal form for $\frac{1}{7}$. How could you quickly find the answers for the decimal expansions of the five fractions above? Describe as many ways as you can.
c. Find some other fractions that behave like $\frac{1}{7}$. In what way is the behavior similar?

6. **a.** Multiply each of the following, giving your answer as a decimal in simplest form.
i. $2 \cdot 0.\overline{3}$ **ii.** $3 \cdot 0.\overline{3}$ **iii.** $3 \cdot 0.\overline{35}$
b. Explain whether the traditional algorithm for multiplying decimals can be used for multiplying repeating decimals.
c. Explain whether repeating decimals can be multiplied.

7. **a.** Does your calculator allow you to enter repeating decimals? If so, in what form?
b. Explain whether repeating decimal arithmetic can be performed on your calculator.

8. Explain whether we would rather have the solution to $3x = 7$ expressed as a fraction or as a repeating decimal.

Cooperative Learning

9. Choose a partner and play the following game. Write a repeating decimal of the form $0.\overline{abcdef}$. Tell your partner that the decimal is of that form but do not reveal the specific values for the digits. Your partner's objective is to find your repeating decimal. Your partner is allowed to ask you for the values of six digits that are at the 100th or greater places after the decimal point but not the digits in consecutive places. For example, your opponent may ask for the 100th, 200th, 300th, . . . digits but may not ask for the 100th and 101st digit. Switch roles at least once. After playing the game, discuss in your group a strategy for asking your partner the least number of questions that will allow you to find your partner's repetend.

Connecting Mathematics to the Classroom

10. A student argues that repeating decimals are of little value because no calculator will handle computations with repeating decimals, and in real life decimals do not have an infinite number of digits. How do you respond?

11. A student asks if a sum of two repeating decimals is always a repeating decimal. How do you respond?

12. A student needs to find $0.3333 \cdot 48$ without using a calculator. She finds $\frac{1}{3}$ of 48 and gets 16 as the answer. How do you respond?

13. Joe says terminating decimals are those with repetend 0 and nonterminating decimals are those with nonzero repetends. Is this correct? Explain.

14. Bob uses a calculator to compute $\frac{1}{7}$ as 0.142857142. He then multiplies this decimal by 2 to get the decimal approximation for $\frac{2}{7}$. Is this correct? How would you help him?

15. A student says that $0.\overline{4} = \frac{4}{9}$ and $0.\overline{44} = \frac{44}{99}$, so $0.\overline{4} \neq 0.\overline{44}$. How do you respond?

Review Problems

16. Jill received a bonus totaled $27,849.50 and deposited it in a new checking account. She wrote checks for $1520.63, $723.30, and $2843.62. What was the balance in her new checking account?

17. The speed of light is approximately 186,000 mi/sec. It takes light from the nearest star, Alpha Centauri, approximately 4 years to reach Earth. How many miles away is Alpha Centauri from Earth? Express the answer in scientific notation.

18. Find the product of 0.22 and 0.35 on a calculator. How does the placement of the decimal point in the answer on the calculator compare with the placement of the decimal point using the rule in this chapter? Explain.

19. **a.** Find a number to add to $^{-}0.023$ to obtain a sum greater than 3 but less than 4.
b. Find a number to subtract from $^{-}0.023$ to obtain a difference greater than 3 but less than 4.
c. Find a number to multiply times 0.023 to obtain a product greater than 3 but less than 4.
d. Find a number to divide into 0.023 to obtain a quotient greater than 3 but less than 4.

7-4 Percents and Interest

7-4 Objectives

Students will be able to understand and explain

- Fractions, decimals, and percents as representations of rational numbers with conversions from one form to another.
- Proportional relationships to solve percent problems.
- Techniques to solve problems including discounts, interest, compound interest, and percent increase and percent decrease.
- Strategies for percent mental computation and estimation.

CCSS **In the grade 7 *Common Core Standards*, we find the following:**

Students develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers. (p. 46)

Percents are very useful in conveying information. People hear that there is a 60% chance of rain or that their savings account is drawing 2% annual interest. Percents are special kinds of fractions—namely, fractions with a denominator of 100. The word **percent** comes from the Latin phrase *per centum*, which means *per hundred*. A bank that pays 2% annual simple interest on a savings account pays $2 for each $100 in the account for one year; that is, it pays 2/100 of whatever amount is in the account for one year. The symbol % indicates percent; 2% means 2 for each 100. Hence, to find 2% of $400, we determine how many hundreds are in 400. There are 4 hundreds in 400, so 2% of 400 is $2 \cdot 4 = 8$. Therefore, 2% of $400 = $8.

Definition of Percent

$$n\% = \frac{n}{100}, \text{ where } n \text{ is any nonnegative number.}$$

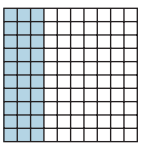
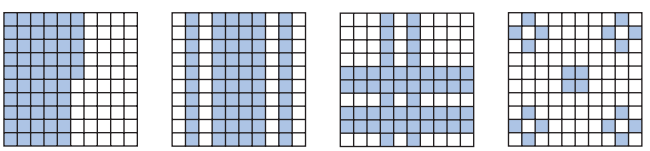


Figure 6

Thus, $n\%$ of a quantity is $\frac{n}{100}$ of the quantity, or $\frac{n}{100}$ times the quantity. Therefore, 1% is one hundredth of a whole, and 100% represents the entire quantity. Similarly, 200% represents $\frac{200}{100}$, or 2, times, the given quantity. Percents can be illustrated by using a hundreds grid. For example, in Figure 6, because 30 out of the 100, or $\frac{30}{100}$, of the squares are shaded, we say that 30% of the grid is shaded. Likewise we say that 70% of the grid is not shaded.

NOW TRY THIS 5

In each part of Figure 7 write the fraction that represents the shaded portion of the large square in lowest terms and its equivalent percent.



(a) (b) (c) (d)

Figure 7

E-Manipulative Activity
For more help with the percents using a hundreds grid, see the *Percents* module.

Because $n\% = \frac{n}{100}$, one way to convert a number to a percent is to write it as a fraction with denominator 100; the numerator gives the amount of the percent. For example, $\frac{3}{4} = \frac{3 \cdot 25}{4 \cdot 25} = \frac{75}{100}$. Hence, $\frac{3}{4} = 75\%$, but $\frac{3}{4}\% = \frac{\frac{3}{4}}{100} = \frac{0.75}{100} = 0.0075$.

Example 13

Write each of the following as a percent.

a. 0.03 **b.** $0.\overline{3}$ **c.** 1.2 **d.** 0.00042
e. 1 **f.** $\frac{3}{5}$ **g.** $\frac{2}{3}$ **h.** $2\frac{1}{7}$

Solution

a. $0.03 = \frac{3}{100} = 3\%$

b. $0.\overline{3} = \frac{33.\overline{3}}{100} = 33.\overline{3}\%$

c. $1.2 = \frac{120}{100} = 120\%$

d. $0.00042 = \frac{0.042}{100} = 0.042\%$

e. $1 = \frac{100}{100} = 100\%$

f. $\frac{3}{5} = \frac{3 \cdot 20}{5 \cdot 20} = \frac{60}{100} = 60\%$

g. $\frac{2}{3} = 100\left[\frac{\left(\frac{2}{3}\right)}{100}\right] = \frac{\left(\frac{200}{3}\right)}{100} = \frac{66.\overline{6}}{100} = 66.\overline{6}\%$

h. $2\frac{1}{7} = 100\left[\frac{\left(2\frac{1}{7}\right)}{100}\right] = \frac{\left(\frac{1500}{7}\right)}{100} = \frac{1500}{7}\%$ or $214\frac{2}{7}\%$.

Still another way to convert a number to a percent is to recall that $1 = 100\%$. Thus, for example, $\frac{3}{4} = \frac{3}{4} \cdot 1 = \frac{3}{4} \cdot 100\% = 75\%$.

The % symbol is crucial in identifying the meaning of a number. For example, $\frac{1}{2}$ and $\frac{1}{2}\%$ are different numbers: $\frac{1}{2} = 50\%$, which is not equal to $\frac{1}{2}\% = 0.5\%$. Similarly, 0.01 is different from 0.01%, which is 0.0001.

Converting percents to decimals can be done by writing the percent as a fraction in the form $\frac{n}{100}$ and then converting the fraction to a decimal. Percents less than 1% and percents greater than 100% are examined in Example 14.

Example 14

Write each of the following percents as a decimal.

a. 5% **b.** 6.3% **c.** 100% **d.** 250% **e.** $\frac{2}{3}\%$ **f.** $33\frac{1}{3}\%$

Solution

a. $5\% = \frac{5}{100} = 0.05$

b. $6.3\% = \frac{6.3}{100} = 0.063$

c. $100\% = \frac{100}{100} = 1$

d. $250\% = \frac{250}{100} = 2.5$

e. $\frac{2}{3}\% = \frac{\frac{2}{3}}{100} = \frac{0.\overline{6}}{100} = 0.00\overline{6}$

f. $33\frac{1}{3}\% = \frac{33\frac{1}{3}}{100} = \frac{33.\overline{3}}{100} = 0.\overline{3}$

Another approach to converting a percent to a decimal is to first convert 1% to a decimal. Because $1\% = \frac{1}{100} = 0.01$, we conclude that $5\% = 5 \cdot 1\% = 5 \cdot 0.01 = 0.05$ and that $6.3\% = 6.3 \cdot 0.01 = 0.063$.

NOW TRY THIS 6

a. Investigate how your calculator handles percents and explain what the calculator does when the [%] key is pushed.

b. Use your calculator to change $\frac{1}{3}$ to a percent.

Applications Involving Percents

CCSS **In grade 6 *Common Core Standard,* we find that students are expected to be able to accomplish the following:**

Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent. (p. 42)

Application problems that involve percents usually take one of the following forms:

1. Finding a percent of a number
2. Finding what percent one number is of another
3. Finding a number when a percent of that number is known

Before we consider examples illustrating these forms, recall what it means to find a fraction "of" a number. For example, $\frac{2}{3}$ of 70 equals $\frac{2}{3} \cdot 70$. Similarly, to find 40% of 70, we have $\frac{40}{100}$ of 70, which equals $\frac{40}{100} \cdot 70$, or $0.40 \cdot 70 = 28$.

A *percent bar* can be used as a model for understanding what 100% of a number means as well as understanding other percents. In Figure 8, consider the percent bar that represents 100% of the whole with 40% of the whole shaded. Note that 100% of the bar represents 70.

Table 3

Percentage	Bar Length
0%	0
10%	
20%	
30%	
40%	?
50%	35
60%	
70%	
80%	
90%	
100%	70

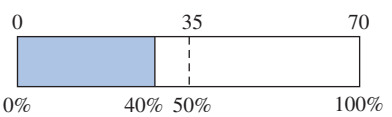


Figure 8

Also, half of the percent bar (50% denoted by the dotted segment) represents half of 70, or 35. Thus, we know that 40% of the bar (or 40% of 70) is less than 35. In fact, if the bottom of the bar is thought of as being marked off in 1% intervals, there are 100 intervals marking whole numbers of percentages. If at the same time the top of the bar is considered to be marked in intervals of 1, there would be only 70 intervals marked at the bottom.

Suppose in Table 3 we know that 0% of 70 is 0, 50% of 70 is 35, and 100% of 70 is 70. What percentages of 70 are 10%, 20%, 30%, and so on? If there are 100 intervals marking percentages compared to only 70 intervals marking the corresponding length, there must be a ratio of $\frac{70}{100}$, or $\frac{7}{10}$. Thus, 10% of 70 is 7, 20% of 70 is $2 \cdot 7$, or 14, and so on. Hence, 40% of 70 is $4 \cdot 7$, or 28.



Percent bars for the three types of percent problems are shown on the partial student page on page 370.

Example 15

A house that sells for \$92,000 requires a 20% down payment. What is the amount of the down payment?

Solution The amount of the down payment is 20% of \$92,000 $= 0.20 \cdot \$92{,}000 = \$18{,}400$.

A percent bar for this example is given in Figure 9.

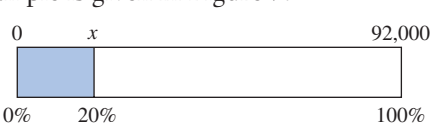


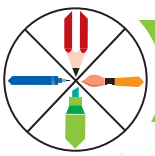
Figure 9

The percent bar leads to the following proportion.

$$\frac{x}{20} = \frac{92{,}000}{100}$$

$$100x = 92{,}000(20)$$

$$x = \$18{,}400$$



School Book Page — Key Concepts

KEY CONCEPTS **Percents and Proportions**

Finding a Percent	Finding a Part	Finding a Whole
What percent of 25 is 5?	What is 20% of 25?	20% of what is 5?
		
$\frac{5}{25} = \frac{n}{100}$	$\frac{n}{25} = \frac{20}{100}$	$\frac{5}{n} = \frac{20}{100}$
$n = 20$	$n = 5$	$n = 25$
5 is 20% of 25.	5 is 20% of 25.	20% of 25 is 5.

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Example 16

If Alberto has 45 correct answers on an 80-question test, what percent of his answers are correct?

Solution Alberto has $\frac{45}{80}$ of the answers correct. To find the percent of correct answers, we need to convert $\frac{45}{80}$ to a percent. We can do this by multiplying the fraction by 100 and attaching the % symbol as follows:

$$\frac{45}{80} = 100 \cdot \frac{45}{80}\%$$
$$= 56.25\%$$

Thus, 56.25% of the answers are correct.

A percent bar for this example is given in Figure 10.

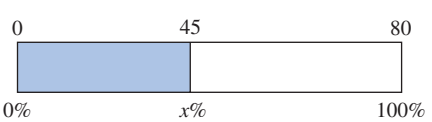


Figure 10

The percent bar leads to the following proportion.

$$\frac{45}{x} = \frac{80}{100}$$
$$80x = 45(100)$$
$$x = 56.25$$

Example 17

Forty-two percent of the parents of the schoolchildren in the Paxson School District are employed at Di Paloma University. If the number of parents employed by the university is 168, how many parents are in the school district?

Solution Let n be the number of parents in the school district. Then 42% of n is 168. We translate this information into an equation and solve for n.

$$\begin{aligned} 42\% \text{ of } n &= 168 \\ \frac{42}{100}n &= 168 \\ 0.42n &= 168 \\ n &= \frac{168}{0.42} = 400 \end{aligned}$$

There are 400 parents in the school district.

A percent bar for this example is given in Figure 11.

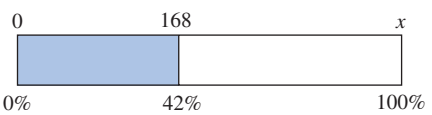


Figure 11

The percent bar leads to the following proportion.

$$\begin{aligned} \frac{168}{42} &= \frac{x}{100} \\ 42x &= 168(100) \\ x &= 400 \end{aligned}$$

We can also solve Example 17 as follows:

$$42\% \text{ of } n \text{ is } 168$$

$$1\% \text{ of } n \text{ is } \frac{168}{42}$$

$$100\% \text{ of } n \text{ is } 100\left(\frac{168}{42}\right)$$

Therefore,

$$n = 100\left(\frac{168}{42}\right), \text{ or } 400.$$

Example 18

Kelly bought a bicycle and a year later sold it for 20% less than what she paid for it. If she sold the bike for \$144, what did she pay for it?

Solution We are looking for the original price P that Kelly paid for the bike. We know that she sold the bike for \$144 and that this included a 20% loss. Thus, we can write the following equation:

$$144 = P - \text{Kelly's loss}$$

Because Kelly's loss is 20% of P, we proceed as follows:

$$\begin{aligned} 144 &= P - 20\% \cdot P \\ 144 &= P - 0.20P \\ 144 &= (1 - 0.20)P \\ 144 &= 0.80P \\ \frac{144}{0.80} &= P \\ 180 &= P \end{aligned}$$

Thus, she paid \$180 for the bike.

A percent bar for this example is given in Figure 12.



Figure 12

The percent bar leads to the following proportion.

$$\begin{aligned} \frac{144}{80} &= \frac{x}{100} \\ 80x &= 144(100) \\ x &= 180 \end{aligned}$$

Alternatively, we can argue that since Kelly lost 20% of the original price, she sold the bike for 100% − 20% or 80% of the original price. Thus,

$$\begin{aligned} 144 &= 80\% \text{ of } P \\ 144 &= 0.8P \\ P &= 180. \end{aligned}$$

We can also approach the problem as follows:

$$80\% \text{ of } P \text{ is } 144.$$

$$1\% \text{ of } P \text{ is } \frac{144}{80}.$$

$$100\% \text{ of } P \text{ is } 100 \cdot \frac{144}{80} \text{ or } 180.$$

Example 19

Westerner's Clothing Store advertised a suit for 10% off, for a savings of \$15. Later, the manager marked the suit at 30% off the original price. What is the amount of the current discount?

Solution A 10% discount amounts to a \$15 savings. We could find the amount of the current discount if we knew the original price P. Thus, finding the original price becomes our *subgoal*. Because 10% of P is \$15, we have the following:

$$\begin{aligned} 10\% \cdot P &= 15 \\ 0.10P &= 15 \\ P &= 150 \end{aligned}$$

To find the current discount, we calculate 30% of \$150. Because $0.30 \cdot \$150 = \$45$, the amount of the 30% discount is \$45.

In the *Looking Back* stage of problem solving, we check the answer and look for other ways to solve the problem. A different approach leads to a more efficient solution and confirms the answer. If 10% of the price is \$15, then 30% of the price is 3 times \$15, or \$45.

Mental Math with Percents

Mental math may be helpful when working with percents. Below we give two techniques.

1. *Using fraction equivalents*
 Knowing fraction equivalents for some percents can make some computations easier. Table 4 gives several fraction equivalents.

Table 4

Percent	25%	50%	75%	$33\frac{1}{3}\%$	$66\frac{2}{3}\%$	10%	1%
Fraction Equivalent	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{10}$	$\frac{1}{100}$

 These equivalents can be used in such computations as the following.

$$50\% \text{ of } 80 = \left(\frac{1}{2}\right)80 = 40$$

$$66\frac{2}{3}\% \text{ of } 90 = \left(\frac{2}{3}\right)90 = 60$$

2. *Using a known percent*
 Frequently, we may not know a percent of something, but we know a close percent of it. For example, to find 55% of 62, we might do the following.

$$50\% \text{ of } 62 = \left(\frac{1}{2}\right)(62) = 31$$

$$5\% \text{ of } 62 = \left(\frac{1}{2}\right)(10\%)(62) = \left(\frac{1}{2}\right)(6.2) = 3.1$$

 Adding, we see that 55% of 62 is $31 + 3.1 = 34.1$.

Estimations with Percents

Estimations with percents can be used to determine whether answers are reasonable. Here are two examples.

1. To estimate 27% of 598, note that 27% of 598 is a little more than 25% of 598, but 25% of 598 is approximately the same as 25% of 600, or $\frac{1}{4}$ of 600, or 150. Here, we adjusted 27% downward and 598 upward, so 150 should be a reasonable estimate. A better estimate might be obtained by estimating 30% of 600 and then subtracting 3% of 600 to obtain 27% of 600, giving $180 - 18$, or 162.
2. To estimate 148% of 500, note that 148% of 500 should be slightly less than 150% of 500. 150% of 500 is $1.5(500) = 750$. Thus, 148% of 500 should be a little less than 750.

Example 20

Laura wants to buy a blouse originally priced at $26.50 but now on sale at 40% off. She has $17 in her wallet, and wonders if she has enough cash. How can she mentally determine this? (Ignore the sales tax.)

Solution It is easier to find 40% of $25 (versus $26.50) mentally. One way is to find 10% of $25, which is $2.50. Now, 40% is 4 times that much, that is, $4 \cdot \$2.50$, or $10. Thus, Laura estimates that the blouse will cost $\$26.50 - \$10$, or $16.50. Since the actual discount is greater than $10 (40% of 26.50 is greater than 40% of 25), Laura will have to pay less than $16.50 for the blouse. She has enough cash.

Sometimes it may not be clear which operations to perform with percent. The following example investigates this.

Example 21

Which of the following statements could be true and which are false? Explain your answers.

a. Leonardo got a 10% raise at the end of his first year on the job and a 10% raise after another year. His total raise was 20% of his original salary.
b. Jung and Dina paid 45% of their first department store bill of $620 and 48% of the second department store bill of $380. They paid $45\% + 48\% = 93\%$ of the total bill of $1000.
c. Bill spent 25% of his salary on food and 40% on housing. Bill spent $25\% + 40\% = 65\%$ of his salary on food and housing.
d. In Bordertown, 65% of the adult population work in town, 25% work across the border, 15% are unemployed, and everyone in town is in exactly one of these categories.
e. In Clean City, the fine for various polluting activities is a certain percentage of one's monthly income. The fine for smoking in public places is 40%, for driving a polluting car is 50%, and for littering is 30%. Mr. Schmutz committed all three polluting crimes in one day and paid a fine of 120% of his monthly income.

Solution

a. In applications, percent has meaning only when it represents part of a quantity. For example, 10% of a quantity plus another 10% of the same quantity is 20% of that quantity. In Leonardo's case, the first 10% raise was calculated based on his original salary, and the second 10% raise was calculated on his new salary. Consequently, the percentages cannot be added, and the statement is false. He received a 21% raise.
b. The last statement does not make sense; 45% of one bill plus 48% of the other bill is not 93% of the total bill because the bills are different.
c. Because the percentages are of the same quantity, the statement is true.
d. Because the percentages are of the same quantity (the number of adults), we can add them: $65\% + 25\% + 15\% = 105\%$. But 105% of the population accounts for more than the town's population (actually 5% more), which is impossible. Hence, the statement is false.
e. Again, the percentages are of the same quantity; that is, the individual's monthly income. Hence, we can add them: 120% of one's monthly income is a stiff fine, but possible.

Computing Interest



In the grade 7 *Common Core Standards*, we find that "students use their understanding of ratios and proportionality to solve a wide variety of percent problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease." (p. 46) We next investigate the topic of interest.

When a bank advertises a $5\frac{1}{2}\%$ interest rate on a loan, the **interest** is the amount of money you will pay for using the bank's money. The original amount borrowed is the **principal**. The percent used to determine the interest is the **interest rate**. Interest rates are given for specific periods of time, such as years, months, or days. Interest computed on the original principal is **simple interest**. For example, suppose we borrow \$5000 from a bank at an annual simple interest rate of 9% for 1 year. The interest we owe on the loan for 1 year is 9% of \$5000, or $0.09 \cdot \$5000 = \$450$. In general, *if a principal, P, is invested at an annual interest rate of r, then the simple interest after 1 year is $Pr \cdot 1$; after t years, it is Prt*. Thus, if I represents simple interest, we have

$$I = Prt.$$

The amount needed to pay off a \$5000 loan at 9% annual simple interest for 1 year is the \$5000 borrowed plus the interest on the \$5000; that is, $5000 + 5000 \cdot 0.09 = \$5450$. In general, *an* **amount** (*or* **balance**) *A is equal to the principal P plus the interest I*:

$$A = P + I = P + Prt = P(1 + rt)$$

Example 22

Vera opened a savings account that pays simple interest at the rate of $5\frac{1}{4}\%$ per year. If she deposits \$2000 and makes no other deposits, find the interest and the final amount for the following time periods:

a. 1 yr
b. 90 days

Solution

a. To find the interest for 1 yr, we proceed as follows.

$$I = \$2000 \cdot 5\frac{1}{4}\% \cdot 1 = \$2000 \cdot 0.0525 \cdot 1 = \$105$$

Her final amount at the end of 1 yr is

$$\$2000 + \$105 = \$2105.$$

b. When the interest rate is annual and the interest period is given in days, we represent the time as a fractional part of a year by dividing the number of days by 365. Thus,

$$I = \$2000 \cdot 5\frac{1}{4}\% \cdot \frac{90}{365}$$

$$= \$2000 \cdot 0.0525 \cdot \frac{90}{365} \approx \$25.89.$$

Hence,

$$A \approx \$2000 + \$25.89$$
$$A \approx \$2025.89.$$

Thus, Vera's amount after 90 days is approximately \$2026.

Example 23

Find the annual interest rate if a principal of \$10,000 increased to \$10,900 at the end of 1 yr.

Solution Let the annual interest rate be $x\%$. We know that $x\%$ of \$10,000 is the increase. Because the increase is \$10,900 − \$10,000 = \$900, we use the strategy of *writing an equation* for x as follows:

$$x\% \text{ of } 10{,}000 = 900$$

$$\frac{x}{100} \cdot 10{,}000 = 900$$

$$x = 9$$

Thus, the annual interest rate is 9%. We can also solve this problem mentally by asking, "What percent of 10,000 is 900?" Because 1% of 10,000 is 100, to obtain 900, we take 9% of 10,000.

Compound Interest

In business transactions, interest is sometimes calculated daily (365 times a year). In the case of savings, the earned interest is added daily to the principal, and each day the interest is earned on a different amount; that is, it is earned on the previous interest as well as the principal. Interest earned in this way is **compound interest**. Compounding is usually done annually (once a year), semiannually (twice a year), quarterly (4 times a year), or monthly (12 times a year). Though common in the past for banks to compound interest quarterly, today banks compound interest monthly, daily, or even continuously. However, even when the interest is compounded, it is given as an annual rate. For example, if the annual rate is 6% compounded monthly, the interest per month is $\frac{6}{12}\%$, or 0.5%. If it is compounded daily, the interest rate per day is $\frac{6}{365}\%$. In general, *the interest rate per period is the annual interest rate divided by the number of periods in a year.*

If we invest \$100 at 8% annual interest compounded quarterly, how much will we have in the account after 1 yr? The quarterly interest rate is $\frac{1}{4} \cdot 8\%$, or 2%. It seems that we would have to calculate the interest 4 times. But we can also reason as follows. If at the beginning of any of the four periods there are x dollars in the account, at the end of that period there will be

$$\begin{aligned} x + 2\% \text{ of } x &= x + 0.02x \\ &= x(1 + 0.02) \\ &= x(1.02) \text{ dollars.} \end{aligned}$$



Hence, to find the amount at the end of any period, we need only multiply the amount at the beginning of the period by 1.02. From Table 5, we see that the amount at the end of the fourth period is $\$100 \cdot 1.02^4$. On a scientific calculator, we can find the amount using 1 0 0 × 1 · 0 2 y^x 4 =. The calculator displays 108.24322. Thus, the amount at the end of 1 yr is approximately \$108.24.

Table 5

Period	Initial Amount	Final Amount
1	100	$100 \cdot 1.02$
2	$100 \cdot 1.02$	$(100 \cdot 1.02)1.02$, or $100 \cdot 1.02^2$
3	$100 \cdot 1.02^2$	$(100 \cdot 1.02^2)1.02$, or $100 \cdot 1.02^3$
4	$100 \cdot 1.02^3$	$(100 \cdot 1.02^3)1.02$, or $100 \cdot 1.02^4$

Finding the final amount at the end of the nth period becomes finding the nth term of a geometric sequence whose first term is $100 \cdot 1.02$ (amount at the end of the first period) and whose ratio is 1.02. Thus, the amount at the end of the nth period is given by $(100 \cdot 1.02)(1.02)^{n-1} = 100 \cdot 1.02^n$. We can generalize this discussion. If the principal is P and the interest rate per period is r, then the amount A after n periods is $P(1 + r)(1 + r)^{n-1} = P(1 + r)^n$. Therefore, the *formula for computing the amount at the end of the nth period is*

$$A = P(1 + r)^n.$$

For convenient comparison, banks are required to report the **effective annual** yield, also called **annual** percentage yield or APY. The effective annual yield on an investment is the simple interest rate that after 1 yr would pay the same amount as the given compound rate compounded at given intervals.

Example 24

Suppose you deposit \$1000 in a savings account that pays 6% annual interest compounded quarterly.

a. What is the balance at the end of 1 yr?
b. What is the *effective annual yield* on this investment?

Solution

a. An annual interest rate of 6% earns $\frac{1}{4}$ of 6%, or an interest rate of $\frac{0.06}{4}$, in 1 quarter. Because there are 4 periods, we have the following:

$$A = 1000\left(1 + \frac{0.06}{4}\right)^4 \approx \$1061.36$$

The balance at the end of 1 yr is approximately \$1061.36.

b. Because the interest earned is \$1061.36 − \$1000.00 = \$61.36, the effective annual yield can be computed by using the simple interest formula, $I = Prt$.

$$61.36 = 1000 \cdot r \cdot 1$$
$$\frac{61.36}{1000} = r$$
$$0.06136 = r$$
$$6.136\% = r$$

The effective annual yield is 6.136%.

Example 25

To save for their child's college education, a couple deposits \$3000 into an account that pays 7% annual interest compounded daily. Find the amount in this account after 8 yr.

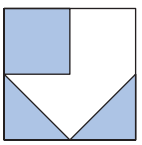
Solution The principal in the problem is \$3000, the daily rate i is 0.07/365, and the number of compounding periods is $8 \cdot 365$, or 2920. Thus, we have

$$A = \$3000\left(1 + \frac{0.07}{365}\right)^{2920} \approx \$5251.74.$$

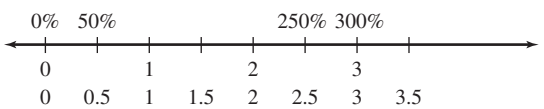
Thus, the amount in the account is approximately \$5251.74.

Assessment 7-4A

1. Estimate the percent of the square that is shaded.



2. Copy the number line and fill in the blanks so that each tick mark is labeled with a percent, a fraction, and a decimal. Write fractions in lowest terms.



3. Express each of the following as a percent.
- **a.** 7.89
- **b.** 193.1
- **c.** $\frac{5}{6}$
- **d.** $\frac{1}{8}$
- **e.** $\frac{5}{8}$
- **f.** $\frac{4}{5}$

4. Convert each of the following percents to a decimal.
- **a.** 16%
- **b.** $\frac{1}{5}$%
- **c.** $13\frac{2}{3}$%
- **d.** $\frac{1}{3}$%

5. Fill in the following blanks to find other expressions for 4%.
- **a.** _____ for every 100
- **b.** _____ for every 50
- **c.** 1 for every _____
- **d.** 8 for every _____
- **e.** 0.5 for every _____

6. Draw a percent bar to represent 125% of 60. Set up a proportion from the percent bar and solve it.

7. Answer each of the following questions.
- **a.** What is 6% of 34?
- **b.** 17 is what percent of 34?
- **c.** 18 is 30% of what number?
- **d.** What is 7% of 49?

8. **a.** Write a fraction representing 5% of x.
- **b.** If 10% of an amount is a, what is the amount in terms of a?

9. Marc had 84 boxes of candy to sell. He sold 75% of the boxes. How many did he sell?

10. Gail received a 7% raise last year. If her salary is now $27,285, what was her salary last year?

11. Joe sold 180 newspapers out of 200. Bill sold 85% of his 260 newspapers. Ron sold 212 newspapers, 80% of those he had.
- **a.** Who sold the most newspapers? How many?
- **b.** Who sold the greatest percentage of his newspapers? What percent?
- **c.** Who started with the greatest number of newspapers? How many?

12. If a dress that normally sells for $35 is on sale for $28, what is the "percent off"? (This could be called a *percent of decrease*, or a *discount*.)

13. Mort bought his house in 2006 for $359,000. It was recently appraised at $195,000. What is the approximate *percent of decrease* in value to the nearest percent?

14. Sally bought a dress marked 20% off. If the regular price was $28.00, what was the sale price?

15. An airline ticket costs $320 without the tax. If the tax rate is 5%, what is the total bill for the airline ticket?

16. Bill got 52 correct answers on an 80-question test. What percent of the questions did he answer incorrectly?

17. If $66\frac{2}{3}$% of 1800 employees favored a new insurance program, how many employees favored the new program?

18. Which represents the greater percent: $\frac{325}{500}$ or $\frac{600}{1000}$? How can you tell?

19. An advertisement reads that if you buy 10 items, you get 20% off your total purchase price. You need 8 items that cost $9.50 each.
- **a.** How much would 8 items cost? 10 items?
- **b.** Is it more economical to buy 8 items or 10 items?

20. John paid $330 for a new mountain bicycle to sell in his shop. He wants to price it so that he can offer a 10% discount and still make 20% of the price he paid for it. At what price should the bike be marked?

21. Solve each of the following using mental mathematics.
- **a.** 15% of $22
- **b.** 20% of $120
- **c.** 5% of $38
- **d.** 25% of $98

22. A crew consists of one apprentice, one journeyman, and one master carpenter. The crew receives a check for $4200 for a job they just finished. A journeyman makes 200% of what an apprentice makes, and a master makes 150% of what a journeyman makes. How much does each person in the crew earn?

23. **a.** In an incoming freshman class of 500 students, only 20 claimed to be math majors. What percent of the freshman class is this?
- **b.** When the survey was repeated the next year, 5% of nonmath majors had decided to switch and become math majors.
 - **i.** How many math majors are there now?
 - **ii.** What percent of the former freshman class do they represent?

24. Ms. Price has received a 10% raise in salary in each of the last 2 yr. If her annual salary this year is $100,000, what was her salary 2 yr ago, rounded to the nearest penny?

25. When the U.S. Congress was sent a $2.57 trillion budget, it was reported that one would have to purchase a $100 item every second for 815 yr to spend that much money.
- **a.** Decide whether or not you believe this report.
- **b.** Assuming that exactly 815 yr are required to spend the entire $2.57 trillion, what percentage of the money is spent each year?

26. If you wanted to spend 25% of your monthly salary on entertainment and 56% of the salary on rent, could those amounts be $500 and $950? Why or why not?

27. An organization has 100,000 members. A bylaw change can be made at the annual business meeting held once each year, and a bylaw change must be approved by a majority of those attending the meeting. The chair of the meeting cannot vote unless there is a tie vote but does count as an attendee at the meeting.
- **a.** With these rules, what is the minimum number required at the meeting to make a bylaw change?
- **b.** Based on your answer to part (a), what percentage of the membership can change the bylaws of the organization?

28. A tip in a restaurant has been typically figured at 15% of the total bill.
- **a.** If the bill is $30, what would be the typical tip?
- **b.** If the patron receiving the bill gave a tip that was half the bill, what is the percentage of the tip?
- **c.** If the patron receiving the bill gave a tip that was equal to the bill, what is the percentage of the tip?

29. Suppose the percent bar below shows the number of students in a school who do not favor dress codes. How many students are in the school?



30. Consider the natural numbers 1 through 50.
- **a.** What percent of these numbers have two digits?
- **b.** What percent are multiples of 9?
- **c.** What percent are even?
- **d.** What percent are prime?
- **e.** What percent are even and prime?

31. Sunscreens block ultraviolet (UV) rays produced by the sun. These rays are harmful and each sunscreen has a Sun Protection Factor (SPF) that tells you how long you can be exposed before you receive 1 minute of UV rays. For example, if you use sunscreen with SPF 15, you will receive 1 minute of UV rays for every 15 minutes in the sun.
- **a.** A sunscreen with SPF 15 blocks $\frac{14}{15}$ of the sun's UV rays. What percent is this?
- **b.** Suppose a sunscreen blocks 80% of the sun's UV rays. What fraction of the sun's UV rays does it block?
- **c.** What is the SPF for the sunscreen in part (b)?
- **d.** Carol bought sunscreen with a SPF 30 label. The label claims it block about 97% of the sun's UV rays. If the SPF 30 label is accurate, is the claim true? Explain.

32. Complete the following compound interest chart.

Compounding Period	Principal	Annual Rate	Length of Time (Years)	Interest Rate per Period	Number of Periods	Amount of Interest Paid	Total Amount in Account
a. Semiannual	$1000	6%	2		4		
b. Quarterly	$1000	8%	3		12		
c. Monthly	$1000	10%	5		60		
d. Daily	$1000	12%	4		1460		

33. Ms. Jackson borrowed $42,000 at 8.75% annual simple interest. If exactly 1 yr later she was able to repay the loan without penalty, how much interest would she owe?

34. Falafel King will need $50,000 in 5 yr for a new addition. To meet this goal, the company deposits money in an account today that pays 3% annual interest compounded quarterly. Find the amount that should be invested to total $50,000 in 5 yr.

35. To save for their retirement, a couple deposits $4000 in an account that pays 5.9% annual interest compounded quarterly. What will be the value of their investment after 20 yr?

36. Adrien and Jarrell deposit $300 on January 1 in a holiday savings account that pays 1.1% per month interest. What is the effective annual yield?

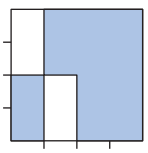
37. An amount of $3000 was deposited in a bank at a rate of 2% annual interest compounded quarterly for 3 yr. The rate then increased to 3% annual interest and was compounded quarterly for the next 3 yr. If no money was withdrawn, what was the balance at the end of this time?

38. The New Age Savings Bank advertises 4% annual interest rates compounded daily, while the Pay More Bank pays 5.2% annual interest compounded annually. Which bank offers a better rate for a customer who plans to leave her money in for exactly 1 yr?

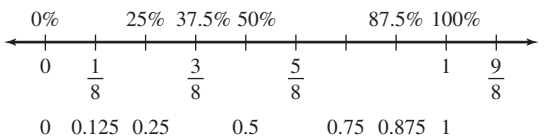
39. Amy is charged 12% annual interest compounded monthly on the unpaid balance of a $2000 loan. She did not make any payments for 2 yr. Her friend said the amount she owed had more than doubled. Is this correct? How much does she now owe?

Assessment 7-4B

1. Estimate the percent of the square that is shaded.

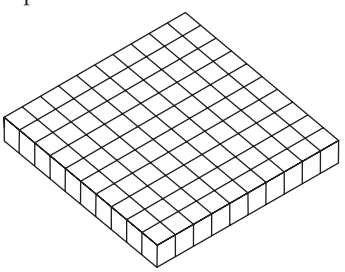


2. Copy the number line and fill in the blanks so that each tick mark is labeled with a percent, a fraction, and a decimal. Write fractions in lowest terms.



3. Express each of the following as a percent.
 a. 0.032
 b. 0.2
 c. $\frac{3}{20}$
 d. $\frac{13}{8}$
 e. $\frac{1}{6}$
 f. $\frac{1}{40}$
4. Convert each of the following percents to a decimal.
 a. $4\frac{1}{2}\%$
 b. $\frac{2}{7}\%$
 c. 125%
 d. $\frac{1}{4}\%$
5. Fill in the following blanks to find other expressions for 5%.
 a. _____ for every 100
 b. _____ for every 50
 c. 1 for every _____
 d. 8 for every _____
 e. 0.5 for every _____
6. Draw a percent bar to represent 150% of 60. Set up a proportion from the percent bar and solve it.
7. Answer each of the following questions.
 a. 63 is 30% of what number?
 b. What is 7% of 150?
 c. 61.5 is what percent of 20.5?
 d. 16 is 40% of what number?
8. a. Write a fraction representing 0.5% of x.
 b. If 0.1% of an amount is a, what is the amount in terms of a?
9. What is the sale price of a softball, if the regular price is $6.80, and there is a 25% discount?
10. Brandy received a 10% raise last year. If her salary is now $60,000, what was her salary last year?
11. A line segment X is 3 in. long. This segment represents 50% of another segment Y. Find the length of each of the following.
 a. A segment that represents 100% of segment Y.
 b. A segment that represents 25% of the segment in part (a).
 c. A segment that represents 150% of segment in part (a).
12. A used car originally cost $1700. One year later, it was worth $1400. What is the percentage of depreciation?
13. On a certain day in Glacier Park, 728 eagles were counted. Five years later, 594 were counted. What was the percentage of decrease in the number of eagles counted?
14. A salesperson earns a weekly salary of $900 plus a commission rate of 4% on all sales. What did the person make for a week with total sales of $1800?
15. If you buy a new bicycle for $380 and the sales tax is 9%, what is your total bill?
16. The number of known living species is about 1.7 million. About 4500 species are mammals. What percent of known living species are mammals?
17. Jim bought two shirts that were originally marked at $40 each. One shirt was discounted 20% and the other was discounted 25%. The sales tax was 4.5%. How much did he spend in all?
18. Without multiplying out find which number represents the greater percent: 0.625 or $(0.625)^2$?
19. a. It is recommended that no more than 30% of your calorie intake should be from fat. If you consumed about 2400 calories daily, what is the maximum amount of fat calories you should consume?
 b. If one cookie contains 140 calories, and 70 calories in the cookie are fat calories, could you eat 3 cookies and not exceed the recommended amount of fat calories for the day?
20. The price of a suit that sells for $200 is reduced by 25%. By what percent must the price of the suit be increased to bring the price back to $200?
21. Solve each of the following using mental mathematics.
 a. 15% of $42
 b. 20% of $280
 c. 5% of $28
 d. 25% of $84
22. If a $\frac{1}{4}$-cup serving of Crunchies breakfast food has 0.5% of the minimum daily requirement of vitamin C, how many cups would you have to eat to obtain the minimum daily requirement of vitamin C?
23. The car Elsie bought 1 yr ago has depreciated by $1116.88, which is 12.13% of the price she paid for it. How much did she pay for the car, to the nearest dollar?
24. If you add 20% of a number to the number itself, what percent of the result would you have to subtract to get the original number back?
25. If we build a 10×10 model with blocks, as shown in the following figure, and paint the entire model, what percent of the cubes will have each of the following?

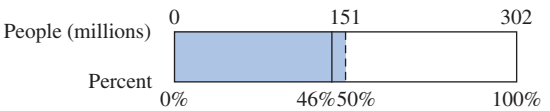
a. Four faces painted **b.** Three faces painted
c. Two faces painted



26. Cartons of milk in a store are labeled "2% milk" or "98% fat free." The data for 1 cup of whole milk shows it has 8 g of fat while 1 cup of 2% milk as 5 g of fat. What is percent of reduction in fat for 1 cup of 2% milk over whole milk?

27. There are 80 coins in a piggy bank of which 20% are quarters. What is the least possible amount of money that could be in the piggy bank?

28. According to a *TV Guide* survey, 46% of people in the United States said they would not stop watching television for anything less than a million dollars. Use the percent bar for the U.S. population to estimate the number of people who would not stop watching television for anything less than a million dollars.



29. Consider the natural numbers 1 through 50.
a. What percent are greater than 9?
b. What percent are factors of 12?
c. What percent have only even digits?
d. What percent have only odd digits?

30. Jake bought an antique clock for $30. The following year the clock had increased in value by 20%.
a. How many dollars did the clock increase in value?
b. What is the clock worth now?
c. You could calculate the worth of the clock by calculating the amount of increase in value and then add it to the original price. How would you calculate the answer in part (b) in only one step? Show the answers are the same and explain why the one-step method works.

31. Complete the following compound interest chart.

Compounding Period	Principal	Annual Rate	Length of Time (Years)	Interest Rate per Period	Number of Periods	Amount of Interest Paid	Total Amount in Account
a. Semiannual	$1000	4%	2				
b. Quarterly	$1000	6%	3				
c. Monthly	$1000	18%	5				
d. Daily	$1000	18%	4				

32. A man collected $28,500 on a loan of $25,000 he made 4 yr ago. If he charged simple interest, what was the rate he charged?

33. Green Energy store will need $100,000 in 5 yr for a new addition. To meet this goal, the company deposits money in an account that pays 3% annual interest compounded quarterly. Find the amount that should be invested to total $100,000 in 5 yr.

34. Sara invested money at a bank that paid 3.5% annual interest compounded quarterly. If she had $4650 at the end of 4 yr, what was her initial investment?

35. A bank pays 2% annual interest compounded daily. What is the value of $10,000 after 15 yr?

36. If a saving account pays 0.5% per month interest, what is the effective annual yield?

37. If a publishing company signed an agreement to allow a textbook (1st edition with 500 pages) to expand over several editions to 1000 pages, and the book was growing at approximately 10% in the number of pages over each edition, how many editions could be published before it reached the contractual limit?

38. Al invests $1000 at 6% annual interest compounded daily and Betty invests $1000 at 7% simple interest. After how many whole years will Al's investments be worth more than Betty's investment?

39. The number of trees in a rain forest decreases each month by 0.5%. If the forest has approximately $2.34 \cdot 10^9$ trees, how many trees will be left after 20 yr?

Mathematical Connections 7-4

Reasoning

1. Use mental math to find 11% of 850. Explain your method.

2. Does 0.4 = 0.4%? Explain.

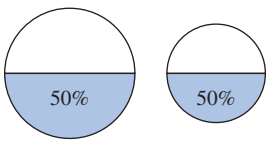
3. What does it mean to reach 125% of your savings goal?

4. Is 4% of 98 the same as 98% of 4? Explain.

5. **a.** If 25% of a number is 55, is the number greater than or less than 55? Explain.
b. If 150% of a number is 55, is the number greater than or less than 55? Explain.

6. Can 35% of one number be greater than 55% of another number? Explain.

7. Why does one picture have so much more shaded area when they both show 50%?



8. Why is it possible to have an increase of 150% in price but not a 150% decrease in price?

9. Two equal amounts of money were invested in two different stocks. The value of the first stock increased by 15% the first year and then decreased by 15% the second year. The second stock decreased by 15% the first year and increased by 15% the second year. Was one investment better than the other? Explain your reasoning.

10. Because of a recession, the value of a new house depreciated 10% each year for 3 yr in a row. Then, for the next 3 yr, the value of the house increased 10% each year. Had the value of the house increased or decreased after 6 yr? Explain.

11. Approximate the number of years it would take for any amount of money to double if it were deposited at a 2% annual interest rate compounded annually. Explain your reasoning.

12. Each year a car's value depreciated 20% from the previous year. Mike claims that after 5 yr the car would depreciate 100% and would not be worth anything. Is Mike correct? Explain why or why not. If not, find the actual percent the car would depreciate after 5 yr.

13. If 70% of the girls in a class wanted to have a prom and 60% of the boys wanted a prom, is it possible that only 50% of the students in the class wanted a prom? Explain your answer.

14. If 70% of the girls in a class wanted to have a prom and 40% of the boys wanted a prom, is it possible that only 50% of the students in the class wanted a prom? Explain your answer.

15. Abe and Barb were shopping at a department store. Abe looked at his receipt and noted that the total before the discount was \$75 but he paid only \$56.25. Barb's total before the discount was \$52 but she paid only \$31.20. Who saved the greater percentage off the original price? Explain.

16. Dan claims that the sales tax going from 5% to 6% is only increasing the sales tax by 1 percentage point. Stan says that the sales tax is going up by 20%. Who is correct? Why?

Open-Ended

17. Write and solve a word problem whose solution involves the following. If any of these tasks is impossible, explain why.
- **a.** Addition of percent
- **b.** Subtraction of percent
- **c.** Multiplication of percent
- **d.** Division of percent
- **e.** A percent whose decimal representation is raised to the second power
- **f.** A percent greater than 100

18. Look at newspapers and magazines for information given in percents.
- **a.** Based on your findings, write a problem that involves social science as well as mathematics.
- **b.** Write a clear solution to your problem in part (a).

19. Write a percentage problem whose answer is the solution of each of the following equations.
- **a.** $\frac{37}{100} = \frac{115}{x}$
- **b.** $\frac{p}{100} = \frac{a}{x}$

20. The effect of depreciation can be computed using a formula similar to the formula for compound interest.
- **a.** Assume depreciation is the same each month. Write a problem involving depreciation and solve it.
- **b.** Develop a general formula for depreciation defining what each variable in the formula represents.

21. Find four large cities around the world and an approximate percentage rate of population growth for the cities. Estimate the population in each of the four cities in 25 yr.

Cooperative Learning

22. Al read that 80% of supermarket prices end in 9 or 5. Investigate this in your group and report percentages to back up your findings.

23. Find the percentage of students in your class who engage in each of the following activities:

a. Studying and Doing Homework

Number of Hours per Week (h)	Percent
$h < 1$	
$1 \le h < 3$	
$3 \le h < 5$	
$5 \le h < 10$	
$h \ge 10$	
Total	

b. Watching TV

Number of Hours per Week (h)	Percent
$h < 1$	
$1 \le h < 5$	
$5 \le h < 10$	
$h \ge 10$	
Total	

- **c.** Did your totals add up to 100% in each table? Why or why not?

24. The federal Truth in Lending Act, passed in 1969, requires lending institutions to quote an annual percentage rate (APR) that helps consumers compare the true cost of loans regardless of how each lending institution computes the interest and adds on costs.
- **a.** Call different banks and ask for their APR on some loans and the meaning of APR.
- **b.** Based on your findings in part (a), write a definition of APR.
- **c.** Use the information given by your credit card (you may need to call the bank) and compute the APR on cash advances. Is your answer the same as that given by the bank? Compare the APR for different credit cards.

Connecting Mathematics to the Classroom

25. Jamal says that since there are 48% girls at Washington School and 52% girls at Jefferson School there must be more girls at Jefferson School. Is he correct? How would you help him?

26. A student claims that if prices at a store can increase 120%, then a candidate should be able to get 120% of the votes in an election. How do you respond?

27. A student says that $3\frac{1}{4}\% = 0.03 + 0.25 = 0.28$. Is this correct? Why?

28. A student argues that a p% increase in salary followed by a q% decrease is equivalent to a q% decrease followed by a p% increase because of the commutative property of multiplication. How do you respond?

29. Noel read that women used to make 75¢ for every dollar that men made. She says that this means that men were paid 25% more than women. Is she correct? Why?

30. A student claims that if the value of an item increases by 100% each year from its value the previous year, and the original price is d dollars, then the value after n years will be $d \cdot 2^n$ dollars. Is the student correct? Why or why not?

Review Problems

31. **a.** Human bones make up 0.18 of a person's body weight. How much do the bones of a 120 lb person weigh?
b. Muscles make up about 0.4 of a person's body weight. How much do the muscles of a 120 lb person weigh?

32. Write each of the following decimals in the form $\frac{a}{b}$, where $a, b \in I, b \neq 0$ and $\frac{a}{b}$ is in simplest terms.
a. 16.72
b. 0.003
c. $^{-}5.07$
d. 0.123

33. Write a repeating decimal equal to each of the following without using more than one zero.
a. 5
b. 5.1
c. $\frac{1}{2}$

34. Write 0.00024 as a fraction in simplest form.

35. Write $0.\overline{24}$ as a fraction in simplest form.

36. Write each of the following as a standard numeral.
a. $2.08 \cdot 10^5$
b. $3.8 \cdot 10^{-4}$

National Assessments

National Assessment of Educational Progress (NAEP) Questions

There were 90 employees in a company last year. This year the number of employees increased by 10 percent. How many employees are in the company this year?

A. 9
B. 81
C. 91
D. 99
E. 100

NAEP, Grade 8, 2005

One store, Price Pleasers, reduces the price *each week* of a $100 stereo by 10 percent of *the original* price.

Another store, Bargains Plus, reduces the price *each week* of the same $100 stereo by 10 percent of *the previous week's* price.

After 2 weeks, how will the prices at the two stores compare?

A. The price will be cheaper at Price Pleasers.
B. The price will be the same at both stores.
C. The price will be cheaper at Bargains Plus.

Explain your reasoning.

NAEP, Grade 8, 2003

Helga's process of calculating the tip to leave in a restaurant starts with the bill for food and drink.

- First, she rounds the bill to the nearest ten cents.
- Then she moves the decimal point in the rounded total one place to the left.
- Finally, she doubles that amount.

Helga's process calculates approximately what percent of the original bill?

A. 2%
B. 5%
C. 10%
D. 15%
E. 20%

NAEP, Grade 8, 2013

Hint for Solving the Preliminary Problem

The first two questions can be answered by direct computations involving percents. The third question can be investigated in several ways. One way is to try the calculations with a specific number such as 100. Another strategy is to write an equation and solve it. The answer can be checked to make sure it is correct.

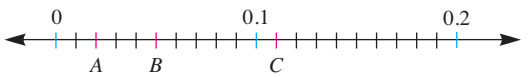
Chapter 7 Summary

KEY CONCEPTS	DEFINITIONS, DESCRIPTIONS, AND THEOREMS
Section 7-1	
Decimal point (p. 330)	A dot used to separate the whole number part from the fractional part of a decimal number. The dot in the decimal 12.618 is the *decimal point*.
Standard (or expanded) form of a decimal (p. 333)	Decimals in standard form can be expanded using place value and powers of 10. The decimal 12.618 in standard form is written as $12.618 = 1 \cdot 10^{1} + 2 \cdot 10^{0} + 6 \cdot 10^{-1} + 1 \cdot 10^{-2} + 8 \cdot 10^{-3}$.
Multiplying or dividing by 10^n (p. 335)	For any decimal and any natural number n: *Multiplying by* 10^n—Move the decimal point n places *to the right*, adjoining zeros as needed. *Dividing by* 10^n—Move the decimal point n places *to the left*, adjoining zeros as needed.
Terminating decimal (p. 336)	Any decimal that can be written with only a finite number of places to the right of the decimal point. *Theorem:* A rational number $\frac{a}{b}$ in simplest form can be written as a terminating decimal if, and only if, the prime factorization of the denominator contains no primes other than 2 or 5.
Comparing terminating decimals (p. 337)	• Write the decimals as rational numbers with a common denominator that is a power of 10, and then compare the numerators. • Line up the decimals by place value (appending zeros as necessary). Starting at the left, find the first place where the digits are different. In comparing these digits, the digit with the greater face value represents the greater of the two decimals.
Section 7-2	
Adding terminating decimals (p. 340)	• Write the decimals as fractions with denominators that are a power of 10, and then add or subtract. • Line up the decimals points in the decimals, and then add or subtract as we do with whole numbers, placing the decimal point of the answers in line with the other decimal points.
Multiplying terminating decimals (p. 342)	Multiply the two numbers, ignoring the decimal points. If there are n digits after the decimal point in the first number and m digits after the decimal point in the second number, then place the decimal point in the resulting product so there are $m + n$ digits after the decimal point.
Scientific notation (p. 344)	Any positive number written as the product of a number greater than or equal to 1 and less than 10 and an integer power of 10. To write a negative number in scientific notation, treat the number as a positive number and adjoin the negative sign in front of the result.
Writing a number in scientific notation (p. 344)	Move the decimal point so that there is only one nonzero digit to the left of the decimal point. If the decimal point is moved *to the left* n places, then multiply the resulting number by 10^n. If the decimal point is moved *to the right* n places, then multiply the resulting number by 10^{-n}.

Section 7-2	
Dividing terminating decimals (p. 345)	Multiply the divisor and dividend by the power of 10 that gives a whole number divisor. Then divide, lining up the decimal point in the quotient with the decimal point in the dividend.
Mental computation algorithms for decimals (p. 348)	• *Breaking and bridging* • *Using compatible numbers* • *Making compatible numbers* • *Balancing with decimals (in subtraction)* • *Balancing with decimals (in division)*
Estimating decimal computations using rounding (p. 349)	Round numbers appropriately to estimate answers for quick computation.
Round-off error (p. 351)	An error that arises when computing with approximate numbers. The final result should not be reported using more significant digits than the number with the fewest significant digits used in the computation.
Section 7-3	
Repeating decimal (p. 357)	Any decimal that has a repeating block of digits, called a *repetend*. The number of digits that repeat is the *period* of the repetend.
Writing a repeating decimal as a fraction (p. 360)	A repeating decimal can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. There are several ways to show that $0.\overline{9} = 1$.
Ordering repeating decimals (p. 362)	• Consider where the repeating decimals might lie on a number line. • Compare the decimals using place value similarly to how terminating decimals are compared.
Section 7-4	
Percent (p. 366)	$n\% = \frac{n}{100}$, where n is a nonnegative number.
Percent application problems (p. 368)	Application problems involving percents take one of the following forms: • *Finding a percent of a number* • *Finding what percent one number is of another number* • *Finding a number when a percent of that number is known*
Percent bar (p. 369)	A graphical model that can be used for understanding what a given percent of a number means.
Mental computation algorithms for percents (p. 373)	• *Using fraction equivalents* • *Using known percents*
Computing interest (p. 374)	*Principal*—the original amount deposited or loaned. *Interest*—the amount of money the bank will pay or charge. *Interest rate*—the percent used to determine the interest. *Simple interest*—the interest computed using the formula $I = Prt$, where P is the principal, r is the annual interest rate, and t is the time in years. *Compound interest*—the interest based on using both the principal and the accrued interest; it is computed using the formula $A = P(1 + r)^n$, where A is the balance, P is the principal, r is the interest rate per period, and n is the number of periods.

Chapter 7 Review

1. **a.** On the number line, find the decimals that correspond to points A, B, and C.
b. Indicate by D the point that corresponds to 0.09 and by E the point that corresponds to 0.15.



2. Write each of the following as a rational number in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ and $\frac{a}{b}$ is in simplest form.
a. 32.012 **b.** 0.00103

3. Describe a test to determine whether a fraction can be written as a terminating decimal without actually performing the division.

4. A board is 442.4 cm long. How many shelves can be cut from it if each shelf is 55.3 cm long? (Disregard the width of the cuts.)

5. Write each of the following as a decimal.
a. $\frac{4}{7}$ **b.** $\frac{1}{8}$ **c.** $\frac{2}{3}$ **d.** $\frac{5}{8}$

6. Write each of the following as a fraction in simplest form.
a. 0.28 **b.** $^{-}6.07$ **c.** $0.\overline{3}$ **d.** $2.0\overline{8}$

7. Round each of the following numbers as specified.
a. 307.625 to the nearest hundredth
b. 307.625 to the nearest tenth
c. 307.625 to the nearest unit
d. 307.625 to the nearest hundred

8. Rewrite each of the following in scientific notation.
a. 426,000 **b.** $324 \cdot 10^{-6}$
c. 0.00000237 **d.** $^{-}0.325$

9. Order the following decimals from greatest to least.
$1.45\overline{19}$, $1.451\overline{9}$, 1.4519, $1.45\overline{19}$, $^{-}0.134$, $^{-}0.13\overline{401}$, $0.13\overline{401}$

10. Write each of the following in scientific notation without using a calculator.
a. 1783411.56
b. $\frac{347}{10^8}$
c. $49.3 \cdot 10^8$
d. $29.4 \cdot \frac{10^{12}}{10^{-4}}$
e. $0.47 \cdot 1000^{12}$
f. $\frac{3}{5^9}$

11. **a.** Find five decimals between 0.1 and 0.11 and order them from greatest to least.
b. Find four decimals between 0 and 0.1 listed from least to greatest, so each decimal starting from the second is twice as large as the preceding one.
c. Find four decimals between 0.1 and 0.2 and list them in increasing order, so the first one is halfway between 0.1 and 0.2, the second halfway between the first and 0.2, the third halfway between the second and 0.2, and similarly for the fourth one.

12. Answer each of the following questions.
a. 6 is what percent of 24?
b. What is 320% of 60?
c. 17 is 30% of what number?
d. 0.2 is what percent of 1?

13. Change each of the following to a percent.
a. $\frac{1}{8}$ **b.** $\frac{3}{40}$
c. 6.27 **d.** 0.0123
e. $\frac{3}{2}$

14. Change each of the following percents to a decimal.
a. 60% **b.** $\frac{2}{3}$% **c.** 100%

15. Sandy received a dividend that equals 11% of the value of her investment. If her dividend was \$1020.80, how much was her investment?

16. Five computers in a shipment of 150 were found to be defective. What percent of the computers were defective?

17. On a mathematics examination, a student missed 8 of 70 questions. What percent of the questions, rounded to the nearest tenth of a percent, did the student answer correctly?

18. A used car costs \$3450 at present. This is 60% of the cost 4 yr ago. What was the cost of the car 4 yr ago?

19. If, on a purchase of one new suit, you are offered successive discounts of 5%, 10%, or 20% in any order you wish, what order should you choose?

20. Jane bought a bicycle and sold it for 30% more than she paid for it. She sold it for \$104. How much did she pay for it?

21. The student bookstore had a textbook for sale at \$89.95. A student found the book on eBay for \$62.00. If the student bought the book on eBay, what percentage of the cost of the bookstore book did she save?

22. When a store had a 60% off sale, Dori had a coupon for an additional 40% off any item and thought she should be able to obtain the dress that she wanted for free. If you were the store manager, how would you explain the mathematics of the situation to her?

23. A store marks up an item by adding 30% to the wholesale price. Later the store has a sale and marks the item down 30% from the retail price. Is the store breaking even on the item, making a profit, or losing money on the item?

24. A company was offered a \$30,000 loan at a 12.5% annual simple interest rate for 4 yr. Find the simple interest due on the loan at the end of 4 yr.

25. A fund pays 14% annual interest compounded quarterly. What is the value of a \$10,000 investment after 3 yr?

26. The crust of a certain pumpkin pie is 25% of the pie. By what percent should the amount of crust be reduced in order to make it constitute 20% of the pie?

Chapter 8

Real Numbers and Algebraic Thinking

8-1 Real Numbers

8-2 Variables

8-3 Equations

8-4 Functions

8-5 Equations in a Cartesian Coordinate System

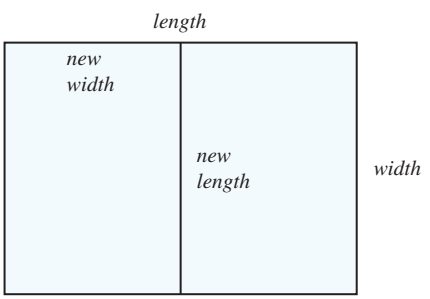


Preliminary Problem

A sheet of paper of the international standard size (used in most countries other than the United States and Canada) has the ratio of length (the longer side) to width (the shorter side) such that when cut or folded widthwise, the halves have the same length to width ratio.

a. What is that ratio?

b. what will the ratio of length to width of a larger sheet of paper be if two such standard sheets are joined along their length?



If needed, see Hint on page 461.

The development of numbers was inspired by such commercial uses as negative numbers to indicate debts and to think about fractions as parts of a whole. The development of real numbers was comparable. The Pythagoreans knew a right triangle with two sides of length 1 had to have a new type of number to describe the length of the third side. Ancient Greeks struggled with finding a number to describe the ratio of the circumference of a circle to its diameter. These examples involve irrational numbers, but the formal definition of a real number came long after such numbers began to be used.

The grade 8 *Common Core Standards* recognize that students need to know that "there are numbers that are not rational, and [how to] approximate them by rational numbers." (p. 54)

In this chapter, we explore real numbers and examine aspects of algebra including the analysis and solution of linear equations, pairs of simultaneous linear equations, and other algebraic notions including functions, and the coordinate system.

These topics are reflected in the grade 8 *Common Core Standards* in the following:

- **Work with radicals and integer exponents**
- **Analyze and solve linear equations and pairs of simultaneous linear equations. (p. 53)**

8-1 Real Numbers

8-1 Objectives

Students will be able to understand and explain

- Real numbers
- Operations on real numbers
- *n*th roots and in particular square root.
- How $a^{1/n}$ and $a^{n/m}$ are defined and why they are defined in this way.
- Expressions involving radicals and radical exponents.

In Chapter 7, we saw that every rational number can be represented either as a terminating decimal or as a repeating decimal. But the ancient Greeks discovered numbers that are not rational, that is, numbers with a decimal representation that neither terminates nor repeats. To find such decimals, consider their characteristics.

1. There must be an infinite number of nonzero digits to the right of the decimal point.
2. There cannot be a repeating block of digits (a *repetend*).

One way to construct a nonterminating, nonrepeating decimal is to devise a pattern of infinite digits so that there is no repetend. For example, consider the number 0.1010010001 If the pattern continues, the next groups of digits are four 0's followed by 1, five 0's followed by 1, and so on. Because this decimal is nonterminating and nonrepeating, it cannot represent a rational number. Numbers that are not rational numbers are **irrational numbers**, and there are infinitely many of them. In the mid-eighteenth century, it was proved that the ratio of the circumference of a circle to its diameter, symbolized by π **(pi)**, is an irrational number. The numbers $\frac{22}{7}$, 3.14, or 3.14159 are rational number approximations of π. The value of π has been computed to over 5 trillion decimal places and the digits appear to be random.

Historical Note



Although the Pythagoreans seem to have known about some irrational numbers, it was not until 1872 that German mathematician Richard Dedekind introduced a method of constructing irrational numbers from sets of rational numbers by showing that an irrational number could be thought of as a cut in a number line that separates the set of rational numbers into two disjoint non-empty subsets, one of which has no least element and the other of which has no greatest element, with all elements of one set less than the elements of the other set. ●

The set of **real numbers** is the union of the set of rational and irrational numbers. Every real number can be represented as a decimal and corresponds to a point on a number line.

Definition of Real Numbers

A **real number** is any number that can be represented by a decimal.

Square Roots

Irrational numbers occur in the study of area. For example, to find the area of a square as in Figure 1(a), we could use the formula $A = s^2$, where A is the area and s is the length of a side of the square. If a side of a square is 3 cm long as in Figure 1(b), then the area of the square is 9 cm^2 (square centimeters).

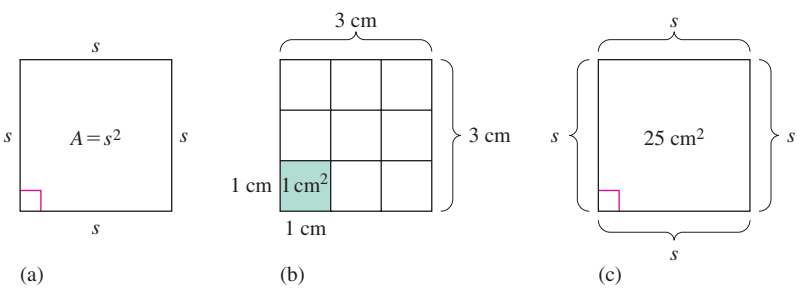


Figure 1

Conversely, given the area, we can use the formula to find the length s of a side of a square. If the area of a square is 25 cm^2 as in Figure 1(c), then $s^2 = 25$ and hence $s = 5$ or $s = -5$.

Each of these solutions is a **square root** of 25. However, because lengths are always non-negative, 5 is the only possible solution to the equation. The non-negative solution of $s^2 = 25$(namely, 5) is the **principal square root** of 25 and is denoted $\sqrt{25}$. Similarly, the principal square root of 2 is denoted $\sqrt{2}$. Also $\sqrt{16} \neq -4$ because -4 is not the principal square root of 16.

Definition of the Square Root and Principal Square Root

1. A **square root** of non-negative number a is a number x, such that $x^2 = a$.
2. If a is any non-negative number, the **principal square root** of a (denoted $\sqrt{a}$) is the non-negative number x such that $x^2 = a$.

REMARK The phrase "the square root of a" is commonly used to mean the principal square root of a.

Example 1

Find the following:

a. The square roots of 144
b. $\sqrt{144}$
c. $\sqrt{\frac{4}{9}}$
d. The solutions of $x^2 = 5$.

Solution

a. The square roots of 144 are 12 and $^-12$.
b. $\sqrt{144} = 12$
c. $\sqrt{\frac{4}{9}} = \frac{2}{3}$
d. $\sqrt{5}$ or $^-\sqrt{5}$

Other Roots

We have seen that the non-negative solution to $s^2 = 25$ is denoted $\sqrt{25}$. Similarly, the non-negative solution to $s^4 = 25$ is denoted $\sqrt[4]{25}$. In general, if n is even, the positive solution to $x^n = 25$ is $\sqrt[n]{25}$ and is the principal ***n*th root** of 25. The number n is the **index**. In the expression $\sqrt{25}$, the index 2 is understood and not written. In general, *the non-negative solution to* $x^n = b$, *where* b *is non-negative, is* $\sqrt[n]{b}$. Thus if b is non-negative, $\sqrt[n]{b}$ is a non-negative number, which raised to the nth power results in b. Hence:

$$\left(\sqrt[n]{b}\right)^n = b$$

If b is negative and n is even, $\sqrt[n]{b}$ is not a real number. For example, $\sqrt[4]{^-16}$. If $\sqrt[4]{^-16} = x$, then $x^4 = ^-16$. Because any nonzero real number raised to the fourth power is positive, there is no real-number solution to $x^4 = ^-16$ and therefore $\sqrt[4]{^-16}$ is not a real number. Similarly, if we are restricted to real numbers, it is not possible to find *any* even root of a negative number. However, the value $^-2$ is the only real number solution to the equation $x^3 = ^-8$. We write, $\sqrt[3]{^-8} = ^-2$. Because there is only one real number solution to $x^3 = ^-8$, $\sqrt[3]{^-8}$ is the cube root of $^-8$, and there is no meaning to "principle cube root of $^-8$." *In general, the odd root of a negative number is a negative number.*

Because $\sqrt{a}$, if it exists, is non-negative by definition, $\sqrt{(^-3)^2} = \sqrt{9} = 3$. In general,

$$\sqrt{a^2} = |a|.$$

Similarly, $\sqrt[4]{a^4} = |a|$ and $\sqrt[6]{a^6} = |a|$, but $\sqrt[3]{a^3} = a$ for all a. (Why?) Notice that when n is even and $b > 0$, the equation $x^n = b$ has two real-number solutions, $\sqrt[n]{b}$ and $^-\sqrt[n]{b}$. If n is odd, the equation has only one real-number solution, $\sqrt[n]{b}$, for any real number b.

Historical Note



Evaluating square roots may have been known by Vedic Hindu scholars before 600 BCE. Sanskrit texts contain incredibly accurate approximations for some square roots. The discovery of irrational numbers by members of the Pythagorean Society was kept secret. Legend has it that a society member was drowned because the secret was relayed to persons outside the society. In 1525, Christoff Rudolff, a German mathematician, became the first to use the symbol $\sqrt{\ }$ for a radical or a root.

Irrationality of Square Roots and Other Roots

Some square roots are rational numbers. Others, like $\sqrt{2}$, are irrational numbers. To see this, we note that $1^2 = 1$ and $2^2 = 4$ and that there is no whole number s such that $s^2 = 2$. Is there a rational number $\frac{a}{b}$ such that $\left(\frac{a}{b}\right)^2 = 2$? To decide, we use the strategy of *indirect reasoning*. If we assume there is such a rational number $\frac{a}{b}$, then the following must be true:

$$\left(\frac{a}{b}\right)^2 = 2$$

$$\frac{a^2}{b^2} = 2$$

$$a^2 = 2b^2$$

If $a^2 = 2b^2$, then by the Fundamental Theorem of Arithmetic, the prime factorizations of a^2 and $2b^2$ must be the same. In particular, the prime 2 appears the same number of times in the prime factorization of a^2 as it does in the factorization of $2b^2$. Because $b^2 = bb$, no matter how many times 2 appears in the prime factorization of b, it appears twice as many times in bb.

Also, a^2 has an even number of 2s for the same reason b^2 does. In $2b^2$, another factor of 2 is introduced, resulting in an odd number of 2s in the prime factorization of $2b^2$. Because $a^2 = 2b^2$, we have 2 appearing both an odd number of times and an even number of times on different sides of the equality, and thus there is a contradiction.

This contradiction could have been caused only by the assumption that $\sqrt{2}$ is a rational number. Consequently, $\sqrt{2}$ cannot be rational and therefore, $\sqrt{2}$ must be an irrational number. We can use a similar argument to show that $\sqrt{3}$ is irrational or $\sqrt{n}$ is irrational, where n is a whole number greater than 1 but not the square of a whole number.

Example 2

Show that if $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers (a, b, c, and d integers), then $\frac{a}{b} + \frac{c}{d}\sqrt{2}$ is an irrational number.

Solution We use *indirect reasoning*. Suppose the expression is a rational number; then $\frac{a}{b} + \frac{c}{d}\sqrt{2} = \frac{e}{f}$, where e and f are integers. We can obtain a contradiction by expressing $\sqrt{2}$ in terms of the integers a, b, c, d, e, and f, which will show that $\sqrt{2}$ is rational. This would be a contradiction to the fact that $\sqrt{2}$ is an irrational number. It would follow then that the assumption that $\frac{a}{b} + \frac{c}{d}\sqrt{2}$ is a rational number is wrong and consequently that it is an irrational number. We proceed according to this plan.

$$\frac{a}{b} + \frac{c}{d}\sqrt{2} = \frac{e}{f} \quad \text{implies}$$

$$\frac{c}{d}\sqrt{2} = \frac{e}{f} - \frac{a}{b} = \frac{eb - af}{bf}$$

We multiply both sides of the above equation by $\frac{d}{c}$ to isolate $\sqrt{2}$.

$$\frac{d}{c} \cdot \frac{c}{d}\sqrt{2} = \frac{d}{c} \cdot \frac{eb - af}{bf}$$

$$\sqrt{2} = \frac{deb - daf}{cbf}$$

Because all the letters are integers, it follows that $\sqrt{2}$ is a rational number, a contradiction. Consequently, $\frac{a}{b} + \frac{c}{d}\sqrt{2}$ is an irrational number.

Many irrational numbers can be interpreted geometrically. For example, we find a point on a number line to represent $\sqrt{2}$ by using the **Pythagorean Theorem**. That is, *if a and b are the lengths of the shorter sides (legs) of a right triangle and* c *is the length of the longer side (hypotenuse), then* $a^2 + b^2 = c^2$, as shown in Figure 2.

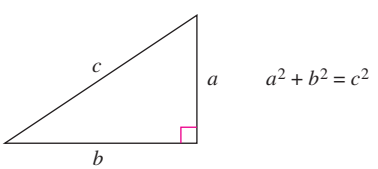


Figure 2

Figure 3 shows a segment 1 unit long constructed perpendicular to a number line at point *P*. Thus two sides of the triangle shown are each 1 unit long. If $a = b = 1$, then $c^2 = 2$ and $c = \sqrt{2}$. To find a point on the number line that corresponds to $\sqrt{2}$, we need to find a point *Q* on the number line such that the distance from 0 to *Q* is $\sqrt{2}$. Because $\sqrt{2}$ is the length of the hypotenuse, the point *Q* can be found by marking an arc with center 0 and radius *c*. The intersection of the positive number line with the arc is point *Q*.

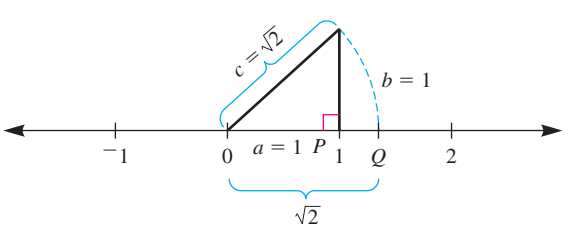


Figure 3

Similarly, other principal square roots can be constructed, as shown in Figure 4 and placed on a number line using the method of Figure 3. Also, their opposites can then be constructed, as shown in Figure 4.

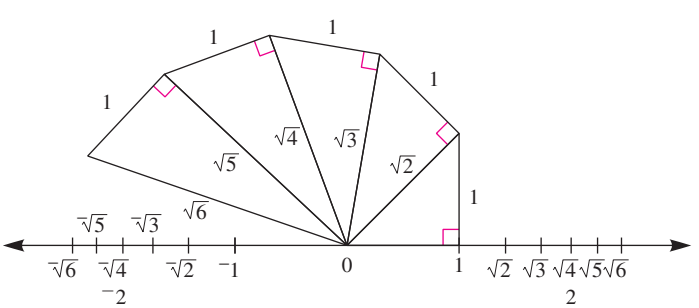


Figure 4

Estimating a Square Root

From Figure 4, we see that $\sqrt{2}$ must have a value between 1 and 2; that is, $1 < \sqrt{2} < 2$. To obtain a closer approximation of $\sqrt{2}$, we attempt to "squeeze" $\sqrt{2}$ between two numbers that are between 1 and 2. Because $(1.4)^2 = 1.96$ and $(1.5)^2 = 2.25$, it follows that $1.4 < \sqrt{2} < 1.5$. Because a^2 can be interpreted as the area of a square with side of length a, this discussion can be pictured geometrically, as in Figure 5.

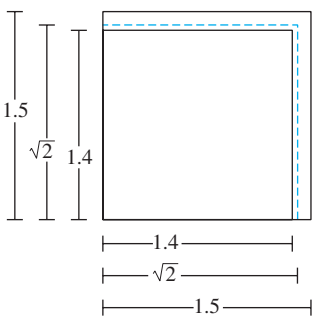


Figure 5

For a more accurate approximation for $\sqrt{2}$, we continue this squeezing process. We see that $(1.4)^2 = 1.96$, is closer to 2 than is $(1.5)^2 = 2.25$, so we choose numbers closer to 1.4 to find the next approximation. We find the following:

$$(1.42)^2 = 2.0164$$
$$(1.41)^2 = 1.9981$$

Thus, $1.41 < \sqrt{2} < 1.42$. We continue this process until we obtain the desired approximation. If a calculator has a square-root key, we can obtain an approximation directly. In fact, $\sqrt{2}$ is approximately 1.41421356.

NOW TRY THIS 1

One algorithm for calculating square roots is sometimes attributed to Archimedes though the Babylonians had a similar method. The method is also referred to as Newton's method. The algorithm finds closer and closer estimates to the square root. To find the square root of a positive number, n, we first make a positive number guess, *Guess*1. Now compute as follows:

Step 1. Divide n by *Guess*1.
Step 2. Now add *Guess*1 to the quotient obtained in step 1.
Step 3. Divide the sum in step 2 by 2. The quotient becomes *Guess*2.
Step 4. Repeat steps 1–3 using *Guess*2 to obtain a successive guess.
Step 5. The process may be repeated until the desired accuracy is achieved.

a. Use the method described to find the square root of 13 to three decimal places.
b. Write the steps for the algorithm in a recursive formula.

The System of Real Numbers, Operations, and Their Properties

Real numbers can be represented as, terminating, repeating, or nonterminating and nonrepeating decimals.

Every integer is a rational number as well as a real number. Every rational number is a real number, but not every real number is rational, as has been shown with $\sqrt{2}$. The relationships among sets of numbers are summarized in the diagram in Figure 6.

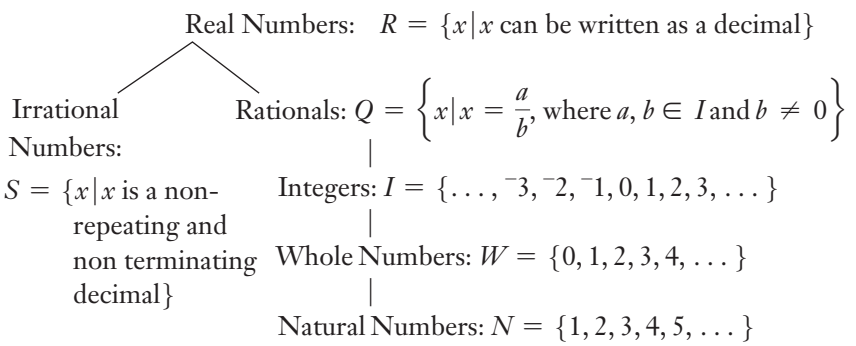


Figure 6

The set of fractions can be extended to be of the form $\frac{a}{b}$, where a and b are real numbers with $b \neq 0$, such as $\frac{\sqrt{3}}{5}$.

Addition, subtraction, multiplication, and division are defined on the set of real numbers in such a way that all the properties of these operations on rationals still hold. The properties are summarized next.

Theorem 8-1: Properties of Operations on Real Numbers

Closure properties	For real numbers a and b, $a + b$ and ab are unique real numbers.
Commutative properties	For real numbers a and b, $a + b = b + a$ and $ab = ba$.
Associative properties	For real numbers a, b, and c, $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$.
Identity properties	The number 0 is the unique additive identity because for any real number a, $a + 0 = a = 0 + a$. The number 1 is the unique multiplicative identity because for any real number a, $a \cdot 1 = a = 1 \cdot a$.
Inverse properties	For every real number a, ${}^{-}a$ is its unique additive inverse; that is, $a + {}^{-}a = 0 = {}^{-}a + a$. For every nonzero real number a, $\frac{1}{a}$ (or a^{-1}) is its unique multiplicative inverse; that is, $a\left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right)a$.
Distributive property of multiplication over addition and subtraction	For real numbers a, b, and c, $a(b + c) = ab + ac$; $a(b - c) = ab - ac$.
Multiplication property of zero	For all real numbers a, $a \cdot 0 = 0 = 0 \cdot a$.
Density property for real numbers	For real numbers a and b with $a < b$, there exists a real number c such that $a < c < b$.
Inverse properties	For every real number a, ${}^{-}a$ is it unique additive inverse; that is $a + {}^{-}a = 0 = {}^{-}a + a$. For every nonzero real number a, $\frac{1}{a}$(or a^{-1}) is its unique multiplicative inverse, that is, $a\left(\frac{1}{a}\right) = 1 = \left(\frac{1}{a}\right)a$.

By the denseness property, we can always find a real number between any two other real numbers. This leads to infinitely many real numbers.

Historical Note



In 1874 Georg Cantor showed that the set of all real numbers cannot be put in a one-to-one correspondence with the set of natural numbers and showed that there are infinities of many different sizes. ●

Order of Operations

The set of real numbers follows the same order of operations given earlier; that is, when addition, subtraction, multiplication, division, and exponentiation appear without parentheses, exponentiation is done first in order from right to left, then multiplications and divisions in the order of their appearance from left to right, and then additions and subtractions in the order of their appearance from left to right. Arithmetic operations that appear inside parentheses must be done first.

Radicals and Rational Exponents



Scientific calculators have a $\boxed{y^x}$ key with which we find the values of expressions like $4^{1/2}$ and $3.41^{2/3}$. What does $4^{1/2}$ mean? By extending the properties of exponents previously developed for integer exponents,

$$4^{1/2} \cdot 4^{1/2} = 4^{(1/2+1/2)} = 4^1$$
$$(4^{1/2})^2 = 4^1 = 4$$

and consequently, $4^{1/2}$ must be a square root of 4. To obtain a unique value, it is defined to be the principal square root, that is:

$$4^{1/2} = \sqrt{4}$$

In general, if x is a nonnegative real number, then $x^{1/2} = \sqrt{x}$. Similarly, $(x^{1/3})^3 = x^{(1/3)3} = x^1$ and $x^{1/3} = \sqrt[3]{x}$. This discussion leads to the following definitions.

Definition of Real Number Raised to Rational Exponent

If x is a real number, n is a positive integer and m is an integer then:

1. $x^{1/n} = \sqrt[n]{x}$, when $\sqrt[n]{x}$ is meaningful.
2. $x^{m/n} = \sqrt[n]{x^m}$, if $\text{GCD}(m, n) = 1$ and $\sqrt[n]{x^m}$ is meaningful.

From part (1) it follows that $(x^m)^{1/n} = \sqrt[n]{x^m}$ if $\sqrt[n]{x^m}$ is meaningful. If n is odd then $\sqrt[n]{x^n} = x$ for all real x, but if n is even then $\sqrt[n]{x^n} = |x|$. For example $\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2$.

More about Properties of Exponents

It can be shown that the properties of integer exponents also hold for rational exponents. These properties are equivalent to the corresponding properties of radicals *if the expressions involving radicals are meaningful.*

Let r and s be any rational numbers, x and y be any real numbers, and n a positive integer. If all the expressions are meaningful then,

a. $x^{-r} = \dfrac{1}{x^r}$.

b. $(xy)^r = x^r y^r$ which implies $(xy)^{1/n} = x^{1/n}y^{1/n}$ and $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$.

c. $\left(\dfrac{x}{y}\right)^r = \dfrac{x^r}{y^r}$ which implies $\left(\dfrac{x}{y}\right)^{1/n} = \dfrac{x^{1/n}}{y^{1/n}}$ and $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$.

d. $(x^r)^s = x^{rs}$ which implies $(x^{1/n})^p = x^{p/n}$ and hence $\left(\sqrt[n]{x}\right)^p = \sqrt[n]{x^p}$, where p is an integer.

The preceding properties can be used to write equivalent and simplified expressions for many roots. For example, $\sqrt{96} = \sqrt{16 \cdot 6} = \sqrt{16}\sqrt{6} = 4\sqrt{6}$. Similarly, $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27}\sqrt[3]{2} = 3\sqrt[3]{2}$.

Example 3

Simplify each of the following if possible.

a. $16^{1/4}$ **b.** $16^{5/4}$ **c.** $(-8)^{1/3}$

d. $125^{-4/3}$ **e.** $(-16)^{1/4}$

Solution

a. $16^{1/4} = \left(2^4\right)^{1/4} = 2^1 = 2$. Also $16^{1/4} = \sqrt[4]{16} = 2$

b. $16^{5/4} = 16^{(1/4)5} = \left(16^{1/4}\right)^5 = 2^5 = 32$

c. $(-8)^{1/3} = \left((-2)^3\right)^{1/3} = (-2)^1 = -2$ or $(-8)^{1/3} = \sqrt[3]{-8} = -2$

d. $125^{-4/3} = \left(5^3\right)^{-4/3} = 5^{-4} = \dfrac{1}{5^4} = \dfrac{1}{625}$

e. Because every real number raised to the fourth power is positive, $\sqrt[4]{-16}$ is not a real number. Consequently, $(-16)^{1/4}$ is not a real number. A simplification is not possible.

NOW TRY THIS 2

Compute $\sqrt[8]{10}$ on a calculator using the following sequence of keys:

$$\boxed{10}\ \boxed{\sqrt{\ }}\ \boxed{\sqrt{\ }}\ \boxed{\sqrt{\ }}$$

a. Explain why this approach works.

b. For what values of n can $\sqrt[n]{10}$ be computed using only the $\boxed{\sqrt{\ }}$ key? Why?

NOW TRY THIS 3

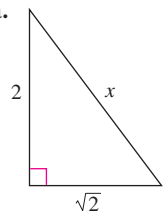
The properties of integer exponents were extended to rational exponents and can be extended to real number exponents. Consider a base with a decimal exponent such as $8^{0.\overline{3}}$ and $8^{0.101001\ldots}$. Write an explanation for a possible meaning of these decimal exponents.

Assessment 8-1A

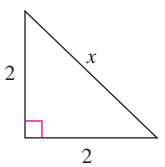
1. Write an irrational number whose digits are 2s and 3s.

2. Use the Pythagorean Theorem to find the value for x.

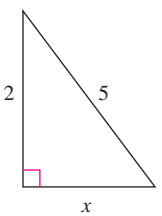
a.



b.



c.



3. Arrange the following real numbers in order from greatest to least.

$$0.9,\quad 0.\overline{9},\quad 0.\overline{98},\quad 0.98\overline{8},\quad 0.99\overline{8},\quad 0.8\overline{98},\quad \sqrt{0.98}$$

4. Determine which of the following represent irrational numbers.

a. $\sqrt{51}$ **b.** $\sqrt{64}$ **c.** $\sqrt{324}$
d. $\sqrt{325}$ **e.** $2 + 3\sqrt{2}$ **f.** $\sqrt{2} \div 5$

5. If possible, find each of the following without using a calculator.

a. The principal square root of 225
b. $\sqrt{169}$
c. $^{-}\sqrt{81}$
d. $\sqrt{625}$
e. $\sqrt{\frac{1}{4}}$
f. $\sqrt{0.0001}$

6. Find the approximate square root of each of the following, rounded to hundredths, using the squeezing method.

a. 7 **b.** 0.0120

7. Classify each of the following statements as true or false. If false, give a counterexample.

a. The sum of any rational number and any irrational number is a rational number.
b. The sum of any two irrational numbers is an irrational number.
c. The product of any two irrational numbers is an irrational number.
d. The difference of two irrational numbers could be a rational number.

8. Find three irrational numbers between each of the following pairs of rational numbers.

a. 1 and 3
b. $0.\overline{54}$ and $0.\overline{55}$
c. $\frac{1}{2}$ and $\frac{1}{3}$

9. Based on your answers in exercise 8, argue that there are infinitely many irrational numbers.

10. If R is the set of real numbers, Q is the set of rational numbers, I is the set of integers, W is the set of whole numbers, and S is the set of irrational numbers, find each of the following:

a. $Q \cup S$ **b.** $Q \cap S$ **c.** $Q \cap R$
d. $S \cap W$ **e.** $W \cup R$ **f.** $Q \cup R$

11. The letters in the following table correspond to the sets listed in exercise 10. Complete the table by placing checkmarks in the appropriate columns. (N is the set of natural numbers.)

	N	*I*	*Q*	*S*	*R*
a. 6.7					
b. 5					
c. $\sqrt{2}$					
d. $^{-}5$					
e. $3\frac{1}{7}$					

12. The letters in following table correspond to the sets listed in exercise 10. Put a checkmark under each set of numbers for which a solution to the problem exists. (N is the set of natural numbers.)

	N	*I*	*Q*	*S*	*R*
a. $x^2 + 1 = 5$					
b. $2x - 1 = 32$					
c. $x^2 = 3$					
d. $\sqrt{x} = {}^{-}1$					
e. $\frac{3}{4}x = 0.\overline{4}$					

13. Determine for what real number values of x, if any, each of the following statements is true.

a. $\sqrt{x} = 8$ **b.** $\sqrt{x} = {}^{-}8$
c. $\sqrt{^{-}x} = 8$ **d.** $\sqrt{^{-}x} = {}^{-}8$
e. $\sqrt{x} > 0$ **f.** $\sqrt{x} < 0$

14. Write each of the following roots in the form $a\sqrt{b}$, where a and b are integers and b has the least value possible.

a. $\sqrt{180}$ **b.** $\sqrt{363}$ **c.** $\sqrt{252}$

15. Write each of the following in the simplest form or as $a\sqrt[n]{b}$, where a and b are integers, $b > 0$, and b has the least value possible.

a. $\sqrt[3]{^{-}54}$ **b.** $\sqrt[5]{96}$
c. $\sqrt[3]{250}$ **d.** $\sqrt[5]{^{-}243}$

16. If each of the following is a part of a geometric sequence, find the missing terms.

a. 5, ____, ____, 10
b. 2, ____, ____, ____, 1
c. ____, 5, ____, 3
d. ____, $^{-}2$, ____, $^{-}3$

17. The expression $2^{10} \cdot 16^t$ approximates the number of bacteria after t hr.
- **a.** What is the initial number of bacteria, that is, the number when $t = 0$?
- **b.** After $\frac{1}{4}$ hr, how many bacteria are there?
- **c.** After $\frac{1}{2}$ hr, how many bacteria are there?

18. Evaluate each of the following without using a calculator.
- **a.** $(0.008)^{\frac{2}{3}}$
- **b.** $(6.25)^{\frac{3}{2}}$
- **c.** $\left(\frac{25}{81}\right)^{-\frac{3}{2}}$
- **d.** $(0.0000128)^{\frac{3}{7}}$
- **e.** $(^{-}27)^{-\frac{4}{3}}$
- **f.** $^{-}(27)^{-\frac{4}{3}}$

19. In each of the following, find all real numbers x that satisfy the given equation. If none satisfy the equation, then state the answer as "None."
- **a.** $3^x = 243$
- **b.** $9^{-x} = 27$
- **c.** $\left(\frac{9}{4}\right)^{3x} = \frac{32}{243}$
- **d.** $\sqrt{^{-}x} = 3\sqrt{2}$
- **e.** $x^{-\frac{3}{4}} = 2$
- **f.** $(x - 1)^2 = 2$

20. Write an equation in x not involving radical signs or fractional exponents with the following as a solution.
- **a.** $\sqrt[5]{20}$
- **b.** $\sqrt[3]{^{-}2}$
- **c.** $\sqrt[3]{10} - 1$
- **d.** $\frac{\sqrt{2}}{\sqrt{3}}$

21. Use the "guess and check" strategy to approximate $\sqrt[3]{103}$ to the nearest whole number.

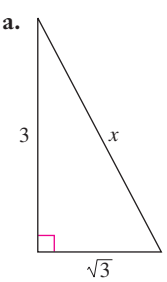
22. Determine which of the following are irrational.
- **a.** $\frac{\sqrt{500}}{\sqrt{20}}$
- **b.** $8^{\frac{1}{3}} + 8^{-\frac{1}{3}}$
- **c.** $\frac{2}{\sqrt{2}} - \sqrt{2}$
- **d.** $\sqrt{1000}$

Assessment 8-1B

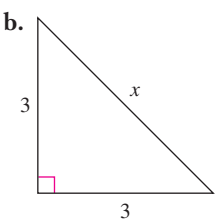
1. Write an irrational number whose digits are 4s and 5s.

2. Use the Pythagorean Theorem to find x.

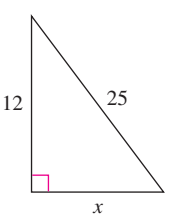
a.



b.



c.



3. Arrange the following real numbers in order from greatest to least.

$$0.8, 0.\overline{8}, 0.\overline{89}, 0.8\overline{89}, \sqrt{0.7744}$$

4. Determine which of the following represent irrational numbers.
- **a.** $\sqrt{78}$
- **b.** $\sqrt{81}$
- **c.** $\sqrt[3]{343}$
- **d.** $3 + \sqrt{81}$
- **e.** $2 \div \sqrt{2}$

5. If possible, find each of the following without using a calculator.
- **a.** The principle square root of 256
- **b.** $\sqrt{324}$
- **c.** $\sqrt[4]{^{-}1}$
- **d.** All x such that $x^2 = 1024$

6. Find the approximate principal square root for each of the following, rounded to hundredths, using the squeezing method.
- **a.** 20.3
- **b.** 1.64

7. Classify each of the following statements as true or false. If false, give a counterexample.
- **a.** The sum of any two rational numbers is a rational number.
- **b.** The difference of any two irrational numbers is an irrational number.
- **c.** The product of any rational number and any irrational number is an irrational number.

8. Find three irrational numbers between each of the following pairs of numbers.
- **a.** 3 and 4
- **b.** $0.\overline{55}$ and $0.\overline{56}$
- **c.** 0.01 and 0.011

9. Describe infinitely many irrational numbers between 1 and 2 in terms of $\sqrt{2}$.

10. If R is the set of real numbers, Q is the set of rational numbers, I is the set of integers, W is the set of whole numbers, N is the set of natural numbers, and S is the set of irrational numbers, simplify or answer the following.
- **a.** $Q \cap I$
- **b.** $S - Q$
- **c.** $R \cup S$
- **d.** Which of the sets could be a universal set for the rest of the sets?
- **e.** If the universal set is R, how would you describe $\overline{S}$?

11. The letters in the table correspond to the sets listed in exercise 10. Complete the following table by placing checkmarks in the appropriate columns.

	N	*I*	*Q*	*S*	*R*
a. $\sqrt{3}$					
b. $4\frac{1}{2}$					
c. $^{-}3\frac{1}{7}$					

12. The letters in the following table correspond to the sets listed in exercise 10. Put a checkmark under each set of numbers for which a solution to the problem exists.

	N	*I*	*Q*	*S*	*R*
a. $x^2 + 2 = 4$					
b. $1 - 2x = 32$					
c. $x^3 = 4$					
d. $\sqrt{x} = {}^-2$					
e. $0.\overline{7}x = 5$					

13. Determine for what real values of x, if any, each of the following statements is true.
a. $\sqrt{x} = 5$ **b.** $\sqrt{x} = {}^-7$
c. $\sqrt{{}^-x} = 5$ **d.** ${}^-\sqrt{x} = 7$
e. ${}^-\sqrt{x} = {}^-5$

14. Write each of the following square roots in the form $a\sqrt{b}$, where a and b are integers and b has the least value possible.
a. $\sqrt{360}$ **b.** $\sqrt{40}$ **c.** $\sqrt{240}$

15. Write each of the following in the simplest form or as $a\sqrt[n]{b}$, where a and b are integers and $b > 0$ and b is as small as possible.
a. $\sqrt[3]{{}^-102}$ **b.** $\sqrt[6]{64}$ **c.** $\sqrt[3]{64}$

16. If each of the following is a part of a geometric sequence, find the missing terms.
a. 4, ____, ____, 8
b. 1, ____, ____, 2
c. ____, ____, 2, $\sqrt{2}$
d. ____, 1, ____, 2

17. In the expression 8^t, let t represents time in hours.
a. What is the value of the expression when $t = 0$?
b. After $\frac{1}{3}$ hour what is the value of the expression?

18. Evaluate each of the following without using a calculator.
a. $(0.008)^{-\frac{2}{3}}$ **b.** $(0.04)^{\frac{3}{2}}$
c. $\left(\frac{{}^-8}{27}\right)^{-\frac{2}{3}}$ **d.** $(0.00000256)^{\frac{3}{8}}$
e. $({}^-32)^{-\frac{4}{5}}$ **f.** ${}^-(32)^{-\frac{4}{5}}$

19. In each of the following, find all real numbers x that satisfy the given equation. If none satisfy the equation, then state the answer as "None."
a. $3^x = {}^-243$ **b.** $81^{{}^-x} = 27$
c. $\left(\frac{4}{9}\right)^{{}^-3x} = \frac{32}{243}$ **d.** $\sqrt[3]{{}^-x} = 2\sqrt[3]{3}$
e. $x^{-\frac{2}{5}} = 2$ **f.** $(x+1)^2 = 2$

20. Write an equation in x not involving radical signs or fractional exponents with the following as a solution.
a. $\sqrt[3]{5}$ **b.** $\sqrt[5]{{}^-2}$
c. $\sqrt{2} - 1$ **d.** $\frac{\sqrt{2}}{\sqrt[3]{2}}$

21. Use the "guess and check" strategy to approximate $\sqrt[3]{2001}$ to the nearest whole number.

22. Determine which of the following irrational.
a. $\frac{\sqrt{320}}{\sqrt{20}}$ **b.** $2\sqrt{50} - 8\sqrt{5}$
c. $\frac{\sqrt{2}}{2} - \sqrt{2}$ **d.** $\sqrt{10^9}$

Mathematical Connections 8-1

Reasoning

1. Use the fact that each prime in the prime factorization of a^3 (a is a whole number) appears a number of times that is a multiple of 3 to prove that $\sqrt[3]{2}$ is irrational.

2. Find the value of $\sqrt{3}$ on a calculator. Explain why this cannot be the exact value of $\sqrt{3}$.

3. Is it true that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ for all a and b? If not, find all the values of a and b for which it is true. Explain.

4. The number $\frac{22}{7}$ can be written as an infinite repeating decimal. Could $\pi = \frac{22}{7}$? Why or why not?

5. **a.** Use the fact that if a prime p appears in the prime factorization of a perfect square then it must appear an even number of times, to prove that for all primes p, $\sqrt{p}$ is irrational.
b. Prove that $\sqrt{\frac{7}{2}}$ is irrational.

6. Without using a calculator, order the following. Explain your reasoning.

$$(4/25)^{-1/3}, (25/4)^{1/3}, (4/25)^{-1/4}$$

7. **a.** To prove that $\frac{2}{\sqrt{3}} < \frac{3}{\sqrt{5}}$ a student wrote the following:

$$\frac{3}{\sqrt{5}} - \frac{2}{\sqrt{3}} = \frac{3\sqrt{3} - 2\sqrt{5}}{\sqrt{15}} = \frac{\sqrt{9}\cdot\sqrt{3} - \sqrt{4}\cdot\sqrt{5}}{\sqrt{15}}$$
$$= \frac{\sqrt{27} - \sqrt{20}}{\sqrt{15}} > 0$$

Explain why this proves that $\frac{2}{\sqrt{3}} < \frac{3}{\sqrt{5}}$.
b. Prove that $\frac{5}{\sqrt{6}} < \frac{6}{\sqrt{7}}$, using the approach in part(a).

Open-Ended

8. The sequence 1, 1.01, 1.001, 1.0001, . . . is an infinite sequence of rational numbers (each term has one more zero between the ones than the preceding term).
a. Describe several other infinite sequences of rational numbers.
b. Describe an infinite sequence of irrational numbers.

9. a. Define three irrational numbers between $\frac{1}{2}$ and $\frac{3}{4}$, each of which has the form $\frac{a}{b}\sqrt{2}$ where a and b are integers.

b. Describe an infinite sequence of irrational numbers between $\frac{1}{2}$ and $\frac{3}{4}$.

Cooperative Learning

10. Let each member of a group choose a number between 0 and 1 on a calculator and check what happens when the $\boxed{x^2}$ key is pressed in succession until it is clear that there is no reason to go on.
- **a.** Compare your answers and write a conjecture based on what you observe.
- **b.** Use other keys on the calculator in a similar way. Describe the process and state a corresponding conjecture.
- **c.** Why do you get the result you do in parts (a) and (b)?

11. A calculator displays the following: $(3.7)^{2.4} = 23.103838$. In your group, discuss the meaning of the expression $(3.7)^{2.4}$ in view of what you know about exponents. Compare your findings with those of other groups.

Connecting Mathematics to the Classroom

12. Jim asked, if $\sqrt{2}$ can be written as $\frac{\sqrt{2}}{1}$, why is it not rational? How would you answer him?

13. A student claims that $(\sqrt{a} + \sqrt{b})^2 = a + b$ because $(\sqrt{a})^2 = a$ and $(\sqrt{b})^2 = b$. How would you help the student?

14. A student wants to know how to write $\sqrt{\sqrt{\sqrt{2}}}$ with a single radical sign. How would you help the student?

15. A student says that she saw online that the solution of $x^2 = 2$ was written as $x = \pm\sqrt{2}$. She claims that if this is true then $\sqrt{2}$ has two values: one positive and one negative. How would you respond?

16. A student noticed that $\sqrt[3]{5} < \sqrt{5}$ and wants to know if whenever $a > 0$, $\sqrt[3]{a} < \sqrt{a}$. How do you respond?

17. Jose says that the equation $\sqrt{-x} = 3$ has no solution, since the square root of a negative number does not exist. How would you help him?

18. A student wants to know what is wrong with the following "proof" that $-1 = 1$. How do you respond?

$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$
$$\text{Let } a = b = -1.$$
$$\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$
$$\text{But also } \sqrt{-1}\sqrt{-1} = -1.$$
$$\text{Thus, } -1 = 1.$$

8-2 Variables

8-2 Objectives

Students will be able to understand and explain

- Variables to translate word phrases into algebraic expressions.
- Solving equations and word problems.
- The formulas for the *n*th term of arithmetic and geometric sequences.

Algebraic thinking is important in mathematics at all levels—from the early grades on. In this section, we focus not only on patterns (introduced in Chapter 1) but on other features of algebraic thinking as well, including solving equations, word problems.

Today, the importance of integrating algebraic thinking and problem solving at all levels begins with kindergarten.

CCSS **In grade 4, *Common Core Standards* emphasizes using algebraic equations and says that students should:**

Multiply or divide to solve word problems involving multiplicative comparison, for instance by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. (p. 29)

Grade 6 *Common Core Standards* suggests that students:

use variables to represent numbers and write expressions when solving a real-world or mathematical problem. . . . (p. 44)

Historical Note



al-Khowarizmi

The word *algebra* comes from the book *Hidab al-jabr wa'l muqabalah*, written by Mohammed ibn Musa al-Khowarizmi (ca. 825 CE). In his book he synthesized Hindu work on the notions of algebra and used the words *jabr* and *muqubalah* to designate two basic operations in solving equations: *jabr* meant to transpose subtracted terms to the other side of the equation; *muqubalah* meant to cancel like terms on opposite sides of the equation.



Fibonacci

Another contributor to the development of algebra was Diophantus (ca. 200–284 CE) whose *Arithmetica* is the most prominent work on algebra in Greek mathematics. About 900 years later, Leonardo di Pisa (Fibonacci) (ca. 1170–1250) introduced algebra to Europe.

A fourth contributor to algebra was François Viète (1540–1603), known as "the father of modern algebra," who introduced the first systematic algebraic notation in his book *In Artem Analyticam Isagoge*. ●

Algebra is a branch of mathematics in which symbols, usually letters, represent numbers or members of a given set. Elementary algebra is used to generalize arithmetic. For example, the fact that $7 + (3 + 5) = (7 + 3) + 5$, or that $9 + (3 + 8) = (9 + 3) + 8$, are special cases of $a + (b + c) = (a + b) + c$, where a, b, and c are numbers from a given set; whole numbers, integers, rational numbers, or real numbers. Similarly, $2 + 3 = 3 + 2$ and $2 \cdot 3 = 3 \cdot 2$ are special cases of $a + b = b + a$ and $a \cdot b = b \cdot a$ for all whole numbers a and b.

A major concept of algebraic thinking is that of **variable**. In basic arithmetic we have fixed numbers, or **constants**, as in $4 + 3 = 7$, but in algebra we have values that vary—hence the term *variable*.

- A variable may stand for a missing element or unknown, as in $x + 2 = 5$.
- A variable can represent more than one thing. For example, in a group of children, we could say that their heights vary with their ages. If h represents height and a represents age, then both h and a can have different values for different children in the group. Here a variable represents a changing quantity.
- A variable can be used in generalizations of patterns.
- A variable can be an element of a set, or a set itself. For example, in the definition of the intersection of two sets $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$, x is any element that belongs to both sets.

To apply algebra in solving problems, we frequently need to translate given information into a mathematical expression involving variables designated by letters or words. In all such examples, we may name the variables as we choose.



In the student page shown on page 402, simple word statements are translated into **algebraic expressions**.

Historical Note



Mary Everest Boole

Mary Everest Boole (1832–1916), born in England and raised in France, was a self-taught mathematician and is most well-known for her works on mathematics and science education. In *Philosophy and Fun of Algebra* (London: C. W. Daniel, LTD, 1909), a book for children, she writes:

> But when we come to the end of our arithmetic we do not content ourselves with guesses; we proceed to algebra—that is to say, to dealing logically with the fact of our own ignorance. . . .
>
> Instead of guessing whether we are to call it nine, or seven, or a hundred and twenty, or a thousand and fifty, let us agree to call it x, and let us always remember that x stands for the Unknown. . . . This method of solving problems by honest confession of one's ignorance is called Algebra. ●

School Book Page

Using Variables to Write Expressions

Step-UP Lesson 2

Common Core

5.OA.2 Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

Using Variables to Write Expressions

How can you write an algebraic expression?

Donnie bought CDs for $10 each. How can you represent the total cost of the CDs?

A variable is a quantity that can change or vary and is often represented with a letter. Variables help you translate word phrases into algebraic expressions.



CDs cost $10 each. The operation is multiplication.

Number of CDs	Total Cost
1	$10 × 1
2	$10 × 2
3	$10 × 3
4	$10 × 4

Use the variable n to represent the number of CDs and write an algebraic expression.

$$\$10 \times n$$

An algebraic expression is a mathematical phrase that has at least one variable and one operation. The total cost of the CDs is represented by

$$10 \times n$$

or $10n$.

The operation is multiplication. The variable is n.

Other Examples

The table shows algebraic expressions for given situations.

Word Phrase	Operation	Algebraic Expression
5 dollars more than cost c	addition	$c + 5$
eleven pencils decreased by a number n	subtraction	$11 - n$
six times a distance d	multiplication	$6 \times d$ or $6d$
b bananas divided by seven	division	$b \div 7$ or $\frac{b}{7}$
four less than two times an amount x	multiplication and subtraction	$2x - 4$

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Example 4

In each of the following descriptions, translate the given information into a symbolic expression involving variables.

a. The cost of renting a car for any number of days if the charge per day is \$40
b. The distance a car traveled at a constant speed of 65 mph for any number of hours
c. One weekend, a store sold twice as many CDs as full-size DVDs and 25 fewer mini DVDs than CDs. If the store sold d full-size DVDs, how many mini DVDs and CDs did it sell?
d. French fries have about 12 calories apiece. A hamburger has about 600 calories. Akiva is on a diet of 2000 calories per day. If he ate f french fries and one hamburger, how many more calories can he consume that day?

Solution

a. If d is the number of days, the cost of renting the car for d days at \$40 per day is $40d$ dollars.
b. If h is the number of hours traveled at 65 mph, the total distance traveled in h hours is $65h$ miles.
c. Because d full-size DVDs were sold, twice as many CDs as full-size DVDs implies $2d$ CDs. Thus, 25 fewer mini DVDs than CDs implies $2d - 25$ mini DVDs.
d. First, find how many calories Akiva consumed eating f french fries and one hamburger. Then, to find how many more calories he can consume, subtract this expression from 2000.

1 french fry $\rightarrow$ 12 calories

f french fries $\rightarrow 12f$ calories

Therefore, the number of calories in f french fries and one hamburger is $600 + 12f$.

The number of calories left for the day is $2000 - (600 + 12f)$, or $2000 - 600 - 12f$, or $1400 - 12f$.

Example 5

A teacher asked her class to do the following:

> Take any real number and add 15 to it. Now multiply that sum by 4. Next subtract 8 and divide the difference by 4. Now subtract 12 from the quotient and tell me the answer. I will tell you the original number.

Analyze the instructions to see whether the teacher can determine the original number.

Solution Translate the information into an algebraic expression as seen in Table 1.

Table 1

Instructions	Discussion	Symbols/Algebraic Expressions
Take any real number.	Since any number is used, we need a variable to represent the number. Let n be that variable.	n
Add 15 to it.	We are told to add 15 to "it." "It" refers to the variable n.	$n + 15$
Multiply that sum by 4.	We are told to multiply "that sum" by 4. "That sum" is $n + 15$.	$4(n + 15)$
Subtract 8.	We are told to subtract 8 from the product.	$4(n + 15) - 8$
Divide the difference by 4.	The difference is $4(n + 15) - 8$. Divide it by 4.	$\dfrac{4(n + 15) - 8}{4}$
Subtract 12 from the quotient and tell me the answer.	We are told to subtract 12 from the quotient.	$\dfrac{4(n + 15) - 8}{4} - 12$

Translating what the teacher asked the class to do results in the algebraic expression $\frac{4(n + 15) - 8}{4} - 12$. We are to determine whether the teacher can take a given student answer and produce the original number. We use the strategy of *working backward* to help with this determination. Suppose a student gives the teacher an answer of r. Think about how r was obtained and reverse the steps in Table 1. Just before obtaining "r," the student had subtracted 12. To reverse that operation, add 12 to obtain $r + 12$. Prior to that, the student had divided by 4. To reverse that, multiply by 4 to obtain $4r + 48$. To get that result, the student had subtracted 8, so now add 8 to obtain $4r + 56$. Just prior to that, the student had multiplied by 4, so now divide $4r + 56$ by 4 to obtain $r + 14$. The first operation had been to add 15, so now subtract 15 from $r + 14$ to get $r - 1$. Thus, the teacher knows when the student tells the final result of r, it is 1 more than the number with which the student started, or the number with which the student started n is the result minus 1.

This analysis can also be shown in a different way as follows:

$$\begin{aligned}\frac{4(n + 15) - 8}{4} - 12 &= \frac{4(n + 15 - 2)}{4} - 12\\ &= (n + 13) - 12\\ &= n + 1\end{aligned}$$

Therefore $n + 1 = r$ and $n = r - 1$.

NOW TRY THIS 4

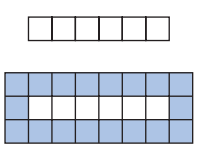


Figure 7

Noah has some white square tiles and some blue square tiles. They are all the same size. He first makes a row of white tiles and then surrounds the white tiles with a single layer of blue tiles, as shown in Figure 7.

How many blue tiles does he need:

a. to surround a row of 100 white tiles?
b. to surround a row of n white tiles?

Generalizations for Arithmetic Sequences

In Chapter 1 we saw examples of arithmetic sequences, and we were able to determine their nth terms. We now generalize this work and find the nth term of any arithmetic sequence.

An arithmetic sequence is determined by its first term and the difference. Suppose the first term of an arithmetic sequence is a_1, and the difference is d. The strategy of *making a table* can be used to find the general term for the sequence $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \ldots$, as shown in Table 2.

Table 2

Number of Term	Term
1	a_1
2	$a_2 = a_1 + d = a_1 + (2 - 1)d$
3	$a_3 = a_1 + 2d = a_1 + (3 - 1)d$
4	$a_4 = a_1 + 3d = a_1 + (4 - 1)d$
5	$a_5 = a_1 + 4d = a_1 + (5 - 1)d$
⋮	⋮
n	$a_n = a_1 + (n - 1)d$

Observe that the number of ds in the given terms is 1 less than the number of the term. This pattern continues since we add d to get the next term. Thus, the *nth term of any arithmetic sequence with first term a_1 and difference d is given by $a_n = a_1 + (n - 1)d$, where n is a natural number.*

For example, in the arithmetic sequence $5, 9, 13, 17, 21, 25, \ldots$, the first term a_1 is 5, and the difference d is 4. Thus, the nth term is given by $a_n = 5 + (n - 1)4$. Simplifying, we obtain $a_n = 5 + (n - 1)4 = 5 + 4n - 4 = 4n + 1$.

Example 6

In an arithmetic sequence with the second term 11 and the fifth term 23, find the 100th term.

Solution We know that in the arithmetic sequence $a_2 = 11$, $a_5 = 23$, and we need to find a_{100}. To find the 100th term, we use the formula developed in Table 2: $a_{100} = a_1 + (100 - 1)d = a_1 + 99d$. Thus to find a_{100}, we need to know a_1 and d. This becomes our *subgoal*. We have $a_2 = 11$ and $a_5 = 23$. Hence, $11 = a_2 = a_1 + d$ and $23 = a_5 = a_1 + (5 - 1)d = a_1 + 4d$. Thus,

$$a_1 + d = 11 \quad \text{and}$$
$$a_1 + 4d = 23.$$

Because we know that $11 = a_1 + d$, we write

$$a_1 + 4d = (a_1 + d) + 3d = 23$$
$$11 + 3d = 23$$
$$3d = 12$$
$$d = 4.$$

To find the first term, we have $11 = a_1 + d$, so

$$11 = a_1 + 4, \text{ or } a_1 = 7.$$

To find the 100th term, we know that $a_{100} = a_1 + (100 - 1)d$, or that $a_{100} = 7 + (100 - 1)4 = 403$.

Generalizations for Geometric Sequences

It is possible to find the nth term, a_n, of any geometric sequence when given the first term and the ratio as mentioned in Chapter 1. For example, in the geometric sequence $3, 12, 48, 192, \ldots$, the first term is 3, and the ratio is 4. To generalize geometric sequences, let the first term be a_1 and the ratio be r, so that the third term is a_1r^2, and the fourth term is a_1r^3 as seen in Table 3.

Table 3

Number of Term	Term
1	a_1
2	a_1r
3	$a_1rr = a_1r^2$
4	$a_1r^2r = a_1r^3$
5	$a_1r^3r = a_1r^4$
⋮	⋮
n	a_1r^{n-1}

The power of r in each term is 1 less than the number of the term. This pattern continues since we multiply by r to get the next term. Thus, *the nth term a_n, of a geometric sequence, is* $a_n = a_1 r^{n-1}$. In the geometric sequence $3, 12, 48, 192, \ldots$, the first term is 3, and the ratio is 4, so the nth term is $a_n = 3 \cdot 4^{n-1}$.

Example 7

a. Find the nth term of the geometric sequence $2, 3, \frac{9}{2}, \frac{27}{4}, \ldots$.

b. If a geometric sequence has first term 3 and ratio $\sqrt{2}$, find its 10th term.

Solution

a. By dividing any term from the 2nd on by its previous term, we can find the ratio r of the geometric sequence. For this sequence, $r = \frac{3}{2}$. Thus, the nth term is

$$a_n = a_1 r^{n-1} = 2\left(\frac{3}{2}\right)^{n-1}.$$

b. The 10th term is computed as follows:

$$a_{10} = 3(\sqrt{2})^{10-1} = 3(\sqrt{2})^9 = 3(2^{9/2}) = 3(2^{8/2})(2^{1/2}) = 3 \cdot 2^4 \cdot 2^{1/2} = 48\sqrt{2}$$

NOW TRY THIS 5

a. Two bacteria are in a dish. The number of bacteria triples every hour. Following this pattern, find the number of bacteria in the dish after 10 hours and after n hours.

b. Suppose that instead of increasing geometrically as in part (a), the number of bacteria increases arithmetically by 3 bacteria each hour. Compare the geometric and arithmetic sequences after 10 hours and after n hours. Comment on the growth of a geometric sequence versus an arithmetic sequence.

The Fibonacci Sequence

Both arithmetic and geometric sequences may be defined *recursively*. A sequence is defined recursively when one or more initial terms are given, and each subsequent term is defined in terms of the preceeding term or terms (see also p. 20.) In a geometric sequence a_1 is given and $a_n = a_{n-1} r$, for $n = 2, 3, \ldots$. In an arithmetic sequence, a_1 is given and $a_n = a_{n-1} + d$, for $n = 2, 3, \ldots$. Similarly the Fibonacci sequence (see Section 1-2) is defined recursively. Fibonacci sequence has F_1 as the first term, F_2 as the second term, and in general, F_n as the nth term. The two terms before the nth, F_n, are F_{n-1} and F_{n-2}. With this notation, the rule for generating the Fibonacci sequence can be written as follows:

$$F_n = F_{n-1} + F_{n-2}, \text{ for } n = 3, 4, \ldots$$

This rule cannot be applied to the first two Fibonacci terms. Because $F_1 = 1$ and $F_2 = 1$, then $F_3 = 1 + 1$, or 2. The beginning terms, or seeds, $F_1 = 1$ and $F_2 = 1$ and the equation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ is a recursive definition because the rule for the sequence defines each term after the first two using previous terms in the same sequence.

More Algebraic Thinking

Algebraic thinking can occur in different ways. One example that uses pictures is seen in Example 8.

Example 8

At a local farmer's market, three purchases were made for the prices shown in Figure 8. What is the cost of each object?

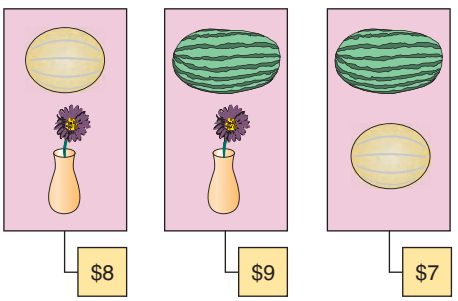


Figure 8

Solution Approaches to this problem may vary. For example, if the objects in the first two purchases are put together, the total cost would be \$8 + \$9, or \$17. That cost would be for two vases with flowers and one each of the cantaloupe and watermelon, as in Figure 9.

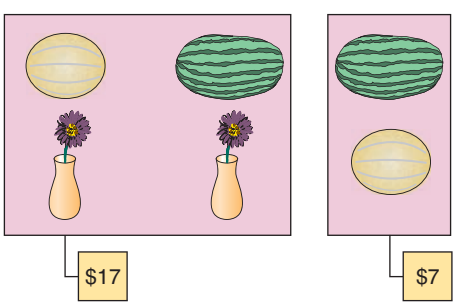


Figure 9

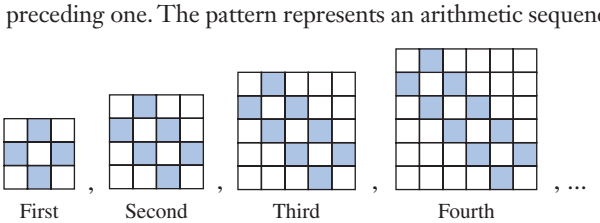
Now if the cantaloupe and watermelon are taken away from that total, then according to the cost of those two objects from the tag on the right, the cost should be reduced to \$10 for two vases with flowers. That means each of the two vases costs \$5. This in turn tells us that the cantaloupe costs \$8 − \$5, or \$3, and the watermelon costs \$9 − \$5, or \$4.

Assessment 8-2A

1. Translate the following phrases into algebraic expressions:
- **a.** The 4th term of an arithmetic sequence whose first term is 10 and whose difference is d
- **b.** 15 less than twice a number
- **c.** 15 times the square of a number
- **d.** The difference between the square of a number and twice the number

2. a. Translate the following instructions into an algebraic expression: Take any number, add $\sqrt{3}$ to it, multiply the sum by 7, subtract 14, and divide the difference by 7. Finally, subtract the original number.
- **b.** Simplify your answer in part (a).

3. In the tile pattern sequence of figures shown, each figure starting from the second has two more blue squares than the preceding one. The pattern represents an arithmetic sequence.



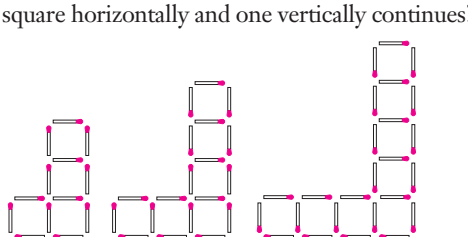
Answer the following questions.
 a. How many blue tiles are there in the nth figure if this arithmetic sequence continues?
 b. If the pattern of large square tiles with shaded blue ones continues, how many white tiles are there in the nth figure?

4. In the following descriptions, write an expression in terms of the given variable that represents the indicated quantity.
 a. The cost of having a plumber spend h hours at your house if the plumber charges \$50 for coming to the house and \$60 per hour for labor
 b. The amount of money in cents in a jar containing d dimes and some nickels and quarters if there are 3 times as many nickels as dimes and 3 times as many quarters as nickels
 c. The sum of three consecutive integers if the least integer is x
 d. The amount of bacteria after n min if the initial amount of bacteria is q and the amount of bacteria doubles every minute (*Hint:* The answer should contain q as well as n.)
 e. The temperature after t hr if the initial temperature is 40°F and each hour it drops by 3°F
 f. Pawel's total earnings after 3 yr if the first year his salary was s dollars, the second year it was \$5000.00 higher, and the third year it was twice as much as the second year
 g. The sum of three consecutive odd natural numbers if the least is x
 h. The sum of three consecutive natural numbers if the middle is m

5. In a college, there are P professors and S students. There are 45 times as many students as professors. Write an algebraic equation that shows this relationship.

6. Suppose a class has g girls and b boys, and there are 5 more girls than boys. Write an algebraic equation that shows this relationship.

7. Ryan is building matchstick sequences as shown. How many matchsticks will he use for the nth figure if the pattern of adding one new square horizontally and one vertically continues?



8. Write an algebraic equation relating the variables described in each of the following situations:
 a. The pay, P, for t hr if you are paid \$8 an hour
 b. The pay, P, for t hr if you are paid \$15 for the first hour and \$10 for each additional hour

9. For a particular event, a student pays \$5 per ticket and a nonstudent pays \$13 per ticket. If x students and 100 nonstudents buy tickets, find the total revenue from the sale of the tickets in terms of x.

10. Suppose a will decreed that three siblings will each receive a cash inheritance according to the following: The eldest receives 3 times as much as the youngest, and twice as much as the middle sibling. Answer the following.
 a. If the youngest sibling receives \$$x$, how much do the other two receive in terms of x?
 b. If the middle sibling receives \$$y$, how much do the other two receive in terms of y?
 c. If the oldest sibling receives \$$z$, how much do the other two receive in terms of z?

11. In each of the following arithmetic sequences, find (i) the 100th term; (ii) the nth term.
 a. $0, -3, -6, \ldots$
 b. $1 - \sqrt{2}, 1 + \sqrt{2}, \ldots$
 c. $\sqrt{3} + 0.5, \sqrt{3} + 2.5, \sqrt{3} + 4.5, \ldots$

12. In each of the following geometric sequences, find an expression for (i) the 17th term; (ii) for the nth term.
 a. $2, 2\sqrt{3}, 6, \ldots$
 b. $2, -2\sqrt{2}, \ldots$
 c. $-\sqrt{5}, 5, \ldots$

13. Find the first term and the difference in an arithmetic sequence, if the 100th term is $\sqrt{2}$ and the 200th term is $\sqrt{3}$.

14. Find the first term in a geometric sequence, if the 11th term is -128 and the ratio is $-\sqrt{2}$.

15. A ball is dropped from a height of 10 ft and bounces 80% of its previous height on each bounce. How high off the ground is the ball at the top of the 5th bounce?

16. Jake bought contact lenses and was told to wear them only 2 hours the first day and to increase the length of time by 20 minutes each day. On what day will Jake be able to wear the contacts for 12 hours?

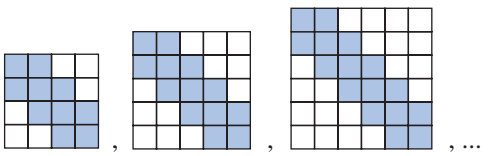
17. In a Fibonacci-like sequence, the first term is $\sqrt{2}$, the second is $\sqrt{3}$, and each subsequent term is the sum of two previous terms. Find the following.
 a. The first 10 terms in simplest form without computing $\sqrt{2}$ or $\sqrt{3}$.
 b. If the terms in the Fibonacci sequence are $F_1, F_2, F_3, \ldots$, and the terms in the Fibonacci-like sequence in this question are $L_1, L_2, L_3, \ldots$, answer the following.
 i. Verify that $L_7 = F_5\sqrt{2} + F_6\sqrt{3}$.
 ii. Write L_8, L_9, and L_{10} using the terms of the Fibonacci sequence, $\sqrt{2}$, and $\sqrt{3}$.
 iii. Use your answers in the previous part to conjecture an expression for L_n in terms of $\sqrt{2}$, $\sqrt{3}$, and Fibonacci terms.

Assessment 8-2B

1. Translate the following phrases into algebraic expressions.
 a. 10 more than a number
 b. 10 less than a number
 c. half a number
 d. The sum of a number and 10
 e. The difference between the cube of a number and the square of the number

2. a. Translate the following instructions into an algebraic expression: Take any number, add 25 to it, multiply the sum by 4, subtract 60, and divide the difference by 4. Finally, add 5.
 b. Simplify your answer in part (a).

3. Discover a possible tile pattern in the following sequence and answer the following.



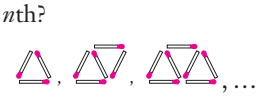
a. How many shaded tiles are there in the nth figure of your pattern if the arithmetic sequence of shaded tiles continues?
b. How many white tiles are there in the nth figure of your pattern?

4. In the following descriptions, write an expression in terms of the given variables that represents the indicated quantity:
a. The cost of having a plumber spend h hours at your house if the plumber charges \$30 for coming to the house and \$$x$ per hour for labor
b. The amount of money in cents in a jar containing some nickels and d dimes and some quarters if there are 4 times as many nickels as dimes and 3 times as many quarters as nickels
c. The sum of three consecutive integers if the greatest integer is x
d. The amount of bacteria after n min if the initial amount of bacteria is q and the amount of bacteria triples every 20 sec. (The answer should contain q as well as n.)
e. The temperature t hr ago if the present temperature is 40°F and each hour it drops by 3°F
f. Pawel's total earnings after 3 yr if the first year his salary was s dollars, the second year it was \$5000 higher, and the third year it was twice as much as the first year
g. The sum of three consecutive even whole numbers if the greatest is x

5. If a school has w women and m men and you know that there are 100 more men than women, write an algebraic equation relating w and m.

6. Suppose there are c chairs and t tables in a classroom and there are 15 more chairs than tables. Write an algebraic equation relating c and t.

7. Ryan is building triangle sequences so that one triangle is added to the right each time, as shown. How many matchsticks will he use for the nth figure and for the figure one before the nth?



8. Write an algebraic equation relating the variables described in each of the following situations:
a. The pay, P, for t hours if you are paid \$$d$ an hour
b. The pay, P, for t hours if you are paid \$25 for the first hour and \$$k$ for each additional hour
c. The total pay, P, for a visit and t hours of gardening if you are paid \$30 for the visit and \$10 for each hour of gardening
d. The total cost, C, of membership in a health club that charges a \$300 initiation fee and \$4 for each of n days attended
e. The cost, C, of renting a midsized car for 1 day of driving m mi if the rent is \$30 per day plus 35¢ per mile

9. For a particular event, students pay \$$d$ per ticket and non-students pay twice as much. If x students and 100 nonstudents buy tickets, find the total revenue from the sale of the tickets in terms of x and d.

10. Matt has twice as many stickers as David. If David has d stickers, and Matt gives David 10 stickers, how many stickers does each have in terms of d?

11. In each of the following arithmetic sequences, find (i) the 100th term; (ii) the nth term.
a. $0, {}^{-}5, {}^{-}10, \ldots$
b. $1 + \sqrt{2}, 1 - \sqrt{2}, \ldots$
c. $x + \sqrt{3}, 2x + 3\sqrt{3}, \ldots$

12. In each of the following geometric sequences, find an expression for (i) the 20th term; (ii) for the nth term.
a. $\sqrt{\frac{3}{2}}, \sqrt{3}, \ldots$
b. $\sqrt{2}, 2, 2\sqrt{2} \ldots$
c. ${}^{-}\sqrt{3}, 3, \ldots$
d. $3, {}^{-}3\sqrt{3}, \ldots$

13. In a geometric sequence, find the first term and the ratio if the 10th term is 25 and the 20th term is 100.

14. **a.** Find the difference in the arithmetic sequence whose nth term is $3n + 5$.
b. Find the ratio of a geometric sequence whose nth term is $3^{\frac{1}{2}n}$.

15. A ball is dropped from a height of 20 meters and bounces 90% of its previous height on each bounce. How high off the ground is the ball at the top of the 4th bounce?

16. Consider the following arithmetic sequences. Determine the first time when a term of the first sequence is greater than the corresponding term of the second.

$0.01, 0.11, 0.21, \ldots$
$100, 100.01, 100.02, \ldots$

17. Find the following.
a. The first seven terms of the Fibonacci-type sequence with seeds 1, 2
b. The sum of the first three terms of the sequence in part (a)
c. The sum of the first four terms of the sequence in part (a)
d. The sum of the first five terms of the sequence in part (a)
e. The sum of the first six terms of the sequence in part (a)
f. The sum of the first seven terms of the sequence in part (a)
g. A pattern for the sums in parts (b)–(f).
h. A rule for the pattern found in part (g) using the notation for Fibonacci terms.

Mathematical Connections 8-2

Reasoning

1. Students were asked to write an algebraic expression for the sum of five consecutive natural numbers. One student wrote $x + (x + 1) + (x + 2) + (x + 3) + (x + 4) = 5x + 10$. Another wrote $(x - 2) + (x - 1) + x + (x + 1) + (x + 2) = 5x$.
 - **a.** Explain who is correct and why.
 - **b.** Does either approach prove that the sum of five consecutive integers equals 5 times the middle integer?
 - **c.** Does the statement in part (b) generalize to the sum of five consecutive terms in every arithmetic sequence? Justify your answer.
2. **a.** Choose seven consecutive terms in any arithmetic sequence and check that the arithmetic average (the sum of the seven terms divided by 7) equals the middle term.
 - **b.** Prove that the result in part (a) is true for every arithmetic sequence.
3. A teacher instructed her class as follows:

 Take any number, multiply it by 4, add 16, and divide the result by 2. Subtract 7 from the quotient and tell me your answer. I will tell you the original number.

 Explain how the teacher was able to tell each student's original number.

Open-Ended

4. A teacher instructed her class to take any number and perform a series of computations using that number. The teacher was able to tell each student's original number by subtracting 1 from the student's answer. Create similar instructions for students so that the teacher needs to do only the following to obtain the student's original number.
 - **a.** Add 1 to the answer.
 - **b.** Multiply the answer by 2.
 - **c.** Multiply the answer by 1.
5. Give an example of a geometric sequence whose first 3 terms are very close to 1 but whose terms are eventually very close to 0. Use a calculator to check that your sequence has the desired property.

Cooperative Learning

6. Examine several elementary school textbooks for grades 1 through 5 and report on which algebraic concepts involving variables are introduced at each grade level.

Connecting Mathematics to the Classroom

7. A student writes $a(bc) = (ab)(ac)$. How do you respond?
8. A student wonders if sets can ever be considered as variables. What do you tell her?
9. A student thinks that if A and B are sets, then the statements $A \cup B = B \cup A$ and $A \cap B = B \cap A$ are set properties in a way similar to the statements $a + b = b + a$ and $ab = ba$ are arithmetic properties of numbers. How do you respond?
10. A student wants to know how can she prove that a sequence whose nth term is $5n + 4$ is arithmetic and a sequence whose nth term is $5 \cdot 3^{\frac{n}{2}}$ is geometric. How do you respond?

Review Problems

11. Find two rational numbers and two irrational numbers between 1.41 and $\sqrt{2}$.
12. Determine which of the following are rational and which are irrational numbers.
 - **a.** $\sqrt[3]{728}$
 - **b.** $\dfrac{2}{\sqrt{2}}$
 - **c.** $\sqrt[3]{2^9 \cdot 3^{12}}$
 - **d.** $\sqrt[3]{2} \cdot \sqrt[3]{4}$
 - **e.** $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$
 - **f.** $(\sqrt{3} - \sqrt{2})^2$
13. Compute the following, if possible. If a number is not a real number, write "not real."
 - **a.** $\left(\dfrac{^{-}8}{27}\right)^{-\frac{2}{3}}$
 - **b.** $\left(\dfrac{^{-}9}{4}\right)^{\frac{3}{2}}$
 - **c.** $^{-}\left(\dfrac{9}{4}\right)^{\frac{3}{2}}$
14. Find all real number values for x, if possible. If there is no real solution, write "None."
 - **a.** $\sqrt{^{-}x} = 5$
 - **b.** $\sqrt{x^2} = {}^{-}x$
 - **c.** $(^{-}x)^{\frac{3}{2}} = 125$
15. Find all integers n such that $n < \sqrt[3]{50} < n + 1$.

National Assessments

National Assessment of Educational Progress (NAEP) Questions

A yellow box holds 72 pencils.

Y is the same as 72.

Two red boxes hold as many pencils as 1 yellow box.

R R is the same as Y.

Three blue boxes hold as many pencils as 1 red box.

B B B is the same as R.

How many pencils does 1 blue box hold?

Answer: _____

Show or explain how you got your answer.

NAEP, Grade 4, 2013

$2 + n$	5
$3 + n$	6
$4 + n$	7
$5 + n$	8

What number does n represent in the table?

A. 2
B. 3
C. 4
D. 5

NAEP, Grade 4, 2009

A. If c and d are different prime numbers less than 10 and the sum $c + d$ is a *composite* number greater than 10, what is one possible pair of values for c and d?

$c =$ _____ $d =$ _____

B. If j and k are different prime numbers less than 10 and the sum $j + k$ is a *prime* number less than 10, what is one possible pair of values for j and k?

$j =$ _____ $k =$ _____

C. If s and t are different prime numbers greater than 10, explain why the sum $s + t$ cannot be a prime number.

NAEP, Grade 8, 2013

8-3 Equations

8-3 Objectives

Students will be able to understand and explain

- Properties of equality and equations.
- Solutions of equations using properties of equations.
- Solutions of word problems.

With variables w and c, we consider equations such as $w + c = \sqrt{7}$. The equal sign indicates that the values on both sides of the equation are the same even though they do not look the same.

To solve equations, we need several properties of equality. Many of these can be discovered by using a balance scale. For example, consider two objects a and b of the same weight on the balances, as in Figure 10(a). If the balance is level, then $a = b$. When we add an equal amount of weight, c, to both sides, the balance is still level, as in Figure 10(b).

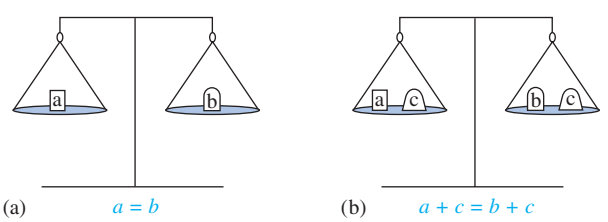


Figure 10

This demonstrates that *if* $a = b$, *then* $a + c = b + c$, which is *the addition property of equality*.

Similarly, if the scale is balanced with amounts a and b, as in Figure 11(a), and we put additional a's on one side and an equal number of b's on the other side, the scale remains level, as in Figure 11(b).

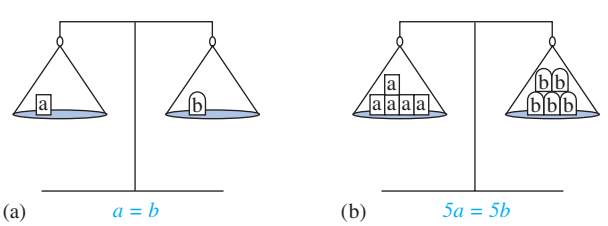


Figure 11

Figure 11 suggests that *if* c *is any natural number and* $a = b$, *then* $ac = bc$. This property can be extended to all real numbers c. The extension is the *multiplication property of equality*. These properties are summarized in the next theorem.

Theorem 8-2: The Addition and Multiplication Properties of Equality

a. For any real numbers a, b, and c, if $a = b$, then $a + c = b + c$.

b. For any real numbers a, b, and c, if $a = b$, then $ac = bc$.

Because $a - c = a + (^{-}c)$ and if $c \neq 0$ then $\frac{a}{c} = a \cdot \frac{1}{c}$, Theorem 8-2 implies the subtraction and division properties of equality.

Equality is not affected if we substitute a number for its equal. This property is referred to as the **substitution property**. Examples of substitution follow:

a. If $a + b = c + d$ and $d = 5$, then $a + b = c + 5$.
b. If $a + b = c + d$, $b = e$, and $d = f$, then $a + e = c + f$.

Using the substitution property, we can see that equations can be added or subtracted as stated in the following theorem.

Theorem 8-3: Addition and Subtraction Property of Equations

If a, b, c, d are real numbers, $a = b$ and $c = d$, then $a + c = b + d$ and $a - c = b - d$.

Theorem 8-2 implies that we may add the same real number to both sides of an equation or multiply both sides of the equation by the same real number without affecting the equality. If $a + c = b + c$ and $ac = bc$, the cancellation properties of equality can be developed by adding ^{-}c to both sides of the first equation and multiplying both sides by $\frac{1}{c}$, where $c \neq 0$, in the second equation.

Theorem 8-4: Cancellation Properties of Equality

a. For any real numbers a, b, and c, if $a + c = b + c$, then $a = b$.
b. For any real numbers a, b, and c, with $c \neq 0$, if $ac = bc$, then $a = b$.

If $ab = 0$, then at least one factor, a or b, must be 0. Thus, we have the following: *For any real numbers a and b, if* $ab = 0$, *then* $a = 0$ *or* $b = 0$.

An algebraic use of the italized statement is seen when we find a solution to an equation like $(x - 3)(x - 5) = 0$. We know that $x - 3 = 0$ or $x - 5 = 0$. Hence $x = 3$ or $x = 5$. The above theorems and properties for real numbers also hold for algebraic expressions.

As mentioned in an earlier chapter, when using the commutative property of multiplication, each of the distributive properties can be written in the equivalent forms:

$$ab + ac = a(b + c)$$
$$ab - ac = a(b - c)$$

When the distributive properties are written in the above equivalent forms, we refer to them as *factoring* and say that a has been "factored out."

Solving Equations with One Variable

Finding solutions to equations is a major part of algebra. Use of tangible objects can increase students' engagement and comprehension when they work with equations. A balance-scale model fosters understanding of the basic concepts used in solving equations and inequalities.

For example, consider Figure 12. If we release the pan on the left, what happens? Upon release, the scale tilts down on the right side and we have an *inequality*, $2 \cdot 3 < 3 + (2 \cdot 2)$.

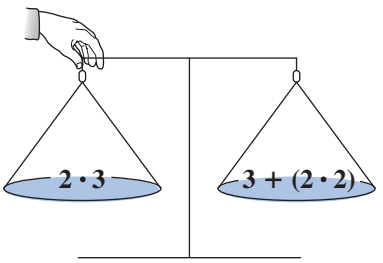


Figure 12

Next consider Figure 13. If we release the pan, then the sides will balance and we have the *equality* $2 \cdot 3 = (1 + 1) + 4$.

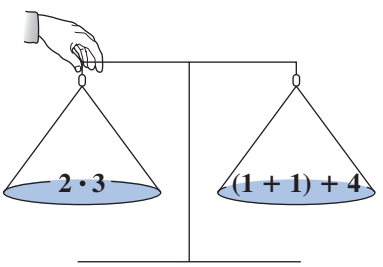


Figure 13

A balance scale can also be used to reinforce the idea of a replacement set for a variable. Name some solutions in Figure 14 that keep the scale balanced. For example, $3(5/2)$ balances $2(15/4)$, $3\sqrt{16}$ balances $2 \cdot 6 = 12$, and so on.

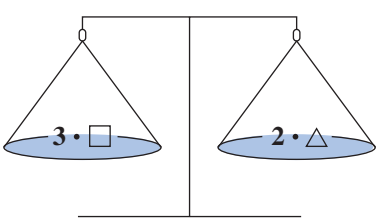


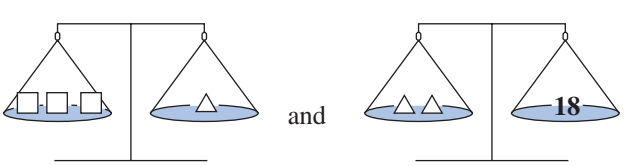
Figure 14

Other types of balance scale problems may help students with algebra. Work through Now Try This 6 before proceeding.

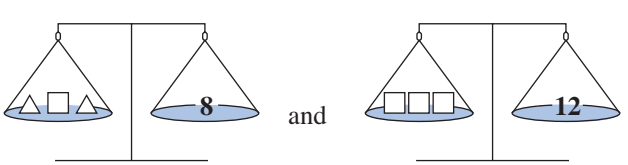
NOW TRY THIS 6

What are the values of □ and △ in parts (a) and (b) below?

a.



b.



To solve equations, we may think of the properties of equality used on a balance pan. Consider $3x - 14 = 1$. Put the equal expressions on the opposite pans of the balance scale. Because the expressions are equal, the pans should be level, as in Figure 15.

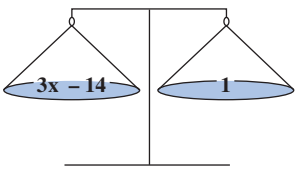


Figure 15

To solve for x, we use the properties of equality to manipulate the expressions on the scale so that after each step, the scale remains level and, at the final step, only an x remains on one side of the scale. The number on the other side of the scale represents the solution to the original equation. To find x in the equation of Figure 15, consider the scales pictured in successive steps in Figure 16, where each successive scale represents an equation that is equivalent to the original equation; that is, each has the same solution as the original. The last scale shows $x = 5$. To check that 5 is the correct solution, we substitute 5 for x in the original equation. Because $3 \cdot 5 - 14 = 1$ is a true statement, 5 is the solution to the original equation.

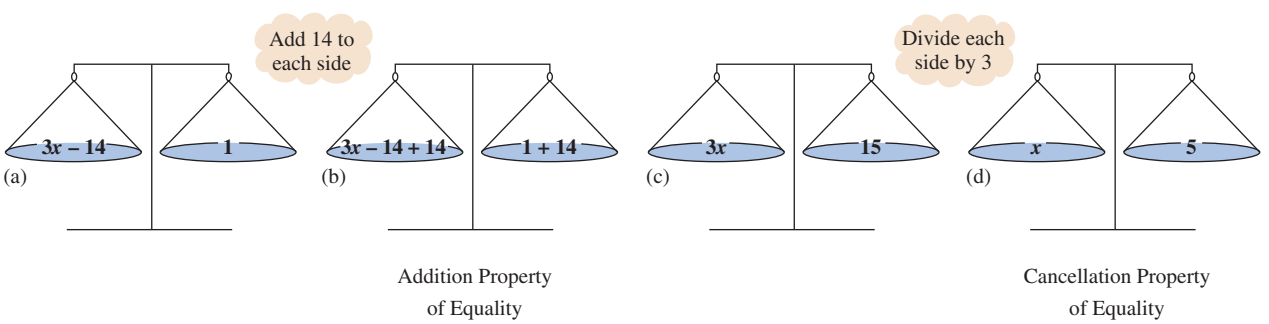


Figure 16

Concrete objects are used in solving equations on the student page shown on page 415.

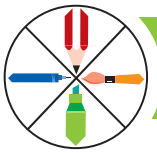
Historical Note



Mary Fairfax Somerville

Born in Scotland, Mary Fairfax Somerville (1780–1872) when she was 13 saw mysterious symbols in a women's fashion magazine. After persuading her brother's tutor to purchase some literature for her on the subject, she began her study of algebra. As a young mother and widow, she obtained books providing her with a background in mathematics. Throughout her life, Somerville distinguished herself as a skilled scientific writer. When she was 89 she published *Molecular and Microscopic Science*. In her autobiography Somerville wrote of how she "was sometimes annoyed when in the midst of a difficult problem" a visitor would enter. Shortly before her death she wrote:

> I am now in my ninety-second year, . . . , I am extremely deaf, and my memory of ordinary events, and especially of the names of people, is failing, but not for mathematical and scientific subjects. I am still able to read books on the higher algebra for four or five hours in the morning and even to solve the problems. Sometimes I find them difficult, but my old obstinacy remains, for if I do not succeed today, I attack them again tomorrow.



School Book Page

Modeling Equations

3-5a Activity Lab — Hands On



Modeling Equations

To solve an equation using models, get the variable by itself on one side.

EXAMPLE **Addition Equations**

Solve $x + 7 = 15$.

$x + 7 = 15$ ← Model the equation.

$x + 7 - 7 = 15 - 7$ ← Remove 7 tiles from each side. This will keep the equation balanced.

$x = 8$ ← Find the solution.

Exercises

Solve each equation by drawing models or using tiles.

1. $x + 2 = 7$ 2. $5 + c = 35$ 3. $7 + m = 21$ 4. $8 = n + 5$

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Example 9

Solve each of the following equations for x.

a. $x + \sqrt[3]{4} = 20$
b. $3x = x + \sqrt{10}$
c. $\frac{4}{3}x = 33x - \frac{1}{3}$
d. $4(x + 3) + 5(x + 3) = 99$

Solution

a.
$$\begin{aligned} x + \sqrt[3]{4} &= 20 \\ \left(x + \sqrt[3]{4}\right) - \sqrt[3]{4} &= 20 - \sqrt[3]{4} \\ x &= 20 - \sqrt[3]{4} \end{aligned}$$

b.
$$\begin{aligned} 3x &= x + \sqrt{10} \\ 3x - x &= (x + \sqrt{10}) - x \\ (3 - 1)x &= (\sqrt{10} + x) - x \\ 2x &= \sqrt{10} \\ x &= \frac{\sqrt{10}}{2} \end{aligned}$$

c.
$$\frac{4}{3}x = 33x - \frac{1}{3}$$
$$3 \cdot \frac{4}{3}x = 3\left(33x - \frac{1}{3}\right)$$
$$4x = 99x - 1$$
$$4x + 1 = 99x - 1 + 1$$
$$4x + 1 = 99x$$
$$^{-}4x + (4x + 1) = {}^{-}4x + 99x$$
$$1 = (^{-}4 + 99)x$$
$$1 = 95x$$
$$\frac{1}{95} = x$$

d.
$$4(x + 3) + 5(x + 3) = 99$$
$$(4 + 5)(x + 3) = 99 \text{ (Why?)}$$
$$9(x + 3) = 99$$
$$x + 3 = 11$$
$$x = 8$$

Application Problems

Figure 17 demonstrates a method for solving application problems with a third-grade example: formulate the problem with a mathematical model, solve that mathematical model, and interpret the solution in terms of the original problem.

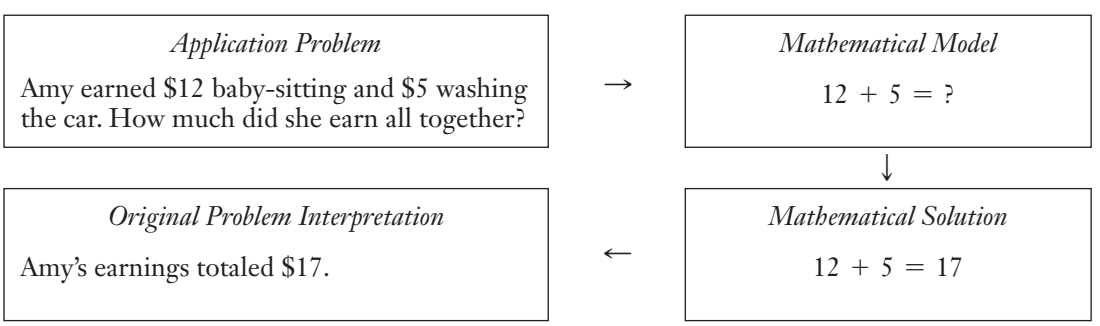


Figure 17

We apply the four-step problem-solving process to solving word problems with algebraic thinking. In Understanding the Problem, we identify what is given and what is to be found. In Devising a Plan, we assign letters to the unknown quantities and try to translate the information in the problem into a model involving equations. In Carrying Out the Plan, we solve the equations or inequalities. In Looking Back, we interpret and check the solution in terms of the original problem.

Problem Solving Overdue Books

Bruno has five books overdue at the library. The fine for overdue books is 10¢ a day per book. He remembers that he checked out an astronomy book a week before he checked out four novels. If his total fine was $8.70, how long was each book overdue?

Understanding the Problem Bruno has five books overdue. The astronomy book was checked out seven days before the four novels. The fine per day for each book is 10¢, and the total fine was $8.70. We need to find out how many days each book is overdue.

Devising a Plan Let d be the number of days that each of the four novels is overdue. The astronomy book was due seven days before the novels so it is $(d + 7)$ days overdue. To *write an equation*

using d, we express the total fine in two ways. The total fine is \$8.70. This fine in cents equals the fine for the four novels, plus the fine for the astronomy book.

$$\text{Fine for each of the novels} = \underbrace{\text{Fine per day}}_{10} \underbrace{\text{times}}_{\cdot} \underbrace{\text{number of overdue days}}_{d}$$

$$\begin{aligned}\text{Fine for the four novels} &= \underbrace{\text{1 day's fine for novels}}_{4 \cdot 10} \underbrace{\text{times}}_{\cdot} \underbrace{\text{number of overdue days}}_{d} \\ &= (4 \cdot 10)d \\ &= 40d\end{aligned}$$

$$\begin{aligned}\text{Fine for the astronomy book} &= \underbrace{\text{Fine per day}}_{10} \underbrace{\text{times}}_{\cdot} \underbrace{\text{number of overdue days}}_{(d+7)} \\ &= 10(d+7)\end{aligned}$$

Because each of the expressions is in cents, we write the total fine of \$8.70 as 870¢ to produce the following:

$$\begin{array}{ccccc}\text{Fine for the four novels} & + & \text{Fine for the astronomy book} & = & \text{Total fine} \\ 40d & + & 10(d+7) & = & 870\end{array}$$

Carrying Out the Plan Solve the equation for d.

$$\begin{aligned}40d + 10(d+7) &= 870 \\ 40d + 10d + 70 &= 870 \\ 50d + 70 &= 870 \\ 50d &= 800 \\ d &= 16\end{aligned}$$

Thus, each of the four novels was 16 days overdue, and the astronomy book was overdue $d + 7 = 23$ days.

Looking Back To check the answer, follow the original information. Each of the four novels was 16 days overdue, and the astronomy book was 23 days overdue. Because the fine was 10¢ per day per book, the fine for each of the novels was $16 \cdot 10$¢, or 160¢. Hence, the fine for all four novels was $4 \cdot 160$¢, or 640¢. The fine for the astronomy book was $23 \cdot 10$¢, or 230¢. Consequently, the total fine was 640¢ + 230¢, or 870¢, which agrees with the given information of \$8.70 as the total fine.

Problem Solving Newspaper Delivery

In a small town, 3 children deliver all the newspapers. Abby delivers 3 times as many papers as Bob, and Connie delivers 13 more than Abby. If the 3 children deliver a total of 496 papers, how many papers does each deliver?

Understanding the Problem The problem asks for the number of papers each child delivers. We are given information comparing the number of papers that each child delivers as well as the total number of papers delivered in the town.

Devising a Plan Let a, b, and c be the number of papers delivered by Abby, Bob, and Connie, respectively. We translate the given information into *algebraic equations* as follows:

Abby delivers 3 times as many papers as Bob: $a = 3b$

Connie delivers 13 more papers than Abby: $c = a + 13$

Total delivery is 496: $a + b + c = 496$

To reduce the number of variables, substitute $3b$ for a in the second and third equations:

$c = a + 13$ becomes $c = 3b + 13$

$a + b + c = 496$ becomes $3b + b + c = 496$

Next, make an equation in one variable, b, by substituting $3b + 13$ for c in the equation $3b + b + c = 496$; solve for b; and then find a and c.

Carrying Out the Plan

$$\begin{aligned} 3b + b + 3b + 13 &= 496 \\ 7b + 13 &= 496 \\ 7b &= 483 \\ b &= 69 \end{aligned}$$

Thus, $a = 3b = 3 \cdot 69 = 207$. Also, $c = a + 13 = 207 + 13 = 220$. So, Abby delivers 207 papers, Bob delivers 69 papers, and Connie delivers 220 papers.

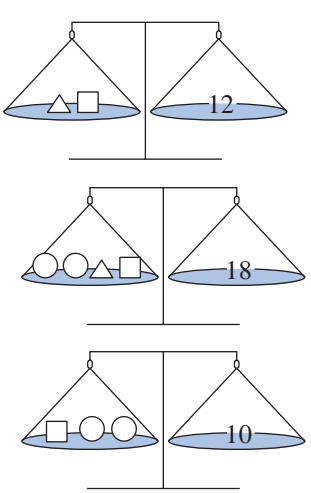
Looking Back To check the answers, follow the original information, using $a = 207$, $b = 69$, and $c = 220$. The information in the first sentence, "Abby delivers 3 times as many papers as Bob," checks, since $207 = 3 \cdot 69$. The second sentence, "Connie delivers 13 more papers than Abby," is true because $220 = 207 + 13$. The information on the total delivery checks, since $207 + 69 + 220 = 496$.

NOW TRY THIS 7

Solve the *Newspaper Delivery* problem above by introducing only one unknown for the number of newspapers Bob delivered.

Assessment 8-3A

1. Consider the balances:



What is the value of each shape?

2. Solve each of the following if x is a real number. If a real number solution does not exist, write "No solution."

a. $x + \sqrt{3} = 2\sqrt{3} - x$

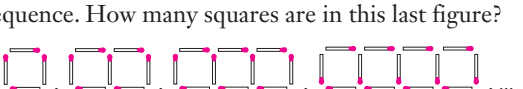
b. $\frac{3}{2}x - 3 = x + \sqrt{2}$

c. $5(2x + \sqrt{2}) + 7(2x + \sqrt{2}) = 12$

d. $3(\sqrt{x} - 3) = 5(\sqrt{x} - 3)$

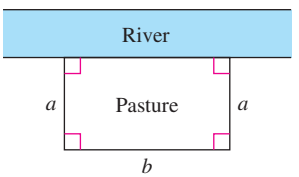
e. $(\sqrt{x} - 5)^2 = 9$

3. Ryan is building matchstick square sequences so that one square is added to the right each time a new figure is formed, as shown. He used 67 matchsticks to form the last figure in his sequence. How many squares are in this last figure?



4. For a particular event, 812 tickets were sold for a total of \$1912. If students paid \$2 per ticket and non-students paid \$3 per ticket, how many student tickets were sold?

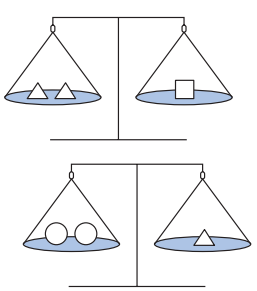
5. An estate of $486,000 is left to three siblings. The eldest receives 3 times as much as the youngest. The middle sibling receives $14,000 more than the youngest. How much did each receive?
6. A 10 ft board is to be cut into three pieces, two equal-length ones and the third 3 in. shorter than each of the other two. If the cutting does not result in any length being lost, how long are the pieces?
7. A box contains 67 coins, only dimes and nickels. The amount of money in the box is $4.20. How many dimes and how many nickels are in the box?
8. Miriam is 10 years older than Ricardo. Two years ago, Miriam was 3 times as old as Ricardo is now. How old are they now?
9. In a college, 20 times as many undergraduate students as graduate students are enrolled. If the total student enrollment at the college is 21,000, how many graduate students are there?
10. A farmer has 1200 yd of fencing to enclose a rectangular pasture for her goats. Since one side of the pasture borders a river, that side does not need to be fenced. Side b must be twice as long as side a. Find the dimensions of the rectangular pasture.



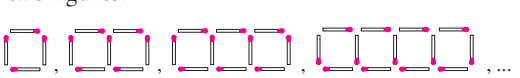
11. The sum of three consecutive terms in the arithmetic sequence 1, 4, 7, 10, . . . is 903; find these three terms.
12. The perimeter of a rectangle (the sum of the lengths of the 4 sides) is 100 feet and the shorter side is $\frac{1}{3}$ of the length of the longer side. Find the length of the diagonal. (Use the Pythagorean Theorem on p. 392.)

Assessment 8-3B

1. Consider the following balances:



 a. Which shape weighs the most? Tell why.
 b. Which shape weighs the least? Tell why.
2. Solve each of the following if x is a real number and if a real number solution does not exist, write "no real solution."
 a. $x - \sqrt{2} = 3\sqrt{2} - x$
 b. $-\frac{3}{2}x - 2 = x - \sqrt{2}$
 c. $5(2x - \sqrt{3}) - 7(2x - \sqrt{3}) = 2$
 d. $2(x^2 - 3) = x^2 - 3$
 e. $(\sqrt{x} - 5)^2 = 9$
3. Ryan is building matchstick square sequences, as shown. He used 599 matchsticks to form the last two figures in his sequence. How many matchsticks did he use in each of the last two figures?

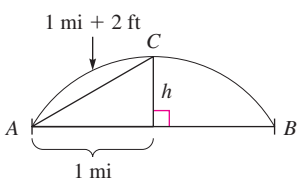


4. At the Out-Rage Benefit Concert, 723 tickets were sold for $3/student and $5/non-student. The benefit raised $2815. How many non-student tickets were sold?
5. An estate of $2,000,000 is left to four siblings. The eldest is to receive 3 times as much as the youngest. The other two siblings are each to receive $16,000 more than the youngest. How much will each receive?
6. Ten years from now Alex's age will be 3 times her present age. Find Alex's age now.
7. Matt has twice as many stickers as David. How many stickers must Matt give David so that they will each have 120 stickers? Check that your answer is correct.
8. Miriam is four years older than Ricardo. Ten years ago Miriam was 3 times as old as Ricardo was then.
 a. How old are they now?
 b. Determine whether your answer is correct by checking that it satisfies the conditions of the problem.
9. In a college there are 13 times as many students as professors. If together the students and professors number 28,000, how many students are there in the college?
10. A farmer has 800 yd of fencing to enclose a rectangular pasture. One side of the pasture borders a river and does not need to be fenced. If side a must be twice as long as side b parallel to the river, what are the dimensions of the rectangular pasture? (see drawing in exercise 10 Assessment 8-3A)
11. The sum of the first two terms of a geometric sequence is 100 times the first term. What is the common ratio?
12. In a rectangle, one side is $\frac{1}{3}$ of the length of the other side and the length of a diagonal is 9 feet, how long are the sides of the rectangle?

Mathematical Connections 8-3

Reasoning

1. Students were asked to find three consecutive whole numbers whose sum is 393. One student wrote the equation $x + (x + 1) + (x + 2) = 393$. Another wrote $(x - 1) + x + (x + 1) = 393$. Explain whether either approach works to find the answer to the question.
2. Explain how to solve the equation $3x + 5 = 5x - 3$ using a balance scale.
3. Rails lengthen in hot weather. Clips anchor the welded rails to concrete or steel sleepers. Without these clips a one mile length of track might expand 2 ft. and buckle in an arc as shown. Use AC as an approximation of half of the bow to estimate the height h of the buckle.



4. **a.** A pipe can fill a pool in 12 hours. Another pipe can fill the pool in 18 hours. How long will it take for the two pipes to fill the pool if they operate simultaneously?
 b. Answer the question in part (a) if the first pipe can fill the pool in a hours and the second in b hours. Write your answer in as simple a form as possible in terms of a and b.

Open-Ended

5. Create an equation with x on both sides of the equation for each of the following.
 a. Every real number is a solution.
 b. No real number is a solution.
 c. 0 is a solution.
6. Create an equation not involving radicals or fractional exponents for each of the following.
 a. $\sqrt{3}$ and $^{-}\sqrt{3}$ are the solutions
 b. $\sqrt[3]{2}$ is the only solution

Cooperative Learning

7. Examine several elementary school textbooks for grades 1 through 5 and report how algebraic concepts involving equations are introduced in each grade level.
8. Each person in your group is to search the Internet for two word problems that seem interesting and somewhat challenging, and bring them back to the group. Next, the group is to discuss the problems and choose which one is most interesting. Each person then is to create a clear, well-written solution. The group should discuss the different solutions and choose the best one. Discuss what makes the chosen solution good.

Connecting Mathematics to the Classroom

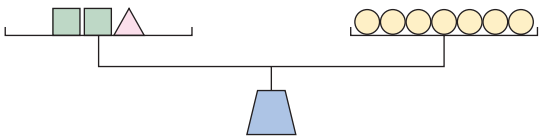
9. A student claims that the equation $3x = 5x$ has no solution because $3 \neq 5$. How do you respond?
10. A student claims that because in the following problem we need to find three unknown quantities, he must set up equations with three unknowns. How do you respond?
 Abby delivers twice as many papers as Jillian, and Brandy delivers 50 more papers than Abby. How many papers does each deliver if the total number of papers delivered is 550?
11. A student was told that in order to check a solution to a word problem like the one in exercise 10, it is not enough to check that the solution found satisfies the equation set up, but rather that it is necessary to check the answer against the original problem. She would like to know why. How do you respond?
12. On a test, a student was asked to solve the equation $4x + 5 = 3(x + 15)$. He proceeded as follows:
 $$4x + 5 = 3x + 45 = x + 5 = 45 = x = 40$$
 Hence, $x = 40$. He checked that $x = 40$ satisfies the original equation; however, he did not get full credit for the problem and wants to know why. How do you respond?

Review Problems

13. If the number of sophomores, juniors, and seniors combined is denoted by x and it is 3 times the number of freshmen, denoted by y, write an algebraic equation that shows the relationship.
14. Write the sum of five consecutive even numbers if the middle one is $2n$. Simplify your answer.
15. If Julie has twice as many CDs as Jack and Tyto has 3 times as many as Julie, write an algebraic expression for the number of CDs each has in terms of one variable.
16. Write an algebraic equation relating the variables described in each of the following.
 a. The total pay P for 3 hours if you are paid \$30 for the first hour and \$$d$ more than the preceding hour for each hour thereafter.
 b. Jimmy's total pay P after 4 years if the first year his salary was d dollars and then each year thereafter his salary is twice as much as in the preceding year
17. **a.** Show that the sum of every three consecutive terms in every arithmetic sequence equals 3 times the middle term.
 b. State and justify a statement similar to the one in part(a) for a product of every three consecutive terms in every geometric sequence.
18. If the first term of a geometric sequence is $\sqrt{5}$ and the ratio is 0.5, find the 6th term of the sequence.
19. In a geometric sequence, the first term is 12 and the 4th term is $\sqrt{5}$. Find the ratio.

National Assessments

National Assessment of Educational Progress (NAEP) Questions



The objects on the scale above make it balance exactly. According to this scale, if △ balances ○○○, then □ balances which of the following?

A. ○
B. ○○
C. ○○○
D. ○○○○

NAEP, Grade 8, 2013

While she was on vacation, Tara sent 14 friends either a letter or a postcard. She spent \$3.84 on postage. If it costs \$0.20 to mail a postcard and \$0.33 to mail a letter, how many letters did Tara send? Show what you did to get your answer.

NAEP, Grade 8, 2013

8-4 Functions

8-4 Objectives

Students will be able to understand and explain

- The concept of a function including domain and range.
- Different representations of functions.
- Derivation of the formulas for the sum of n terms of arithmetic and geometric sequences.
- The concept of a relation.

IMAP Video

Watch students in a 4th grade classroom develop a rule for summing consecutive natural numbers.

CCSS **The concept of a function is central to all of mathematics and particularly to algebra, as elaborated in the following excerpt from grade 8 *Common Core Standards*. We find that students should:**

- **Define, evaluate, and compare functions.**
- **Use functions to model relationships between quantities. (p. 53)**

Functions can model many real-world phenomena. In this section, we explore different ways to represent functions—as *rules*, *machines*, *equations*, *arrow diagrams*, *tables*, *ordered pairs*, *and graphs*. It is important that students see a variety of ways of representing functions.

Functions as Rules Between Two Sets

The following problem is an example of a game called "guess my rule," and often used to introduce the concept of a function.

> When Tom said 2, Noah said 5. When Dick said 4, Noah said 7. When Mary said 10, Noah said 13. When Liz said 6, what did Noah say? What is Noah's rule?

The answer to the first question may be 9, and the rule could be "Take the original number and add 3"; that is, for any number n, Noah's answer is $n + 3$.

Example 10

Guess the teacher's rule for the following responses:

a.

Student	Teacher
1	3
0	0
4	12
10	30

b.

Student	Teacher
2	5
3	7
5	11
10	21

c.

Student	Teacher
2	0
4	0
7	1
21	1

Solution

a. The teacher's rule could be "Multiply the given number n by 3"; that is, for any number n the teacher's answer is $3n$.

b. The teacher's rule could be "Double the original number n and add 1;" that is, for any number n the teacher's answer is $2n + 1$.

c. The teacher's rule could be "If the number n is even, answer 0; if the number is odd, answer 1." Another possible rule is "If the number is less than 5, answer 0; if greater than or equal to 5, answer 1."

Note that in Example 10, the rule connects the set describing what the student says to the set describing the teacher's responses.

Functions as Machines

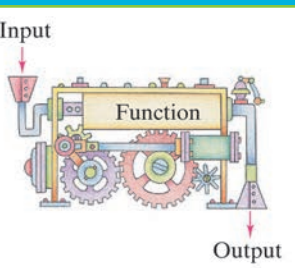
Another way to prepare students for the concept of a function is by using a "function machine." Functions are commonly given letter names, in particular f or g. The following partial student page shows an example of a function machine. What goes into the machine is referred to as *input* and what comes out as *output*. On the student page below, if the input to the function, f, is 3, the output is 12. For any input element x, the output could be denoted as $f(x)$, read "f of x." For the function in the example on the student page, one possibility is to write it as $f(x) = 4x$, where x is a real number.



School Book Page Why Learn This?

Why Learn This?

Pretend you have a machine. You can put any number, or input, into the machine. The machine performs an operation on the number and provides a result, or output. A **function** is a rule that assigns exactly one output value to each input value.



Suppose you tell the machine to multiply by 4. A function table, such as the one at the right, shows the input and output values.

Input	Output
3	12
−7	−28

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Historical Note



Leonhard Euler

The Babylonians of Mesopotamia (ca. 2000 BCE) developed a precursor to a function. To them, a function was a table or a correspondence.

In his book *Geometry* (1637), René Descartes (1596–1650) used functions to describe many mathematical relationships. Almost 50 years after the publication of Descartes's book, Gottfried Wilhelm Leibniz (1646–1716) introduced the term *function*. Function was further formalized by Leonhard Euler (pronounced "oiler," 1707–1783), who introduced the notation $y = f(x)$. In the early twenty-first century, on most graphing calculators, Y1, Y2, Y3, . . . serve as function notations where Y1 acts like $f(x)$ if the function rule is written in terms of x.

Example 11

Consider the function machine in Figure 18. For the function named f, what happens when the numbers 0, 1, 3, 4, and 6 are input?

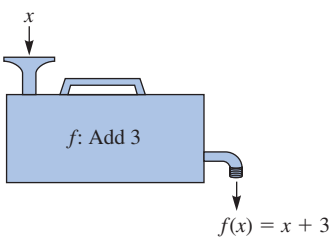


Figure 18

Solution For the given values of x, the corresponding values $f(x)$ are described in Table 4.

Table 4

x	$f(x)$
0	3
1	4
3	6
4	7
6	9

Functions as Equations

The equation $f(x) = x + 3$ describes the function in Example 11. The output values can be obtained by substituting the values 0, 1, 3, 4, and 6 for x in $f(x) = x + 3$, as shown:

$$f(0) = 0 + 3 = 3$$
$$f(1) = 1 + 3 = 4$$
$$f(3) = 3 + 3 = 6$$
$$f(4) = 4 + 3 = 7$$
$$f(6) = 6 + 3 = 9$$

In many applications, both the inputs and the outputs of a function machine are numbers. However, inputs and outputs can be any objects. For example, consider a particular candy machine that accepts only 25¢, 50¢, and 75¢ and outputs one of three types of candy with costs of 25¢, 50¢, and 75¢, respectively. A function machine associates *exactly one output with each input*. If we enter some element x as input and obtain $f(x)$ as output, then every time we enter the same x as input, we obtain the same $f(x)$ as output. The idea of a function machine associating exactly one output with each input according to some rule leads to a definition of a function as a relation between two sets as seen below.

E-Manipulative Activity
Additional practice with function machines can be found in *Function Machine* on the E-Manipulative Disk.

Definition of Function

A **function** from set A to set B is a correspondence from A to B in which each element of A is paired with one, and only one, element of B.

From the definition it follows that f is a function from set A to set B if for every $x \in A$, x is paired with a single element in B, denoted by $f(x)$ the image of x. Later in this section we will see that functions are sets of ordered pairs.

The set, A, of all allowable inputs in the definition of a function is the **domain** of the function. Normally, *if no domain is given to describe a function, then the domain is assumed to contain all elements for which the rule is meaningful*. Set B, the **codomain**, is any set that includes all the possible outputs. The set of all actual outputs is the **range** of the function. Set B includes the range and could be the range itself. The distinction is made for convenience sake, since sometimes the range is not easy to find. Students frequently have trouble with the language of functions (for example, *image*, *domain*, *range*, and *one-to-one*), which subsequently impacts their ability to work with graphical representations of functions.

A typical calculator contains many functions. For example, the $\boxed{\pi}$ button always displays an approximation for π, such as 3.1415927; the $\boxed{+/-}$ button either displays a negative sign in front of a number or removes an existing negative sign; and the $\boxed{x^2}$ and $\boxed{\sqrt{\ }}$ buttons square numbers and take the principal square root of numbers, respectively.

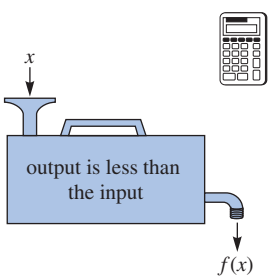


Figure 19

Not all input-output machines are function machines. Consider the machine in Figure 19. For any *natural-number* input x, the machine outputs a number that is less than x. If, for example, the number 10 is input, the machine may output 9, since 9 is less than 10. If 10 is input again, the machine may output 3, since 3 is less than 10. Such a machine is not a function machine because the same input may have different outputs; it is an example of a *relation*.

Example 12

A bicycle manufacturer incurs a daily fixed cost of \$1400 for overhead expenses and a cost of \$500 per bike manufactured.

a. Find the total cost $C(x)$ of manufacturing x bikes in a day.
b. If the manufacturer sells each bike for \$700 and the profit (or loss) in producing and selling x bikes in a day is $P(x)$, find $P(x)$ in terms of x.
c. Find the break-even point, that is, the number of bikes x produced and sold that will result in neither a profit nor a loss.

Solution

a. Since the cost of producing each bike is \$500, the cost of producing x bikes is $500x$ dollars. Because of the fixed cost of \$1400 per day, the total cost, $C(x)$ in dollars, of producing x bikes in a given day is $C(x) = 500x + 1400$.

b. $P(x) =$ Income from selling x bikes $-$ total cost of manufacturing x bikes

$$P(x) = 700x - (500x + 1400)$$
$$= 200x - 1400$$

c. We need the number of bikes x to be produced so that $P(x) = 0$; that is, we need to solve $200x - 1400 = 0$.

$$200x - 1400 = 0$$
$$200x = 1400$$
$$x = \frac{1400}{200} = 7$$

Thus, the manufacturer needs to produce and sell 7 bikes to break even.

Functions as Arrow Diagrams

Arrow diagrams can be used to examine whether a correspondence represents a function. This representation is normally used when sets A and B are finite sets with few elements. Example 13 shows how arrow diagrams can be used to examine both functions and nonfunctions.

Example 13

Which, if any, of the parts of Figure 20 exhibit a function from A to B? If a correspondence is a function from A to B, find the range of the function.

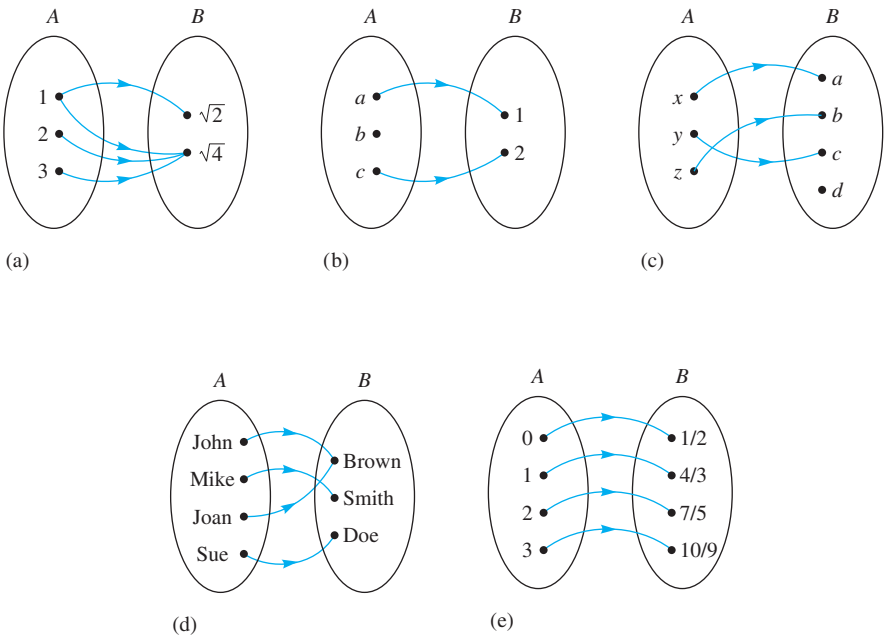


Figure 20

Solution

a. Figure 20(a) does not define a function from A to B, because the element 1 is paired with both $\sqrt{2}$ and $\sqrt{4}$.

b. Figure 20(b) does not define a function from A to B, since the element b is not paired with any element of B. (It is a function from a subset of A to B.)

c. Figure 20(c) does define a function from A to B, since there is one, and only one, arrow leaving each element of A. The fact that d, an element of B, is not paired with any element in the domain does not violate the definition. The range is $\{a, b, c\}$ and does not include d because d is not an output of this function, as no element of A is paired with d.

d. Figure 20(d) illustrates a function from A to B, since there is one, and only one, arrow leaving each element in A. It does not matter that an element of set B, Brown, has two arrows pointing to it. The range is {Brown, Smith, Doe}.

e. Figure 20(e) illustrates a function from A to B whose range is $\left\{\frac{1}{2}, \frac{4}{3}, \frac{7}{5}, \frac{10}{9}\right\}$.

Figure 20(e) also illustrates a **one-to-one correspondence between A and B**. In fact, *any one-to-one correspondence between A and B defines a function from A to B as well as a function from B to A.*

NOW TRY THIS 8

Determine which of the following are functions from the set of natural numbers to $\{0, 1\}$.

a. For every natural-number input, the output is 0.

b. For every natural-number input, the output is 0 if the input is an even number, and the output is 1 if the input is an odd number.

Functions as Tables and Ordered Pairs

One way to describe a function is with a table. Consider the information in Table 5 relating the amounts spent on advertising and the resulting sales in a given month for a small business. The information is given in thousands of dollars. We could define a function between the set of amounts of dollars, A, spent in Advertising and the set of amounts of dollars in Sales, S, or we could simply define the function as follows. If $A = \{0, 1, 2, 3, 4\}$ and $S = \{1, 3, 5, 7, 9\}$, the table describes a function from A to S, where A represents thousands of dollars in advertising and S represents thousands of dollars in sales. For example, $(2, 5)$ means \$2000 was spent on advertising resulting in \$5000 in sales.

Table 5

Amount of Advertising (in \$1000s)	Amount of Sales (in \$1000s)
0	1
1	3
2	5
3	7
4	9

The function could be given using *ordered pairs* (discussed in Chapter 2) as $\{(0, 1), (1, 3), (2, 5), (3, 7)$, and $(4, 9)\}$. The first component in each ordered pair is the *input* and the second is the *output*.

Example 14

Which of the following sets of ordered pairs represent functions? If a set represents a function, give its domain and range. If it does not, explain why.

a. $\{(1, 2), (1, 3), (2, 3), (3, 4)\}$

b. $\left\{\left(1, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{4}\right), \left(4, \frac{1}{5}\right)\right\}$

c. $\{(1, 0), (2, 0), (3, 0), (4, 4)\}$

d. $\{(a, b) \mid a \in N \text{ and } b = 2a\}$

Solution

a. This is not a function because the input 1 has two different outputs.

b. This is a function with domain $\{1, 2, 3, 4\}$. Because the range is the set of outputs corresponding to these inputs set of outputs, the range is $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$.

c. This is a function with domain $\{1, 2, 3, 4\}$ and range $\{0, 4\}$. The output 0 appears more than once, but this does not contradict the definition of a function in that each input corresponds to only one output.

d. This is a function with domain N and range, the set of all even natural numbers.

Functions as Graphs

A sequence can be viewed as a function whose domain, is the set of natural numbers. For example in Figure 21(a), we see a partial *graph* of the arithmetic sequence: $3, 5, 7, 9, 11, \ldots, 2n + 1, \ldots$. The graph of this sequence is depicted on the grid with points given as $(1, 3), (2, 5), (3, 7), (4, 9)$, and $(5, 11)$. Each ordered pair (a, b) is paired with a point on the grid. The horizontal axis in this case is used for the inputs (the numbers of the terms) and the vertical scale depicts the outputs (the terms of the sequence).

In general, to plot an ordered pair (a, b), we move to the point a (the **abscissa**) on the horizontal scale, and then move (up or down) to the point b (the **ordinate**) along the vertical line that goes through a.

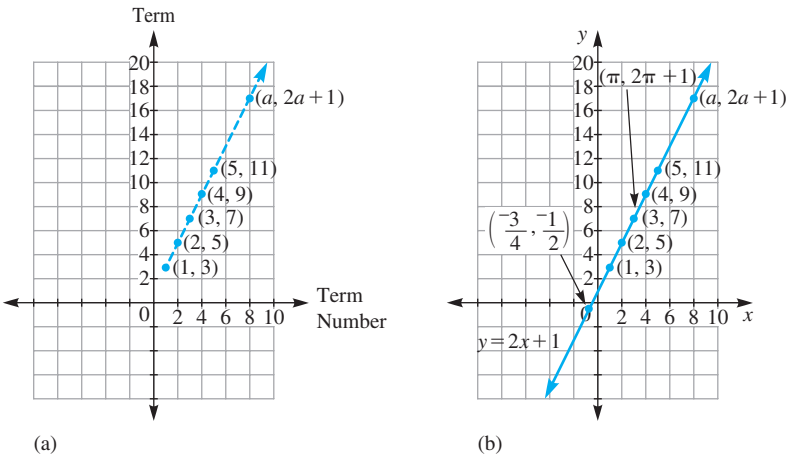


Figure 21

For example, to mark the point corresponding to $(1, 3)$, we start at 1 on the horizontal scale and move up 3 units on the vertical grid line through 1. Marking the point that corresponds to an ordered pair is referred to as **graphing** the ordered pair. The set of all points corresponding to all ordered pairs is the **graph** of the function or relation. In the graph of 21(a) the points are connected by a dashed ray to emphasize that they lie on a straight line, but not every point on the ray belongs to the graph.

Using all real numbers in the domain of the function, $y = 2x + 1$ is depicted in Figure 21(b) and the graph is drawn as a solid line because for every real number a, there is a corresponding real number $2a + 1$ resulting in the ordered pair $(a, 2a + 1)$, which lies on that line. For example, $(\pi, 2\pi + 1)$ and $(^{-}3/4, ^{-}1/2)$ are points on the line. The domain and range in this case are each the set of real numbers. Additionally, the horizontal scale in this case is the ***x*-axis**; the vertical scale is the ***y*-axis**.

Example 15

Explain why a telephone company would not set rates for telephone calls as depicted on the graph in Figure 22.

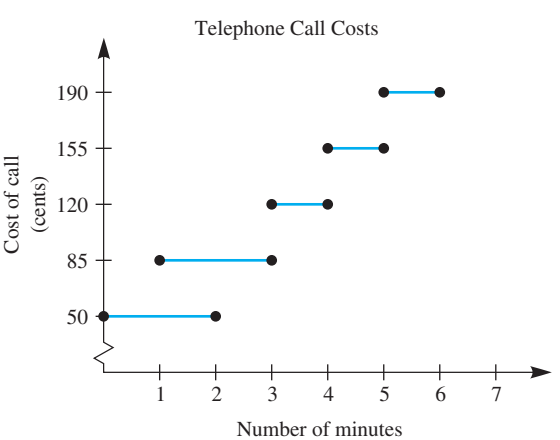


Figure 22

Solution The graph does not depict a function. For example, a customer could be charged either \$0.50 or \$0.85 for a 2-min call; hence, not every input has a unique output.

Sequences as Functions

As mentioned earlier, arithmetic, geometric, and other sequences can be thought of as functions whose inputs are natural numbers and whose outputs are the terms of the particular sequences. For example, an arithmetic sequence 2, 4, 6, 8, . . . , whose nth term a_n is $2n$ can be described as a function from the set N (natural numbers) to the set E (even natural numbers) using the rule $f(n) = 2n$, where n is a natural number representing the number of the term and $f(n)$ stands for the value of the nth term, a_n. Thus, $f(n) = a_n$.

Example 16

If $f(n)$ denotes the nth term of a sequence, find $f(n)$ in terms of n for each of the following:

a. An arithmetic sequence whose first term is 3 and whose difference is 3.
b. A geometric sequence whose first term is 3 and whose ratio is 3.

Solution

a. $a_n = a_1 + (n - 1)d = 3 + (n - 1)3 = 3 + n \cdot 3 - 3 = 3n$. Thus, $f(n) = 3n$, where n is a natural number.
b. $a_n = a_1 r^{n-1} = 3 \cdot 3^{n-1} = 3^n$. Hence, $f(n) = 3^n$, where n is a natural number.

Sums of Sequences as Functions

Gauss's Problem in Chapter 1 to find the sum of $1 + 2 + 3 + \ldots + 100$ is an example of finding the sum of an arithmetic sequence with 100 terms. Recall that one way to find this sum was to write it twice as follows:

$$\begin{aligned} S &= \quad 1 + \quad 2 + \quad 3 + \ldots + 100 \\ S &= 100 + 99 + 98 + \ldots + \quad 1 \end{aligned}$$

Adding equals to equals:

$$\begin{aligned} S + S &= (1 + 100) + (2 + 99) + (3 + 98) + \ldots + (100 + 1) \\ 2S &= 100 \cdot 101 \\ S &= \frac{100 \cdot 101}{2} = 5050 \end{aligned}$$

The same process can be used to find the sum of the first n terms of any arithmetic sequence. $a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \ldots + (a_1 + (n - 1)d)$. We observe that the sum could also be written backwards as $a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \ldots + (a_n - (n - 1)d)$. (Why?) Using $S(n)$ as a function notation to represent the sum of n terms, we have.

$$\begin{aligned} S(n) &= a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n - 1)d) \\ S(n) &= a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n - 1)d) \end{aligned}$$

Adding as before:

$$S(n) + S(n) = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n)$$

Thus, $2S(n) = n(a_1 + a_n)$ and therefore

$$S(n) = \left(\frac{n}{2}\right)(a_1 + a_n) = \left(\frac{a_1 + a_n}{2}\right)n.$$

This proves Theorem 8-5.

Theorem 8-5

The sum S_n of the first n terms of an arithmetic sequence with first term a_1 and nth term a_n is given by $S(n) = \left(\frac{a_1 + a_n}{2}\right)n = \frac{(a_1 + a_n)n}{2}$.

Because $a_n = a_1 + (n - 1)d$, we may substitute for a_n to obtain an equivalent expression for the sum:

$$S(n) = \left(\frac{a_1 + a_1 + (n - 1)d}{2}\right)n = \left(\frac{2a_1 + (n - 1)d}{2}\right)n = a_1 n + \frac{n(n - 1)}{2}d$$

The sum can be thought of as a function of n where n is the number of terms of the sequence. In this case the domain is the set of natural numbers and the output for input n is the sum of the first n terms of the sequence. There is no need to memorize the above two formulas for $S(n)$. Knowing Theorem 8-5 and the expression $a_n = a_1 + (n - 1)d$ is sufficient.

Example 17

Find the sum of the first 100 terms of the following arithmetic sequence: 3, 7, 11, 15, 19, ...

Solution In this arithmetic sequence $a_1 = 3$ and $d = 7 - 3 = 4$.

From Theorem 8-5, we have $S(100) = \left(\frac{a_1 + a_{100}}{2}\right)100 = \left(\frac{3 + a_{100}}{2}\right)100$. Thus, our *subgoal* is to compute a_{100}. We have $a_{100} = 3 + (100 - 1)4 = 399$. Thus, $S(100) = \left(\frac{3 + 399}{2}\right)100 = 20{,}100$.

Similarly, we can find the sum of the first n terms of a geometric sequence. Consider the geometric sequence $a_1, a_1r, a_1r^2, a_1r^3, \ldots, a_1r^{(n-1)}$. Let

$$S(n) = a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1}.$$

Because the terms are very similar, suppose we multiply both sides of the equation by r obtaining $rS_n = a_1r + a_1rr + a_1r^2r + \ldots + a_1r^{n-1}r = a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^n$. Then:

$$\begin{aligned} S(n) &= a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1} \\ rS(n) &= \quad a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^n \end{aligned}$$

Subtracting, we obtain:

$$\begin{aligned} S(n) - rS(n) &= a_1 - a_1r^n \\ S(n)(1 - r) &= a_1(1 - r^n) \\ S(n) &= a_1\frac{1 - r^n}{1 - r} \end{aligned}$$

Note that in the above formula, $r \neq 1$ because division by 0 is not defined.

This proves Theorem 8-6.

Theorem 8-6

The sum of the first n terms of a geometric sequence whose first term is a_1, and whose ratio is $r \neq 1$ is $S(n) = a_1\frac{1 - r^n}{1 - r}$.

If the sum of n terms of a geometric sequence is multiplied by $(^{-}1)/(^{-}1)$, it can be written as $S(n) = a_1\left(\frac{r^n - 1}{r - 1}\right)$. As with the arithmetic sequence sum, the sum of the first n terms of a geometric sequence is a function of n, the number of terms. It is common to write the sum of n terms of a sequence as S_n, we used $S(n)$ to emphasize that the sum is a function of n.

Example 18

Find the sum of the first 10 terms of the geometric sequence: $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$.

Solution The geometric sequence has first term 3 and ratio $\frac{-\frac{3}{2}}{3} = -\frac{1}{2}$. Thus, the sum of the first 10 terms is $S(10) = 3\left(\frac{1 - \left(-\frac{1}{2}\right)^{10}}{1 - \left(-\frac{1}{2}\right)}\right) = 3\left(\frac{1 - \left(\frac{1}{2}\right)^{10}}{\frac{3}{2}}\right) = 2\left(1 - \left(\frac{1}{2}\right)^{10}\right) = \frac{1023}{512} \doteq 1.998.$

Problem Solving **Computer Chips Problem**

Employees working at the same rate produce an order for computer chips. If all were to start at the same time they would complete the job in 3 hours. However, the employees did not start at the same time; each (starting from the second) came to work a fixed number of hours after the first employee so that the one who arrived first worked 11 times longer than the one who arrived last. If they all stayed working until the job was done, how many hours did the first employee work?

Understanding the Problem We know that the first employee worked the most, 11 times more than the last. We designate the number of hours that the last employee worked by b. Then the first worked $11b$ hours. Each employee thereafter worked a certain number of hours, d less than the proceeding one. Thus, the number of hours that each employee worked is an arithmetic sequence. We subtract a fixed number to see how much the next employee worked; that is add ^{-}d. We also know that if all employees were to start the job at the same time, they would have completed it in 24 hours.

Devising a Plan Suppose there were n employees. We write expressions for the total number of working hours in terms of the unknowns introduced. On the one hand, the total number of working hours is $3n$. (Each of the n employees would work 3 hours until the job was finished.) On the other hand, the total working hours is the sum of the hours that each employee worked. Because the hours that each employee worked form an arithmetic sequence, we know that the sum is $\frac{(b + 11b)n}{2}$. We now write an equation and solve it for b.

Carrying Out the Plan Equating the total number of working hours to complete the job we get:

$$3n = \frac{(b + 11b)n}{2}$$

$$3n = \frac{12bn}{2}$$

$$3n = 6bn$$

We divide both sides of the equation by n:

$$3 = 6b$$

$$\frac{1}{2} = b$$

Thus, $11b = 11 \cdot \frac{1}{2} = 5\frac{1}{2}$ and therefore the first employee worked $5\frac{1}{2}$ hours.

Looking Back We notice that the answer to the problem is the same regardless of the number of employees. We also did not have to know d, the number of hours that an employee, starting from the second one, worked less than the preceding one. However, d can be found if n is known. Show that $d = \frac{5}{n-1}$.

Composition of Functions

Consider the function machines in Figure 23. If 2 is entered in the top machine, then $f(2) = 2 + 4 = 6$. The number 6 is then entered in the second machine and $g(6) = 2 \cdot 6 = 12$. The functions in Figure 23 illustrate the **composition of two functions**. In the composition of two functions, the range of the first function must be a subset of the domain of the second function.

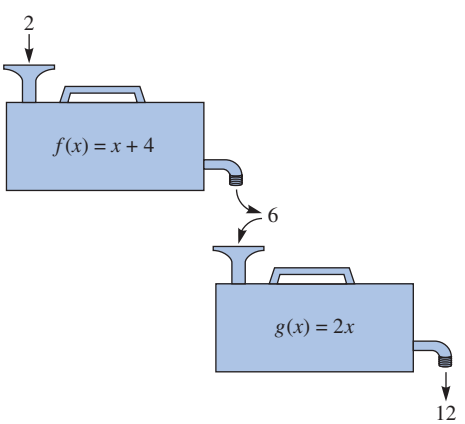


Figure 23

If the first function f is followed by a second function g, as in Figure 23, we symbolize the composition of the functions as $g \circ f$. If we input 3 in the function machines of Figure 23, then the output is symbolized by $(g \circ f)(3)$. Because f acts first on 3, to compute $(g \circ f)(3)$ we find $f(3) = 3 + 4 = 7$ and then $g(7) = 2 \cdot 7 = 14$. Hence, $(g \circ f)(3) = 14$. Observe that $(g \circ f)(3) = g(f(3))$, which is read as "g of f of 3." Also $(g \circ f)(x) = g(f(x)) = 2f(x) = 2(x + 4)$ and hence $g(f(3)) = 2(3 + 4) = 14$.

Example 19

If $f(x) = 2x + 3$ and $g(x) = x - 3$, find the following outputs:

a. $(f \circ g)(3)$ **b.** $(g \circ f)(3)$ **c.** $(f \circ g)(x)$ **d.** $(g \circ f)(x)$

Solution

a. $(f \circ g)(3) = f(g(3)) = f(3 - 3) = f(0) = 2 \cdot 0 + 3 = 3$
b. $(g \circ f)(3) = g(f(3)) = g(2 \cdot 3 + 3) = g(9) = 9 - 3 = 6$
c. $(f \circ g)(x) = f(g(x)) = 2g(x) + 3 = 2(x - 3) + 3 = 2x - 6 + 3 = 2x - 3$
d. $(g \circ f)(x) = g(f(x)) = f(x) - 3 = (2x + 3) - 3 = 2x$

Example 19 shows that composition of functions is not commutative, since $(f \circ g)(3) \neq (g \circ f)(3)$.

Calculator or Computer Representation of a Function

A function can be represented in a variety of ways: pictures of sets with arrows, function machines, tables, equations, or graphs. Depending on the situation, one representation may be more useful than another. For example, if the domain of a function has many elements, a table is not a convenient representation. Graphing calculators or computer graphing applications can be used to display a graph of most functions given by equations with specified domains.

A sketch of the function $f(x) = 2x + 1$ is shown in Figure 24 using a graphing calculator downloaded on a computer. Note that on this graphing calculator $f(x)$ is depicted as y_1, and the domain was automatically chosen.

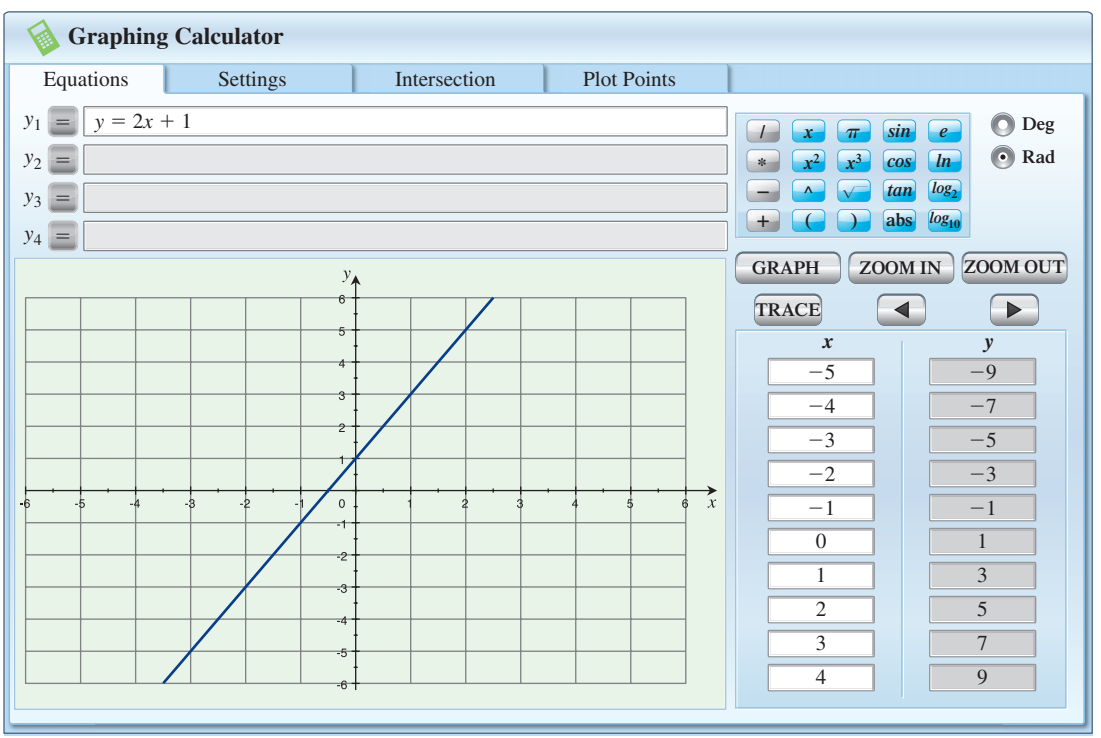


Figure 24

Because $f(x) = 2x + 1$ in Figure 24 and x is the only variable; we say that f is a function of *single variable*. However, we can also have functions of two or more variables. For example, the function that assigns to each ordered pair (x, y) the number $x + y$ is written $f(x, y) = x + y$, and is a function of two variables. If a particle moves at a speed V miles per hour for t hours, the distance S that it travels is Vt. If we let V as well as t vary, then $S(V, t) = Vt$ is a function of two variables. The function $f(x, y, z) = (x + y)z$ is an example of a function of three variables.

Relations

A function from set A to set B is a form of a *relation from set A to set B*. In a relation from A to B, there is a correspondence between elements of A and elements of B, but we do not require that each element of A be paired with one, and only one, element of B. Consequently, any set of ordered pairs is a relation. With appropriate set definitions, examples of relations include the following:

"is a daughter of" "is the same color as"
"is less than" "is greater than or equal to"

Consider the relation "is a sister of". Figure 25 illustrates this relation among children on a playground, with letters A through J representing the children. An arrow from I to J indicates that I "is a sister of" J. The arrows from F to G and from G to F indicate that F is a sister of G and G is a sister of F. This implies that F and G are girls. On the other hand, the absence of an arrow from J to I implies that J is not a sister of I. Thus, I is a girl and J is a boy.

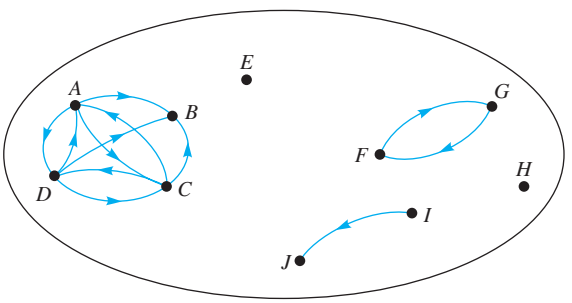


Figure 25

NOW TRY THIS 9

All sister relationships are indicated in Figure 25.

a. Which children are boys and which are girls?
b. For which children is there not enough information to determine gender?

Another way to depict the arrow relation "A is a sister of B" is as an ordered pair (A, B). Using this notation, the relation "is a sister of" can be described for the children on the playground in Figure 26 as the set

$$\{(A,B),(A,C),(A,D),(C,A),(C,B),(C,D),(D,A),(D,B),(D,C),(F,G),(G,F),(I,J)\}.$$

This set is a subset of the Cartesian product $\{A, B, C, D, E, F, G, H, I, J\} \times \{A, B, C, D, E, F, G, H, I, J\}$.

This observation motivates the following definition of a relation.

Definition of a Relation from Set *A* to Set *B*

Given any two sets A and B, a **relation** from A to B is a subset of $A \times B$; that is, R is a relation from set A to set B if, and only if, $R \subseteq A \times B$.

In the definition, if $A = B$, we say that the **relation is on *A***. Using the concept of a relation a function from set A to set B can be defined as a relation from set A to set B such that:

- for each $a \in A$, there exist some element $b \in B$ such that (a, b) is in the relation and,
- two different ordered pairs in the relation cannot have the same first component.

Properties of Relations

Figure 26 represents a set of children in a small group. The children have drawn all possible arrows representing the relation "has the same first letter in his or her name as." Three properties of relations are illustrated in Figure 26.

Definition of the Reflexive Property

A relation R on a set X is **reflexive** if, and only if, for every element $a \in X$, a is related to a; that is, for every $a \in X$, $(a, a) \in R$.

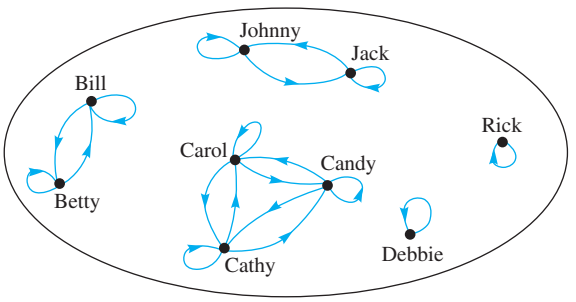


Figure 26

In the diagram, there is a loop at every point. For example, Rick has the same first initial as himself, namely R. A relation such as "is taller than" is not reflexive because people cannot be taller than themselves.

Definition of the Symmetric Property

A relation R on a set X is **symmetric** if, and only if, for all elements a and b in X, whenever a is related to b, then b also is related to a; that is, if $(a, b) \in R$, then $(b, a) \in R$.

In terms of the diagram, every pair of points that has an arrow headed in one direction also has a return arrow. For example, if Bill has the same first initial as Betty, then Betty has the same first initial as Bill. A relation such as "is a brother of" is not symmetric since Dick can be a brother of Jane, but Jane cannot be a brother of Dick.

Definition of the Transitive Property

A relation R on a set X is **transitive** if, and only if, for all elements a, b, and c of X, whenever a is related to b and b is related to c, then a is related to c. That is, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.

In the definitions for the symmetric and transitive properties, a, b, and c are variables and do not have to be different.

The relation in Figure 26 is transitive. For example, if Carol has the same first initial as Candy, and Candy has the same first initial as Cathy, then Carol has the same first initial as Cathy. A relation such as "is the father of " is not transitive since, if Tom Jones is the father of Tom Jones, Jr. and Tom Jones, Jr. is the father of Joe Jones, then Tom Jones is not the father of Joe Jones. He is, instead, the grandfather of Joe Jones.

The relation "is the same color as" is reflexive, symmetric, and transitive. In general, relations that satisfy all three properties are **equivalence relations**.

Definition of the Equivalence Relation

An **equivalence relation** is any relation R that satisfies the reflexive, symmetric, and transitive properties.

The most natural equivalence relation encountered in elementary school is "is equal to" on the set of all numbers. The relation on W (the set of whole numbers) defined by "have the same remainder upon division by m $(m \geq 2)$" is an equivalence relation. (Why?) For example if $m = 3$, W is partitioned into three disjoint subsets by this relation, all the elements of W that have remainder 0 when divided by 3, remainder 1 when divided by 3, and those that have remainder 2 when divided by 3.

Equivalence relations are useful in mathematics because of the property that an equivalence relation on a set S partitions the set into disjoint subsets whose union is S. In subsequent chapters, we will see other examples of equivalence relations, such as congruence and similarity in geometry.

Assessment 8-4A

1. The following sets of ordered pairs are functions. Give a rule that could describe each function.
- **a.** $\{(^-1, 2), (0, 0), (1, ^-2), (2^{\frac{1}{2}}, ^-2^{\frac{3}{2}})\}$
- **b.** $\{(^-6, 0), (0, 6), (5, 11), (\sqrt{3}, \sqrt{3} + 6)\}$
- **c.** $\{(^-1, 1), (1, 1), (2, 4), (\sqrt{3}, 3)\}$

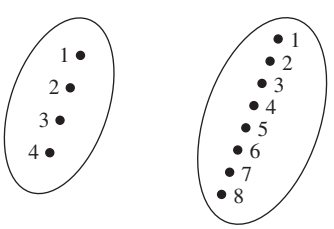
2. Which of the following are functions from the set $\{1, 2, 3\}$ to the set $\{a, b, c, d\}$? If the set of ordered pairs is not a function, explain why not.
- **a.** $\{(1, a), (2, b), (3, c), (1, d)\}$
- **b.** $\{(1, a), (2, b), (3, a)\}$
- **c.** $\{(1, a), (2, a), (3, a)\}$
- **d.** $\{(1, a), (2, b), (3, b)\}$

3. **a.** Draw an arrow diagram of a function with domain $\{1, 2, 3, 4, 5\}$ and range $\{a, b\}$.
- **b.** How many possible functions are there for part (a)?

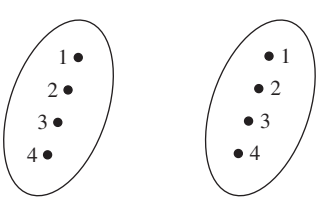
4. Suppose $f(x) = 2x + 1$ and the domain is $\{0, 1, \sqrt{2}, \sqrt{3}, 4\}$. Describe the function in the following ways.
- **a.** Draw an arrow diagram involving two sets.
- **b.** Use ordered pairs.
- **c.** Make a table.
- **d.** Draw a graph to depict the function.

5. Determine which of the following are functions from the set of real numbers, R, or a subset of R, to R. If your answer is that it is not a function, explain why not.
- **a.** $f(x) = 2$ for all $x \in R$
- **b.** $f(x) = \sqrt{x}$
- **c.** $\{(x, y) | x = y^2, x \geq 0\}$
- **d.** $\{(x, y) | x = y^3\}$

6. **a.** Make an arrow diagram for each of the following.
- **i.** Rule: "when doubled is"



- **ii.** Rule: "is greater than"



- **b.** Which, if any, of the parts in (a) exhibits a function from A to B? If it is a function, tell why and find the range of the function.

7. The dosage of a certain drug is related to the weight of a child as follows: 50 mg of the drug and an additional 15 mg for each 2 lb or fraction of 2 lb of body weight above 30 lb. Sketch the graph of the dosage as a function of the weight of a child for children who weigh between 20 and 40 lb.

8. For each of the following, guess what might be Latifah's rule. In each case, if n is your input and $L(n)$ is Latifah's answer, express $L(n)$ in terms of n.

a.

You	Latifah
3	8
4	11
5	14
10	29

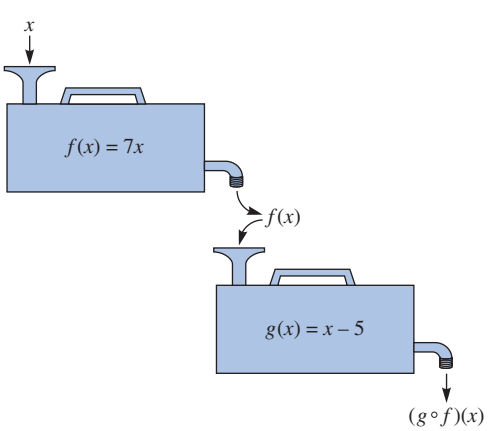
b.

You	Latifah
0	1
3	10
5	26
8	65

9. Quick-Talk advertises monthly cellular phone service for \$0.50 a minute for the first 60 minutes but only \$0.10 a minute for each minute thereafter. Quick-Talk charges for the exact amount of time used. Answer the following.
- **a.** Make one graph showing the cost per minute as a function of number of minutes and the other showing the total cost for calls as a function of the number of minutes up to 100 min.
- **b.** If you connect the points in the second graph in part (a), what kind of assumption needs to be made about the way the telephone company charges phone calls?
- **c.** Why does the total cost for calls consist of two line segments? Why is one part steeper than the other?
- **d.** The function representing the total cost for calls as a function of number of minutes talked can be represented by two equations. Write these equations.

10. For each of the following sequences (either arithmetic or geometric), find a possible function $f(n)$ whose domain is the set of natural numbers and whose outputs are the terms of the sequence.
- **a.** $3, 8, 13, 18, 23, \ldots$
- **b.** $3, 9, 27, 81, 243, \ldots$
- **c.** $3, 3, 3, 3, \ldots$
- **d.** $3, 3^{\frac{4}{3}}, 3^{\frac{5}{3}}, 3^2, \ldots$

11. Consider the following two function machines. Find the final output for each of the following inputs.
- **a.** 6
- **b.** 10
- **c.** $\sqrt{10}$
- **d.** 0
- **e.** $\frac{5}{8}$
- **f.** n



12. In problem 11, find all inputs for which the final output equals twice the input.

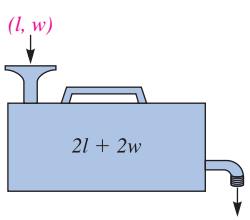
13. **a.** Repeat problem 11 when the bottom machine $g(x) = x - 5$ is on top and the top machine $f(x) = 7x$ is at the bottom. (In this case, the final output is $(f \circ g)(x)$.)
b. Find all x for which $(f \circ g)(x) = x$.

14. Consider the function $f(x) = \bar{}x + b$, where b is a constant, and find the following.
a. $f(f(0))$ **b.** $f(f(2))$
c. All inputs x for which the final output is x, that is, $f(f(x)) = x$.

15. Let $t(n)$ represent the nth term of a sequence for $n \in N$. Answer the following.
a. If $t(n) = 4n - 3$, determine which of the following are output values of the function:
i. 1 **ii.** 385 **iii.** 389 **iv.** 392
b. If $t(n) = n^2$, determine which of the following are output values of the function.
i. 0 **ii.** 25 **iii.** 625 **iv.** 1000 **v.** 90
c. If $t(n) = n(n - 1)$, determine which of the following are in the range of the function.
i. 0 **ii.** 2 **iii.** 20 **iv.** 999

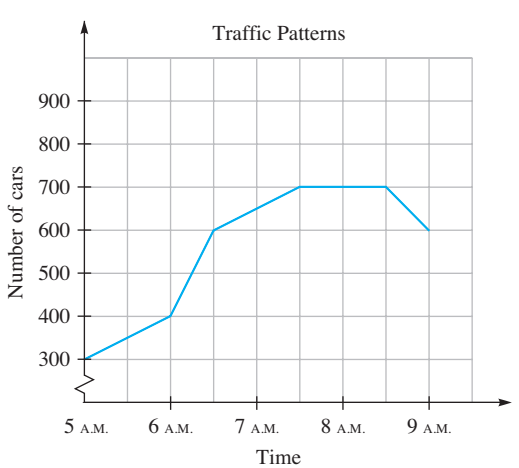
16. **a.** If $f(x) = x^2$ and the output is 2 find all possible inputs.
b. Repeat (a) if $f(x) = x^3$

17. Consider a function machine that accepts inputs as ordered pairs. Suppose the components of the ordered pairs are natural numbers and the first component is the length of a rectangle and the second is its width. The following machine computes the perimeter (the distance around a figure) of the rectangle. Thus, for a rectangle whose length, l, is 3 and whose width, w, is 2, the input is (3, 2) and the output is $2 \cdot 3 + 2 \cdot 2$, or 10.



a. For each of the following inputs, find the corresponding output: (1, 7), (2, 6), (6, 2), $(\sqrt{5}, \sqrt{5})$.
b. Find the set of all the inputs for which the output is 20.
c. What is the domain and the range of the function?

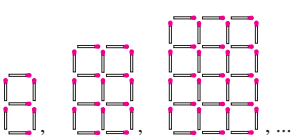
18. The following graph shows the relationship between the number of cars on a certain road and the time of day for times between 5:00 A.M. and 9:00 A.M.:



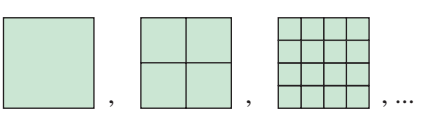
a. What was the increase in the number of cars on the road between 6:30 A.M. and 7:00 A.M.?
b. During which half hour was the increase in the number of cars the greatest?
c. What was the increase in the number of cars between 8:00 A.M. and 8:30 A.M.?
d. During which half hour(s) did the number of cars decrease? By how much?
e. The graph for this problem is composed of segments rather than just points. Why do you think segments are used here instead of just points?

19. A ball is shot out of a cannon at ground level. Its height H in feet after t sec is given by the function $H(t) = 128t - 16t^2$.
a. Find $H(2), H(6), H(3)$, and $H(5)$. Why are some of the outputs equal?
b. Graph the function and from the graph find at what instant the ball is at its highest point. What is its height at that instant?
c. How long does it take for the ball to hit the ground?
d. What is the domain of H?
e. What is the range of H?

20. In the following sequence of matchstick figures assume a possible pattern and, let $S(n)$ be the function giving the total number of matchsticks in the nth figure. Find $S(n)$ in terms of n.

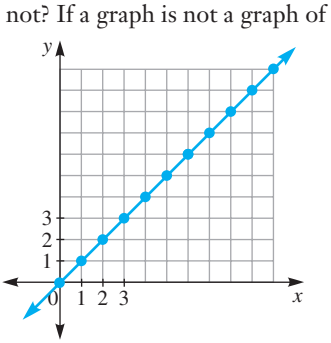


21. Assume the pattern continues for the following sequence of square tile figures; that is each square is divided into four squares in the subsequent figure. Let $S(n)$ be the function giving the total number of tiles in the nth figure. Find a formula for $S(n)$ in terms of n.

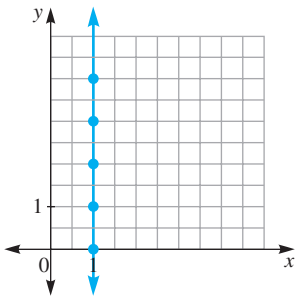


22. Which of the following equations or inequalities represent functions and which do not? In each case x and y are real numbers; x is in the domain and y in the range. If your answer is "not a function" explain why not.
a. $x + y = 2$ **b.** $x - y < 2$
c. $y = x^3 + x$ **d.** $xy = 2, x \neq 0$
e. $y = |x|$ **f.** $|y| = x$
g. $x^2 + y^2 = 1$ **h.** $y = \sqrt{\bar{}x}$

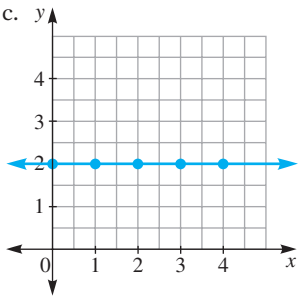
23. Which of the following are graphs of functions and which are not? If a graph is not a graph of function explain why not.
a.



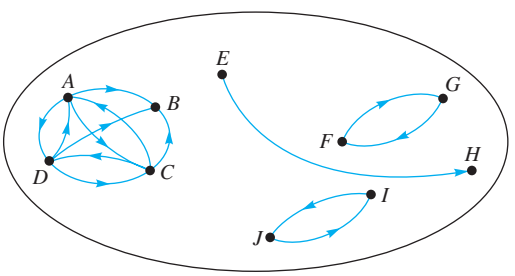
b.



c.



24. Suppose each point and letter in the figure represents a child on a playground, and an arrow going from I to J means that I is the sister of J.



a. Based on the information in the figure, who are definitely girls and who are definitely boys?

b. Suppose we write "A is the sister of B" as an ordered pair (A, B). Based on the information in the diagram, write the set of all such ordered pairs.

c. Is the set of all ordered pairs in (b) a function with the domain equal to the set of all first components of the ordered pairs and with the range equal to the set of all second components?

25. a. Consider the relation consisting of ordered pairs (x, y) such that y is the biological mother of x. Is this a function whose domain is the set of all people born on January 1, 2015?

b. Use the same information as in part (a) but now y is a brother of x. Is the relation a function from the set of all boys to the set of all boys?

26. Tell whether each of the following is reflexive, symmetric, or transitive on the set of all people. Which are equivalence relations?

a. "Is a parent of"
b. "Is the same age as"
c. "Has the same last name as"
d. "Is the same height as"
e. "Is married to"
f. "Lives within 10 mi of"
g. "Is older than"

Assessment 8-4B

1. The following sets of ordered pairs are functions. Give a rule that could describe each function.

a. $\{(^-1, 3), (0, 0), (1, ^-3), (3^{-\frac{1}{2}}, ^-3^{\frac{1}{2}})\}$
b. $\{(^-5, 0), (0, 5), (5, 10), (\sqrt{5}, \sqrt{5} + 5)\}$
c. $\left\{(0, 0), (\sqrt{3}, 3), (^-\sqrt{5}, 5), \left(\frac{1}{\sqrt{6}}, \frac{1}{6}\right)\right\}$

2. Which of the following are functions from the set $\{1, 2, 3\}$ to the set $\{a, b, c, d\}$? If the set of ordered pairs is not a function, explain why not.

a. $\{(1, c), (3, d)\}$ **b.** $\{(1, a), (1, b), (1, c)\}$
c. $\{(1, b), (2, b), (3, b)\}$ **d.** $\{(1, a), (2, b), (2, c), (3, d)\}$

3. a. Draw an arrow diagram of a function with domain $\{1, 2, 3\}$ and range $\{a, b\}$.

b. How many possible functions are there for part (a)?

4. Suppose $f(x) = 2(x - 1)$ and the domain is $\{0, 1, \sqrt{2}, \sqrt{3}, 4\}$. Describe the function in the following ways.

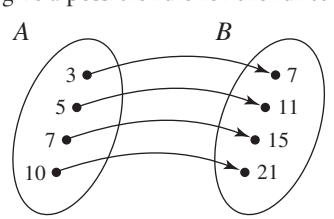
a. Draw an arrow diagram involving two sets.
b. Use ordered pairs.
c. Make a table.
d. Draw a graph to depict the function.

5. Determine which of the following are functions from the set of real numbers, R, or a subset of R, to R. If your answer is that it is not a function, explain why not.

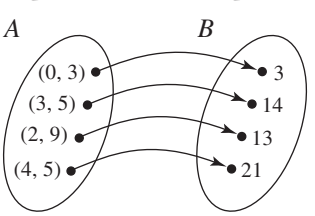
a. $f(x) = 0$ if $x \in \{0, 1, 2, 3\}$, and $f(x) = 3$ if $x \notin \{0, 1, 2, 3\}$
b. $f(x) = 0$ for all $x \in R$ and $f(x) = 1$ if $x \in \{3, 4, 5, 6, \ldots\}$
c. $\{(x, y) \mid y = |x|\}$
d. $\{(x, y) \mid x = \sqrt{y^2}\}$

6. Given the following arrow diagrams for functions from A to B, give a possible rule for the function.

a.



b.



7. According to wildlife experts, the rate at which crickets chirp is a function of the temperature; specifically, $C = T - 40$, where C is the number of chirps every 15 sec and T is the temperature in degrees Fahrenheit.

a. How many chirps does the cricket make per second if the temperature is 70°F?
b. What is the temperature if the cricket chirps 40 times in 1 min?

8. For each of the following, guess what might be Latifah's rule. In each case, if n is your input and $L(n)$ is Latifah's answer, express $L(n)$ in terms of n.

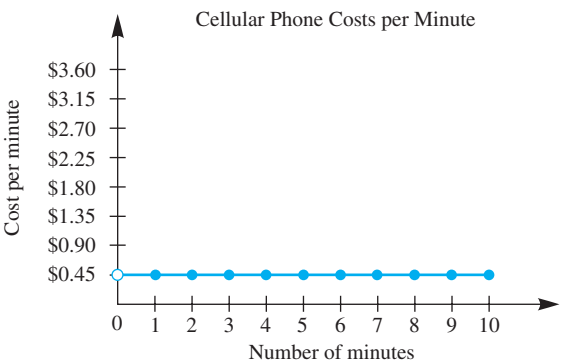
a.

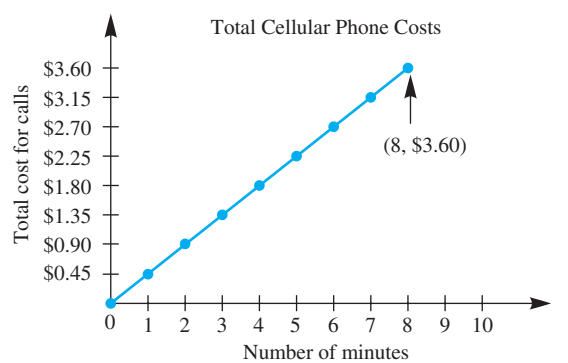
You	Latifah
6	42
0	0
8	72
2	6

b.

You	Latifah
0	1
1	2
5	32
6	64
10	1024

9. ChitChat charges \$0.45 a minute for cellular phone calls. The cost per minute does not change, but the total cost changes as the telephone is used.





a. When the number of minutes is 6, what do the values of the corresponding point on each graph represent?

b. What kind of assumption about the charges needs to be made to allow the connection of the points on each graph? Explain.

c. If the time in minutes is t and the cost in dollars for calls is c, write c as a function of t for each graph.

10. For each of the following (arithmetic or geometric) sequences, discover the pattern and find a function whose domain is the set of natural numbers and whose outputs are the terms of the sequence:

a. 2, 4, 6, 8, 10, . . .

b. 1, 3, 9, 27, 81, . . .

c. $\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, \ldots$

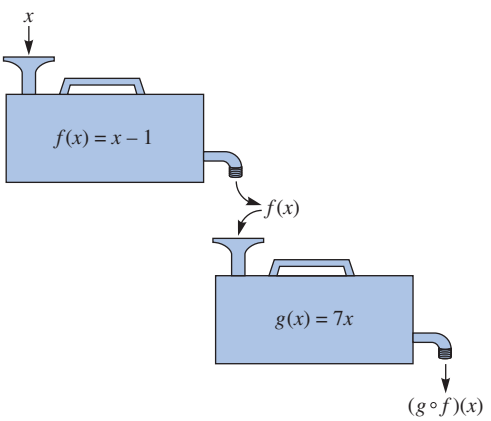
d. $3^{-1}, 3^{-\frac{1}{2}}, 1, 3^{\frac{1}{2}}, 3, \ldots$

11. Consider two function machines that are placed as shown. Find the final output for each of the following inputs.

a. 5 **b.** $\sqrt{3}$

c. 10 **d.** a

e. $\frac{7}{6}$



12. In problem 11, find all inputs for which the final output equals twice the input.

13. **a.** Repeat problem 11 when the bottom machine $g(x) = 7x$ is on top and the top machine $f(x) = x - 1$ is at the bottom. (In this case, the final output is $(f \circ g)(x)$.)

b. Find all x for which $(f \circ g)(x) = x$.

14. Consider the function $f(x) = \overline{\ }(x + b)$, where b is a constant, and find the following (simplify your answers).

a. $f(f(0))$

b. $f(f(\sqrt{2}))$

c. All inputs x for which the final output is x, that is, $f(f(x)) = x$

15. Consider the function $f(x) = \dfrac{2x - 1}{2x + 1}$ and find, all possible, inputs for the following outputs.

a. 2 **b.** 0

c. $-\frac{1}{2}$ **d.** 1

16. Let $t(n)$ represent the nth term of a sequence for $n \in N$.

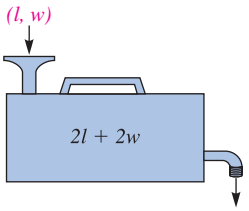
a. If $t(n) = n^2$, determine which of the following are output values of the function:

i. 1 **ii.** 4 **iii.** 9 **iv.** 10 **v.** 900

b. If $t(n) = n(n + 1)$, determine which of the following are in the range of the function:

i. 2 **ii.** 12 **iii.** 2550 **iv.** 2600

17. Consider a function machine that accepts inputs as ordered pairs. Suppose the components of the ordered pairs are natural numbers and the first component is the length of a rectangle and the second is its width. The following machine computes the perimeter (the distance around a figure) of the rectangle. Thus, for a rectangle whose length, l, is 3 and whose width, w, is 1, the input is (3, 1) and the output is $2 \cdot 3 + 2 \cdot 1$, or 8.

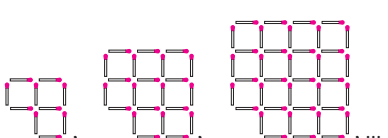


a. For each of the following inputs, find the corresponding output: $(1, 4), (2, 1), (1, 2), (\sqrt{3}, \sqrt{3}), (x, y)$.
b. Find the set of all the inputs for which the output is 20.
c. Is (2, 2) a possible output? Explain your answer.

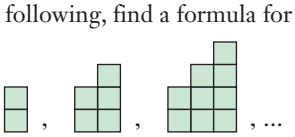
18. A health club charges a one-time initiation fee of \$100.00 plus a membership fee of \$40.00 per month.
a. Write an expression for the cost function $C(x)$ that gives the total cost for membership at the health club for x months.
b. Draw the graph of the function in (a).
c. The health club decided to give its members an option of a higher initiation fee but a lower monthly membership charge. If the initiation fee is \$300 and the monthly membership fee is \$30, use a different color and draw on the same set of axes the cost graph under this plan.
d. Determine after how many months the second plan is less expensive for the member.

19. A ball is shot straight up at ground level. Its height H in feet after t sec is given by the function $H(t) = 128t - 16t^2$.
a. Graph the function and from the graph find at what instant the ball is at its highest point. What is its height at that instant?
b. Find from the graph all t such that $H(t) = H(1)$.
c. Use your graph to find how long it takes the ball to hit the ground.
d. What is the domain of H?
e. What is the range of H?

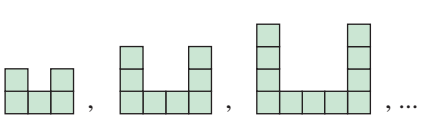
20. In the following sequence of matchstick figures, assume that your discovered pattern continues and let $S(n)$ be the function giving the total number of matchsticks in the nth figure.
a. Find the total number of matchsticks in the fourth figure.
b. Find a formula for $S(n)$ in terms of n.



21. Assume the pattern continues for each of the following sequences of square tile figures. Let $S(n)$ be the function giving the total number of tiles in the nth figure. For each of the following, find a formula for $S(n)$ in terms of n.
a.

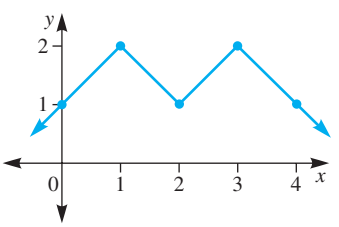


b.

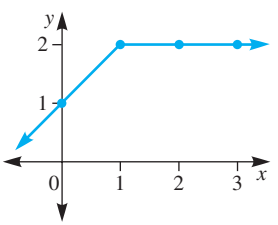


22. Which of the following equations or inequalities represent functions and which do not? In each case x and y are real numbers; x is in the domain and y in the range. If your answer is "not a function" explain why not.
a. $x - y = 2$ **b.** $x + y < 20$
c. $y = 2x^2$ **d.** $y = x^3 - 1$
e. $y = -|x|$ **f.** $|y| = |x|$
g. $x^2 - y^2 = 1$ **h.** $y = -\sqrt{x}$

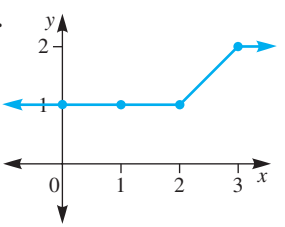
23. Which of the following are graphs of functions and which are not? Justify your answers.
a.



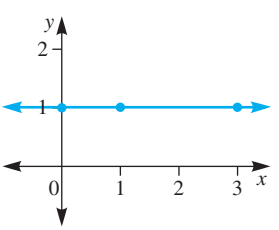
b.



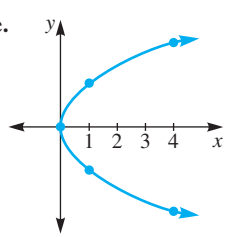
c.



d.

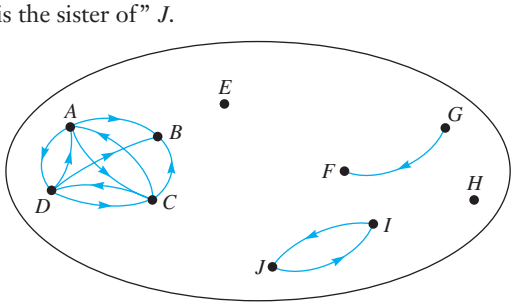


e.



24. **a.** Which of the following relations from the set W of whole numbers to W have the symmetric property? Justify your answers.
i. $x + y = 10$
ii. $x - y = 100$
iii. $xy = 100$
iv. $y = x$
v. $y = x^2$
b. Which of the relations in part (a) are functions? Justify your answer.

25. Suppose each point and letter in the figure represents a child on a playground, and an arrow going from I to J means that I "is the sister of" J.



a. Based on the information in the figure, who are definitely girls and who are definitely boys?

b. Suppose we write "A is the sister of B" as an ordered pair (A, B). Based on the information in the diagram, write the set of all such ordered pairs.
c. Is the set of all ordered pairs in (b) a function with the domain equal to the set of all first components of the ordered pairs and with the range equal to the set of all second components?

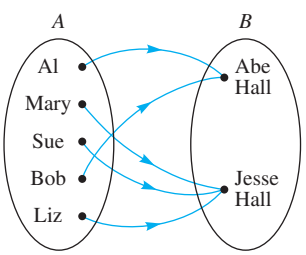
26. Which of the following are functions and which are relations but not functions from the set of first components of the ordered pairs to the set of second components?
a. {(Montana, Helena), (Oregon, Salem), (Illinois, Springfield), (Arkansas, Little Rock)}
b. {(Pennsylvania, Philadelphia), (New York, Albany), (New York, Niagara Falls), (Florida, Ft. Lauderdale)}
c. $\{(x, y) \mid x$ resides in Birmingham, Alabama, and x is the mother of y, where y is a U.S. resident$\}$
d. $\{(1, 1), (2, 4), (3, 9), (4, 16)\}$
e. $\{(x, y) \mid x$ and y are natural numbers and $x + y$ is an even number$\}$

27. Tell whether each of the following is reflexive, symmetric, or transitive on the set of subsets of a nonempty set. Which are equivalence relations?
a. "Is equal to"
b. "Is a proper subset of"
c. "Is not equal to"
d. "Has the same cardinal number as"

Mathematical Connections 8-4

Reasoning

1. Does the diagram define a function from A to B? Why or why not?



2. Is a one-to-one correspondence a function? Explain your answer and give an example.

3. Which of the following are functions from A to B? If your answer is "not a function," explain why not.
a. A is the set of mathematics faculty at the university. B is the set of all mathematics classes. To each mathematics faculty member, we associate a class that person is teaching during a given term.
b. A is the set of mathematics classes at the university. B is the set of mathematics faculty. To each mathematics class, we associate the teacher who is teaching the class.
c. A is the set of all U.S. senators and B is the set of all senate committees. We associate each senator to a committee of which the senator is chairperson.

4. If S is the set of students in Ms. Carmel's class, and A is any subset of S, we define: $f(A) = \overline{A}$ (where $\overline{A}$ is the complement of A). Notice that the input in this function is a subset of S and the output is a subset of S. Answer the following.
a. Explain why f is a function and describe the domain and the range of f.
b. If there are 20 children in the class, what are the number of elements in the domain and the number in the range? Explain.
c. Is the function in this question a one-to-one correspondence? Justify your answer.

5. Answer the following about $f(x) = 2^x$, an exponential function.
a. What do you think $f(x) = 2^{3.14}$ means?
b. What could $2^{\sqrt{3}}$ mean?
c. What is the domain of $f(x)$? Why?
d. What is the range of $f(x)$? Why?

6. A sum of money is invested for a certain length of time in an account earning interest at a certain rate. Explain how the balance in the account at the end of that time can be considered a function of three variables. Clearly identify the variables and derive a formula for the balance in terms of these three variables.

7. Is every set of ordered pairs a function whose domain is the set of first components and whose range is the set of second components? Justify your answer.

Open-Ended

8. Examine several newspapers and magazines and describe at least three examples of functions that you find. What is the domain and range of each function?

9. Give at least three examples of functions from A to B, where neither A nor B is a set of numbers.

10. Draw a sequence of matchstick figures and describe the pattern in words. Find as simple an expression as possible for $S(n)$, the total number of matchsticks in the nth figure.

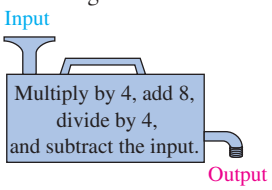
11. A function whose output is always the same regardless of the input is a *constant function*. Give several examples of constant functions from real life.

12. A function whose output is the same as its input is an *identity function*. Give several concrete examples of identity functions.

13. Define three functions $f(x)$, $g(x)$, and $h(x)$ that are placed in a way similar to the two machines in exercise 11 of Assessment 8-4B. Draw a figure depicting the machines and find $(h \circ g \circ f)(x)$ for the following inputs.
a. 1
b. $\sqrt{3}$
c. n

Cooperative Learning

14. Each person in a group picks a natural number and uses it as an input in the following function machine:

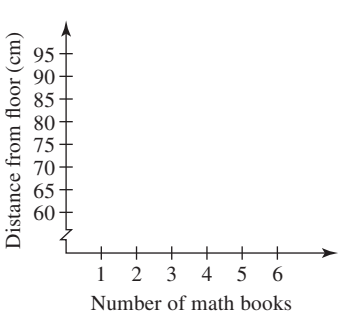


- **a.** Compare your answers. Based on the answers, make a conjecture about the range of the function.
- **b.** Based on your answer in (a), graph the function.
- **c.** Write the function in the simplest possible way using $f(x)$ notation.
- **d.** Justify your conjecture in (a).
- **e.** Make up similar function machines and try different inputs in your group.
- **f.** Devise a function machine in which the machine performs several operations but the output is always the same as the input. Exchange your answer with someone in the group and check that the other person's function machine performs as required.

15. In a group of four, work through the following. You will need a metric tape or meterstick.

- **a.** Place your mathematics book on the edge of a desk and measure the distance (to the nearest centimeter) from the floor to the top of the book. Record the distance.
- **b.** Place a second mathematics book on top of the first and measure the distance (to the nearest centimeter) from the floor to the top of the second book. Record the distance.
- **c.** Continue this procedure for all four of your mathematics books and complete the following table and graph:

Number of books	Distance from floor
1	
2	
3	
4	

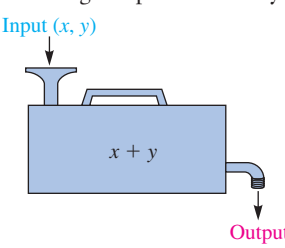


- **d.** Without measuring, what is the distance from the floor with 0 books? 5 books?
- **e.** Write a rule or function for $d(x)$, where $d(x)$ is the distance above the floor to the top of the stack of books and x is the number of books.
- **f.** Suppose the distance from the floor to the ceiling is 2.5 m. If you stack the books as described above, how many books would be needed to reach the ceiling?
- **g.** The function $h(x) = 34x + 70$ represents the height of another stack of x mathematics books (in centimeters) of the same thickness in a cabinet. What does the function tell you about the height of the cabinet?
- **h.** Suppose that a table with a stack of similar mathematics books (more than 10) is 200 cm high. If the top mathematics book is removed, the height is 197 cm. If a second book is removed, the height is 194 cm. What is the height if 5 books are removed?
- **i.** Write a function $h(x)$ for the height of the stack in part (h) after x books are removed.

16. As a teacher you noticed that several students in your class assumed that if f is a function, then $f(a + b) = f(a) + f(b)$ whenever a, b, and $a + b$ are in the domain of the function. How do you respond?

Connecting Mathematics to the Classroom

17. A student claims that the following machine does not represent a function machine because it accepts two inputs at once rather than a single input. How do you respond?



18. A student claims that the following does not represent a function, since all the values of x correspond to the same number.

x	0	1	2	3	4	5
y	1	1	1	1	1	1

How do you respond?

19. A student asks if the function $f(x) = 3x + 5$ with domain the set of real numbers is a one-to-one correspondence between the sets of real numbers. How do you respond?

20. A student wants to know why sometimes it is incorrect to connect points on the graph of a function. How do you respond?

21. A student does not understand why $\{(x,y) \mid y = x^2, x \in R\}$ is a function but $\{(x,y) \mid x = y^2, x \in R, x \geq 0\}$ is not. How do you respond?

Review Problems

22. Solve the following equations for x.

- **a.** $2\sqrt{2} - \sqrt{2}x = x$
- **b.** $\frac{3}{4}x = \frac{\sqrt{2}}{2}x + 3$
- **c.** $(x - 1)^2 = 2$
- **d.** $\sqrt{2}(2x - 3) = \sqrt{3}(2x - 3)$
- **e.** $\sqrt{(x - 1)^2} = x - 1$
- **f.** $\sqrt{(x - 1)^2} = 1 - x$

23. Solve the following problem by setting up an appropriate equation:

Two cars, each traveling at a constant speed—one 60 mph and the other 70 mph—start at the same time from the same point traveling in the same direction. After how many hours will the distance between them be 40 mi?

24. **a.** Find two rational numbers between $\sqrt{3}$ and 2.

b. Find two irrational numbers between $\frac{11}{13}$ and $\frac{12}{13}$.

25. Show that the set of irrational numbers is not closed under addition.

National Assessments

National Assessment of Educational Progress (NAEP) Questions

In	Out
2	5
3	7
4	9
5	11
15	31
38	

The table shows how the "In" numbers are related to the "Out" numbers. When 38 goes in, what number comes out?
A. 41
B. 51
C. 54
D. 77

NAEP, Grade 4, 2007

Each figure in the pattern below is made of hexagons that measure 1 centimeter on each side.

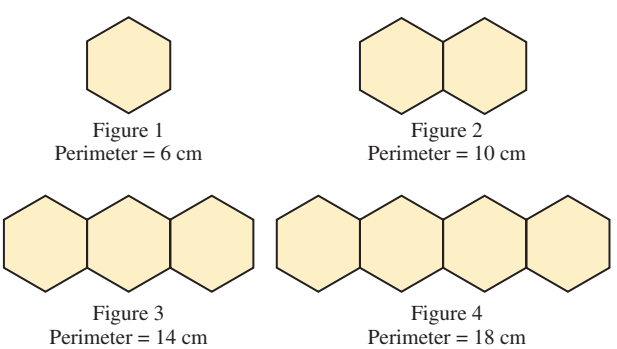


Figure 1
Perimeter = 6 cm

Figure 2
Perimeter = 10 cm

Figure 3
Perimeter = 14 cm

Figure 4
Perimeter = 18 cm

If the pattern of adding one hexagon to each figure is continued, what will be the perimeter of the 25th figure in the pattern? Show how you got your answer.

NAEP, Grade 8, 2007

Old Faithful is one of hundreds of geysers in Yellowstone National Park. Predicting when Old Faithful will erupt next can be done by timing the previous eruption.

If an eruption lasts t minutes, then the next eruption will occur approximately $12.5t + 33$ minutes after the eruption ends. If the previous eruption lasted 6 minutes and ended at 1:23 P.M., when is the next eruption expected to occur?

Answer: _____

Show how you found your answer.

NAEP, Grade 8, 2012

8-5 Equations in a Cartesian Coordinate System

8-5 Objectives

Students will be able to understand and explain

- Equations of lines.
- The slope of a line and the slope formula.
- Graphic and algebraic solutions for systems of linear equations.
- Ways to determine if a system of linear equations has a unique solution, infinitely many solutions, or no solution.

The Cartesian coordinate system (named for René Descartes) enables us to combine geometry and algebra. The **Cartesian coordinate system**, is constructed by placing two number lines perpendicular to each other, as shown in Figure 27. The intersection point of the two lines is the **origin**, the horizontal line is the x-axis, and the vertical line is the y-axis. The location of any point P can be described by an ordered pair of numbers (a, b), where a perpendicular from P to the x-axis intersects at a point with coordinate a and a perpendicular from P to the y-axis intersects at a point with coordinate b; the point is identified as $P(a, b)$. A line is **perpendicular** to another line if they form a 90° angle (right angle). There is a one-to-one correspondence between all the points in the plane and all the ordered pairs of real numbers. For example, in Figure 27, Q has coordinates $(^{-}4, ^{-}3)$, written $Q(^{-}4, ^{-}3)$.

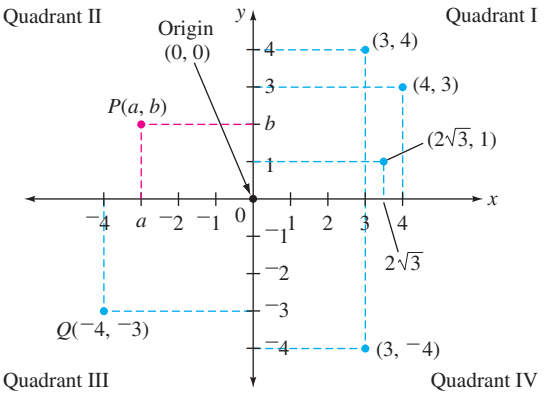


Figure 27

In Figure 27, the axes separate the plane into four **quadrants** numbered counterclockwise.

Equations of Vertical and Horizontal Lines

Every point on the x-axis has a y-coordinate of zero. Thus, the x-axis can be described as the set of all points (x, y) such that $y = 0$. The *x-axis has the equation $y = 0$*. Similarly, *the y-axis can be described as the set of all points (x, y) such that $x = 0$ and y is an arbitrary real number*. Thus, *$x = 0$ is the equation of the y-axis*. Using set notation, the x-axis is the set of points $\{(x, 0) \mid x \text{ is a real number}\}$ and the y-axis is the set of points $\{(0, y) \mid y \text{ is a real number}\}$.

Example 20

Sketch the graph for each of the following on a Cartesian coordinate system.

a. $x = 2$
b. $y = 3$
c. $\{(x, y) \mid x < 2 \text{ and } y = 3\}$

Solution

a. The equation $x = 2$ represents the set of all points (x, y) for which $x = 2$ and y is any real number, as in Figure 28(a).

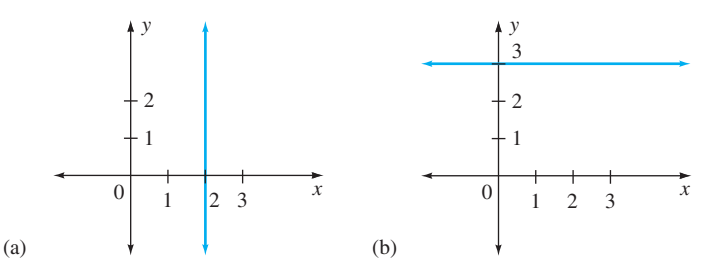


Figure 28

b. The equation $y = 3$ represents the set of all points (x, y) for which $y = 3$ and x is any real number, as in Figure 28(b).

c. The statements in part (c) represent the set of all points (x, y) for which $x < 2$, but $y = 3$. The set is part of a line, as shown in Figure 29. Note that the hollow dot at $(2, 3)$ indicates that this point is not included in the solution set.

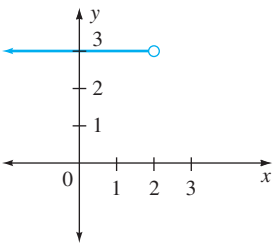


Figure 29

As seen in Example 20(a) and (b) *the graph of the equation* $x = a$, *where* a *is some real number, is the line perpendicular to the x-axis through the point with coordinates* $(a, 0)$. Similarly, *the graph of the equation* $y = b$ *is the line perpendicular to the y-axis through the point with coordinates* $(0, b)$.

Equations of Lines

The arithmetic sequence $4, 7, 10, 13, \ldots$ in Table 6 has xth term $3x + 1$. If the number of the term is the x-coordinate and the corresponding term the y-coordinate, the set of points appear to lie on a line that is parallel to neither the x-axis nor the y-axis, as in Figure 30.

Table 6

Number of Term	Term
1	4
	3
2	7
	3
3	10
	3
4	13
⋮	⋮
x	$3x + 1$

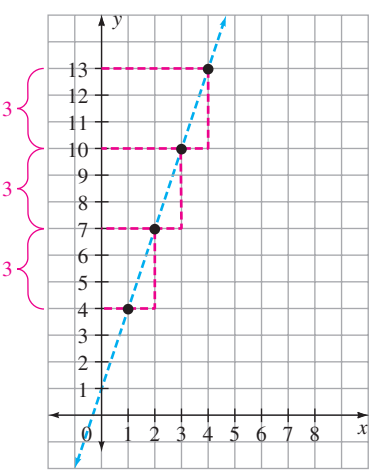


Figure 30

In Table 6, we see that there is a difference of 3 in the y-coordinates of the marked points. Correspondingly, there is a difference of 1 in the number of the term on the x-axis. Hence, we might say that the rate of change of the term with respect to the number of the term is 3, or the ratio is 3 to 1. Does something comparable happen with other arithmetic sequences?

To help answer this question, we consider the following sequences in Table 7.

Table 7

Number of Term x	$1x = x$	$2x$	$\frac{1}{2}x$	$(^-1)x = {}^-x$	$(^-2)x = {}^-2x$
1	1	2	$\frac{1}{2}$	$^-1$	$^-2$
2	2	4	1	$^-2$	$^-4$
3	3	6	$\frac{3}{2}$	$^-3$	$^-6$
4	4	8	2	$^-4$	$^-8$
5	5	10	$\frac{5}{2}$	$^-5$	$^-10$
6	6	12	3	$^-6$	$^-12$
⋮	⋮	⋮	⋮	⋮	⋮
x	x	$2x$	$\frac{1}{2}x$	^-x	^-2x

Figure 31 shows the sets of ordered pairs (x, y) in Table 7 plotted on a graph so that the number of the term is the x-coordinate and the corresponding term is the y-coordinate. Again the separate sets of points appear to lie on straight lines.

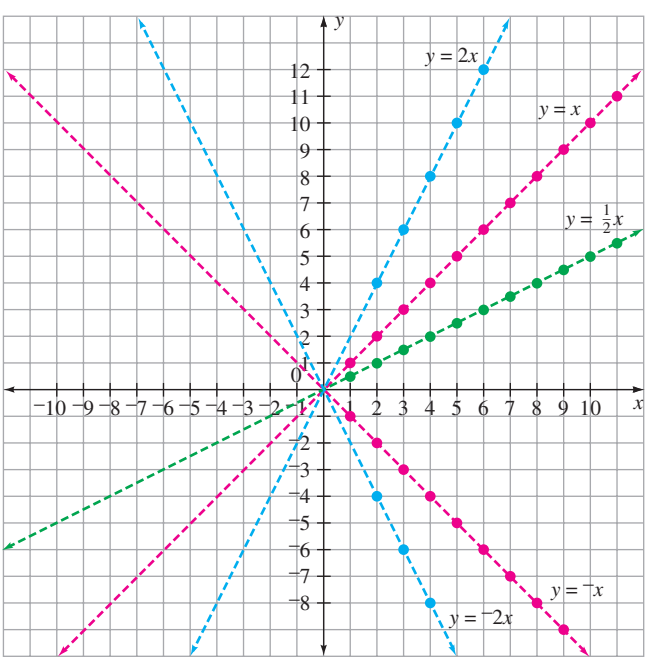


Figure 31

All dashed lines in Figure 31 have equations of the form $y = mx$, where m takes the values 1, 2, $\frac{1}{2}$, $^-1$, and $^-2$. If we consider the equations but allow x to be any real number, then all the points (x, y) that satisfy the equation lie along a given dashed line and all the points on that line satisfy the equation. The number m is a measure of steepness and is the **slope** of the line whose equation is $y = mx$. The graph goes up from left to right (y increases as x increases) if m is positive, and it goes down from left to right (y decreases as x increases) if m is negative.

NOW TRY THIS 10

a. In the equation $y = mx$, if m is 0, what happens to the line?
b. What happens to the line as m increases from 0?
c. What happens to the line when m decreases from 0?

All dashed lines in Figure 31 pass through the origin. This is true for any line whose equation is $y = mx$. If $x = 0$, then $y = m \cdot 0 = 0$, and $(0, 0)$ is a point on the graph of $y = mx$. Conversely, it is possible to show that any nonvertical line passing through the origin has an equation of the form $y = mx$ for some value of m.

Example 21

Find the equation of the line that contains $(0, 0)$ and $(2, 3)$.

Solution The line goes through the origin so its equation has the form $y = mx$. To find the equation of the line, we find the value of m. The line contains $(2, 3)$, so we substitute 2 for x and 3 for y in the equation $y = mx$ to obtain $3 = m \cdot 2$, and thus $m = \frac{3}{2}$. The required equation is $y = \frac{3}{2}x$.

Next, we consider equations of the form $y = mx + b$, where b is a real number. We do this by first examining the graphs of $y = x$, $y = x + 2$ and $y = x - 2$. Given the graph of $y = x$, we obtain the graph of $y = x + 2$ by "raising" each point on the graph of $y = x$ by 2 units. This is shown in Figure 32(a). Similarly, to sketch the graph of $y = x - 2$, we first draw the graph of $y = x$ and then lower each point vertically by 2 units, as shown in Figure 32(a).

The graphs of $y = x$, $y = x + 2$, and $y = x - 2$ are parallel lines. In general, *for a given value of m, the graph of* $y = mx + b$ *is a straight line through* $(0, b)$ *and parallel to the line whose equation is* $y = mx$.

Thus, the graph of the line $y = mx + b$ can be obtained from the graph of $y = mx$ by sliding $y = mx$ up b units if $b > 0$ and down $|b|$ units if $b < 0$, as shown in Figure 32(b). In general, *any two parallel lines have the same slope it will be shown later in this section that* vertical lines have no slope.

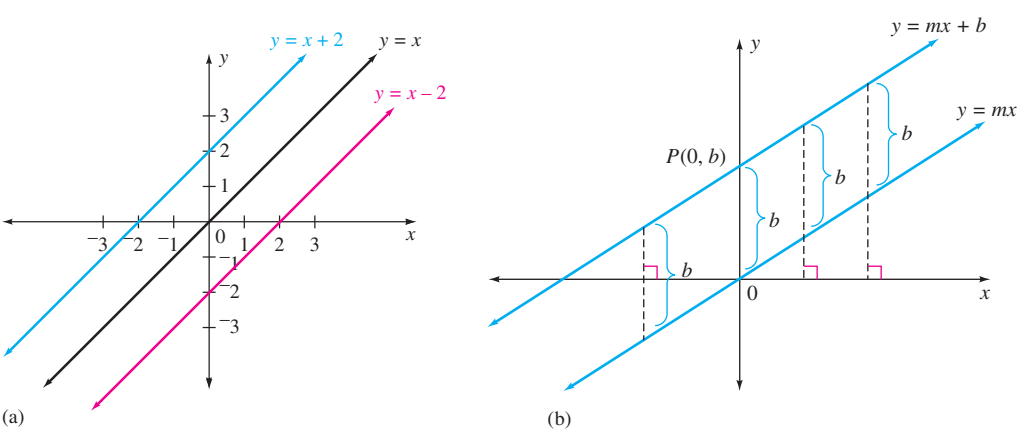


Figure 32

The graph of $y = mx + b$ in Figure 32(b) crosses the y-axis at point $P(0, b)$. The value of y at the point of intersection of any line with the y-axis is the ***y*-intercept**. Thus, b is the y-intercept of $y = mx + b$; this form of an equation of a straight line is the **slope-intercept form**. Similarly, the value of x at the point of intersection of a line with the x-axis is the ***x*-intercept**.

Example 22

Given the equation $y - 3x = {}^{-}6$, answer the following.

a. Find the slope of the line.
b. Find the y-intercept.
c. Find the x-intercept.
d. Sketch the graph of the equation.

Solution

a. We write the equation in the form $y = mx + b$ by adding $3x$ to both sides of the given equation: $y = 3x + ({}^{-}6)$. Hence, the slope is 3.
b. The form $y = 3x + ({}^{-}6)$ shows that $b = {}^{-}6$, which is the y-intercept. The y-intercept can also be found directly by substituting $x = 0$ in the equation and finding the corresponding value of y.
c. The x-intercept is the x-coordinate of the point where the graph intersects the x-axis. At that point, $y = 0$. Substituting 0 for y in $y = 3x - 6$ gives 2 as the x-intercept.
d. Knowing the y-intercept and the x-intercept gives us two points, $(0, {}^{-}6)$ and $(2, 0)$, on the line. We may plot these points and draw the line through them to obtain the desired graph in Figure 33.

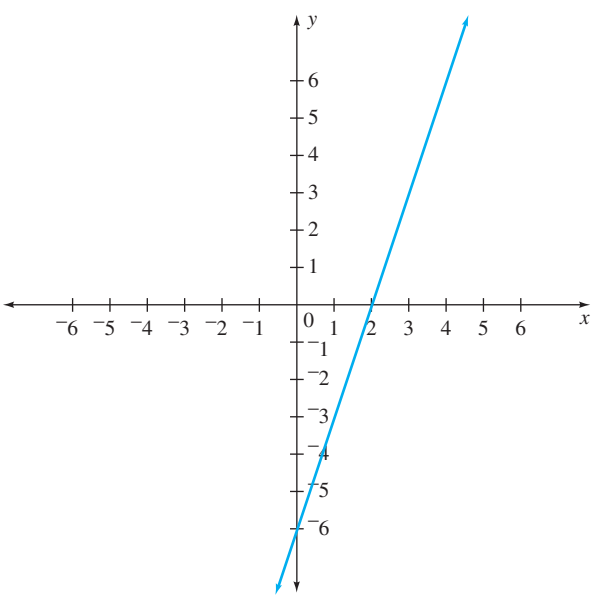


Figure 33

Any vertical line has equation $x = a$ for some real number a, and cannot be written in slope-intercept form. A horizontal line has slope of 0, and can be written in a slope intercept form as $y = 0 \cdot x + b = b$. In general, *every line has an equation of the form either* $y = mx + b$ or $x = a$. Any equation that can be put in one of these forms is a **linear equation**. Equations of the form $ax + by + c = 0$, where a, b, c are real numbers, are linear equations and may be written in slope intercept form if $b \neq 0$. (Why?) If $b = 0$ and $a \neq 0$, we get the equation $x = \frac{{}^{-}c}{a}$, which is an equation of a vertical line.

Theorem 8-7: Equation of a Line

Every line has an equation of the form either $y = mx + b$ or $x = a$, where m is the slope and b is the ***y*-intercept**. In the equation $x = a$, a is the ***x*-intercept**.

NOW TRY THIS 11

The nth term of an arithmetic sequence is given as $a_n = a_1 + (n - 1)d$. Explain why this is considered a *linear relationship*.

Determining Slope

We defined the slope of a line with equation $y = mx + b$ to be m. The slope is a measure of steepness of a line. A different way to discuss the steepness of a line is to consider the rate of change in y-values in relation to their corresponding x-values. In Figure 34, line k has a greater rate of change than line ℓ. In other words, line k rises faster than line ℓ for the same horizontal run.

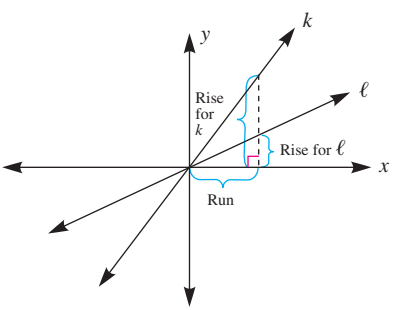


Figure 34

Thus, we could express the rate of change, or steepness, as the ratio $\frac{\text{change in } y\text{-values}}{\text{corresponding change in } x\text{-values}}$. The rate is frequently expressed as $\frac{rise}{run}$.

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$ the difference $x_2 - x_1$ is the *run*, and the difference $y_2 - y_1$ is the *rise*. The values of *rise* and *run* depend on the points chosen on the line and hence vary. However, the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ is always the same, regardless of which two points on a given nonvertical line are chosen, and equals the slope of the line. This can be shown as follows:

$$\begin{aligned}\frac{y_2 - y_1}{x_2 - x_1} &= \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} \\ &= \frac{mx_2 - mx_1 + b - b}{x_2 - x_1} \\ &= \frac{m(x_2 - x_1)}{x_2 - x_1} \\ &= m\end{aligned}$$

Thus, as seen in Figure 35, we can define the slope of a nonvertical line as $\frac{rise}{run}$.

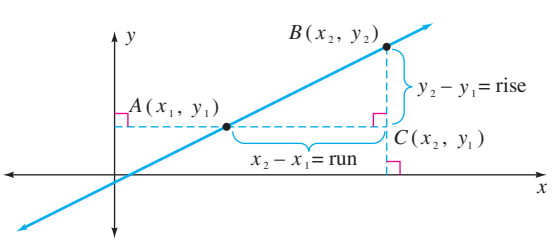


Figure 35

The discussion of slope is summarized in the following definition for the slope of a line.

Definition of Slope (Slope Formula)

Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$ with $x_1 \neq x_2$, the slope m of the line through A and B is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

In the above definition, if we multiply both the numerator and the denominator on the right side of the slope formula by $^-1$, we obtain

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(y_2 - y_1)(^-1)}{(x_2 - x_1)(^-1)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Thus, it does not matter which point is named (x_1, y_1) and which is named (x_2, y_2); *the order of the coordinates in the subtraction must be the same.*

NOW TRY THIS 12

a. Use the slope formula to find the slope of any horizontal line.
b. What happens when we attempt to use the slope formula for a vertical line? What is your conclusion about the slope of a vertical line?

When a line is inclined downward from the left to the right, the slope is negative. This is illustrated in Figure 36, where the graph of the line $y = ^-2x$ is shown.

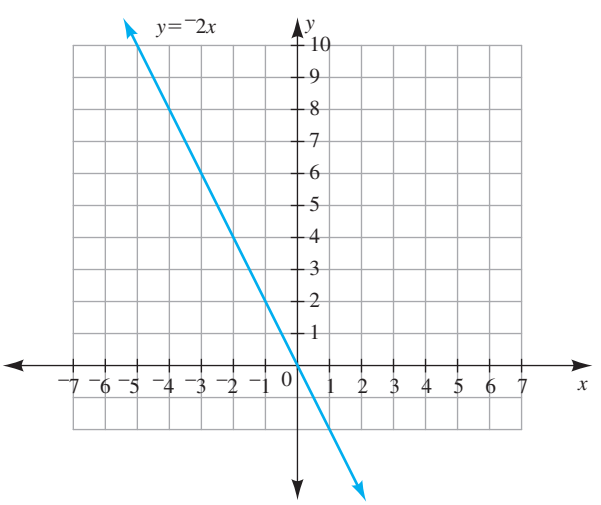


Figure 36

Example 23

a. Given $A(3, 1)$ and $B(5, 4)$, find the slope of $\overleftrightarrow{AB}$ (the line through A and B).
b. Find the slope of the line passing through the points $A(^-3, 4)$ and $B(^-1, 0)$.
c. In Figure 37, find the slope of $\overleftrightarrow{OA}$ (the line through O and A).

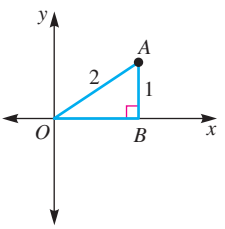


Figure 37

Solution

a. $m = \dfrac{4-1}{5-3} = \dfrac{3}{2}$

b. $m = \dfrac{4-0}{^-3-(^-1)} = \dfrac{4}{^-2} = {}^-2$

c. OB (the distance from O to B) is the run while AB, or 1, is the rise.
$(OB)^2 + 1^2 = 2^2$ in right triangle OAB, using the Pythagorean theorem.
$(OB)^2 + 1 = 4$
$(OB)^2 = 4 - 1 = 3$
$OB = \sqrt{3}$

The slope of $\overleftrightarrow{OA}$ (the line through O and A) is $\dfrac{1}{\sqrt{3}}$, which is sometimes written as $\dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$.

Given two points on a nonvertical line, we use the slope formula to find the slope of the line and its equation as demonstrated in the following example.

Example 24

In Figure 38, the points $(^-4, 0)$ and $(1, 4)$ are on the line ℓ. Find:

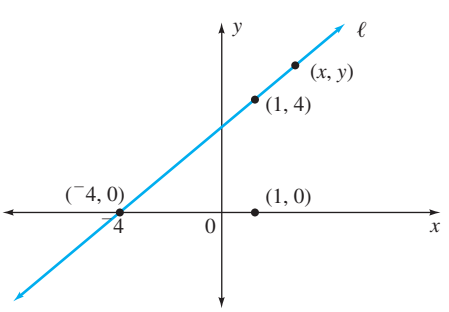


Figure 38

a. The slope of the line
b. The equation of the line

Solution

a. $m = \dfrac{4-0}{1-(^-4)} = \dfrac{4}{5}$

b. Any point (x, y) on ℓ different from $(^{-}4, 0)$ can be found using the slope $\frac{4}{5}$ determined by points $(^{-}4, 0)$ and (x, y). Thus,

$$\frac{y - 0}{x - (^{-}4)} = \frac{4}{5}$$

$$\frac{y}{x + 4} = \frac{4}{5}$$

$$y = \frac{4}{5}(x + 4).$$

The equation of the line is $y = \frac{4}{5}x + \frac{16}{5}$.

Systems of Linear Equations



The mathematical descriptions of many problems involve more than one equation, each having more than one unknown. To solve such problems, we must find a common solution to the equations, if it exists. An example is given below and on the student page shown on page 452.

Example 25

May Chin paid \$18 for three soyburgers and twelve orders of fries. Another time she had paid \$12 for four soyburgers and four orders of fries. Assume the prices have not changed. Set up a system of equations with two unknowns representing the prices of a soyburger and an order of fries, respectively.

Solution Let x be the price in dollars of a soyburger and y be the price in dollars of an order of fries. Three soyburgers cost $3x$ dollars, and twelve orders of fries cost $12y$ dollars. Because May paid \$18 for her entire order, we have $3x + 12y = 18$, or, after dividing each side by 3, $x + 4y = 6$. Similarly, $4x + 4y = 12$ is equivalent to $x + y = 3$. The system of equations is $x + 4y = 6$ and $x + y = 3$.

An ordered pair satisfying the two linear equations in Example 25 is a point that belongs to each of the lines. Figure 39 shows the graphs of $x + 4y = 6$ and $x + y = 3$. The price x of the soyburger and the price y of an order of fries satisfy both equations; that is, (x, y) is a point on both lines. The two lines appear to intersect at $(2, 1)$. Thus, $(2, 1)$ seems to be the solution of the given system of equations. This solution can be checked by substituting 2 for x and 1 for y in each equation. Because two distinct lines intersect in only one point, $(2, 1)$ is the only solution to the system. Therefore, in Example 25 a soyburger costs \$2 and fries cost \$1.

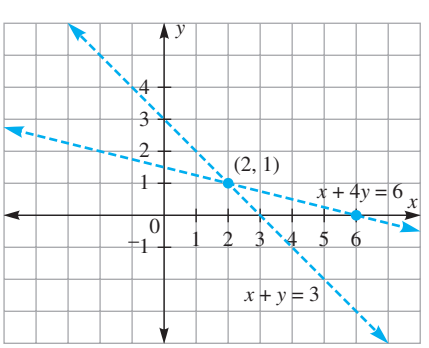


Figure 39

Because costs of the soyburger and an order of fries cannot be negative, parts of the lines are dashed.

School Book Page

Linear Functions

GPS Guided Problem Solving

Linear Functions

For each rental plan, represent the relationship between the number of miles (from 20 to 45) and the cost. Use a linear function, a table of ordered pairs, and a graph (using the same coordinate grid). What conclusions can you draw about these plans?



What You Might Think

What You Might Write

What do I know? What do I want to find out?

Plan 1 costs $15 plus $.25 per mile. Plan 2 costs $8 plus $.45 per mile. I want to compare the two plans using function rules, tables, and graphs.

How can I write a function rule for each plan?

Let m = the number of miles driven. Let C = the cost of the rental in dollars.

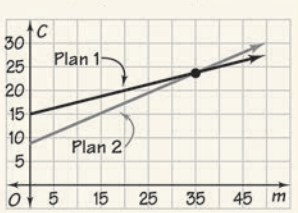
Plan 1: $C_1 = \$.25m + 15$

Plan 2: $C_2 = \$.45m + 8$

How can I make a table and graph?

I can use the function rule to get data points for the table. Then I can graph those points.

m	C_1	C_2
20	20	17
30	22.5	21.5
35	23.75	23.75
45	26.25	28.25



What conclusions can be stated?

The lines intersect at (35, 23.75). Plan 2 is better if you drive less than 35 miles.

Plan 1 is better if you drive more than 35 miles. The costs of Plans 1 and 2 are equal if you drive exactly 35 miles.

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Substitution Method

The graphing method for solving systems of equations is not practical when the point of intersection of the lines cannot be accurately identified. There are algebraic methods for solving systems of equations. One such method is the **substitution method**. The equations

$$x + 4y = 6 \quad \text{and} \quad x + y = 3$$

can be solved by finding x in terms of y (or vice versa) from one of the equations and substituting in the other. From the second equation, we have $x = 3 - y$. Substituting for x in the first equation we get the equivalent equations: $3 - y + 4y = 6, 3 + 3y = 6, 3y = 3, y = 1$. Substituting $y = 1$ in $x + y = 3$, we get $x + 1 = 3$ and hence $x = 2$. Thus, $x = 2$ and $y = 1$ is the solution to the system. Example 26 demonstrates the substitution method in another system of equations.

Example 26

Solve the following system using the substitution method.

$$3x - 4y = 5$$
$$2x + 5y = 1$$

Solution First, rewrite one equation, expressing y in terms of x.

$$y = \frac{3x - 5}{4}$$

Because we are looking for x and y that satisfy each equation, we substitute $\frac{3x - 5}{4}$ for y in the other original equation and solve the resulting equation for x.

$$2x + 5\left(\frac{3x - 5}{4}\right) = 1$$
$$4\left[2x + 5\left(\frac{3x - 5}{4}\right)\right] = 4 \cdot 1$$
$$8x + 5(3x - 5) = 4$$
$$8x + 15x - 25 = 4$$
$$23x = 4 + 25$$
$$x = \frac{29}{23}$$

Substituting $\frac{29}{23}$ for x in $y = \frac{3x - 5}{4}$ gives $y = \frac{-7}{23}$. Hence, $x = \frac{29}{23}$ and $y = \frac{-7}{23}$. This can be checked by substituting the values for x and y in the original equations.

Elimination Method

The **elimination method** for solving two equations with two unknowns is based on eliminating one of the variables by adding or subtracting the original or equivalent equations. For example, consider the following system:

$$x - y = {}^{-}3$$
$$x + y = 7$$

By adding the two equations, we can eliminate the variable y. The resulting equation can then be solved for x.

$$\begin{aligned} x - y &= {}^{-}3 \\ x + y &= 7 \\ \hline 2x &= 4 \\ x &= 2 \end{aligned}$$

Substituting 2 for x in the first equation (either equation may be used) gives $y = 5$. Checking this result shows that $x = 2$ and $y = 5$, or $(2, 5)$, is the solution to the system.

Often, another operation is required before equations are added so that an unknown can be eliminated. For example, consider the following system:

$$\begin{aligned} 3x + 2y &= 5 \\ 5x - 4y &= 3 \end{aligned}$$

Neither adding nor subtracting the equations eliminates an unknown. However, if the first equation contained $4y$ rather than $2y$, the variable y could be eliminated by adding. To obtain $4y$ in the first equation, we multiply both sides of the equation by 2 to obtain the equivalent equation $6x + 4y = 10$. Adding the equations in the equivalent system gives the following:

$$\begin{aligned} 6x + 4y &= 10 \\ 5x - 4y &= 3 \\ \hline 11x &= 13 \\ x &= \frac{13}{11} \end{aligned}$$

To find the corresponding value of y, we substitute $\frac{13}{11}$ for x in either of the original equations and solve for y. We get $y = \frac{8}{11}$.

Consequently, $x = \frac{13}{11}$ and $y = \frac{8}{11}$, or equivalently $\left(\frac{13}{11}, \frac{8}{11}\right)$ is the solution of the original system. This solution, as always, should be checked by substitution in the original equations.

Solutions to Various Systems of Linear Equations

All examples thus far have had unique solutions. However, other situations may arise. Geometrically, a system of two linear equations can be characterized as follows:

1. *The system has a unique solution if, and only if, the graphs of the equations intersect in a single point.*
2. *The system has no solution if, and only if, the equations represent distinct parallel lines.* Two distinct lines in a plane are parallel if they do not intersect.
3. *The system has infinitely many solutions if, and only if, the equations represent the same line.* A line is considered parallel to itself.

Consider the following system and assume it has a solution.

$$\begin{aligned} 2x - 3y &= 1 \\ {}^{-}4x + 6y &= 5 \end{aligned}$$

In an attempt to solve for x, we multiply the first equation by 2 and then add as follows:

$$\begin{aligned} 4x - 6y &= 2 \\ {}^{-}4x + 6y &= 5 \\ \hline 0 &= 7 \end{aligned}$$

A false statement results from the assumption that the system had a solution. That assumption caused a false statement; therefore, the assumption itself must be false. Hence, the system has no solution. In other words, the solution set is $\varnothing$. This situation arises if, and only if, the corresponding lines are parallel and different. The lines are parallel because their equations can be written as $y = \frac{2}{3}x - \frac{1}{3}$ and $y = \frac{2}{3}x + \frac{5}{6}$, which shows that the slope of each line is $\frac{2}{3}$.

Next, consider the following system:

$$\begin{aligned} 2x - 3y &= 1 \\ {}^{-}4x + 6y &= {}^{-}2 \end{aligned}$$

To solve this system, we multiply the first equation by 2 and add as follows:

$$\begin{aligned} 4x - 6y &= 2 \\ {}^{-}4x + 6y &= {}^{-}2 \\ \hline 0 &= 0 \end{aligned}$$

The resulting statement, $0 = 0$, is always true. In the original system, if we multiply by $^{-}2$ both sides of the first equation, the result would be the second equation. Thus, the graphs are exactly the same or the graph is a single line. All pairs of x and y that correspond to points on the line $2x - 3y = 1$ also satisfy the equivalent equation $^{-}4x + 6y = {}^{-}2$.

Thus, one way to determine that a system of two equations in two unknowns has infinitely many solutions is by checking whether each of the original equations represents the same line. It can be shown that a system has infinitely many solutions if, and only if, we get $0 = 0$ by trying to eliminate one of the variables and adding or subtracting the equations side by side. The statement $0 = 0$ originated from $0 \cdot x + 0 \cdot y = 0$, which is satisfied by all x and y.

Example 27

Identify each of the following systems as having a unique solution, no solution, or infinitely many solutions:

a. $\begin{aligned} 2x - 3y &= 5 \\ \tfrac{1}{2}x - y &= 1 \end{aligned}$

b. $\begin{aligned} \frac{x}{3} - \frac{y}{4} &= 1 \\ 3y - 4x + 12 &= 0 \end{aligned}$

c. $\begin{aligned} 6x - 9y &= 5 \\ {}^{-}8x + 12y &= 7 \end{aligned}$

Different methods are illustrated in the solutions.

Solution One method is to attempt to solve each system. Another method is to write each equation in the slope-intercept form and interpret the system geometrically.

a. *First method.* To eliminate x, multiply the second equation by $^{-}4$ and add the equations.

$$\begin{aligned} 2x - 3y &= 5 \\ {}^{-}2x + 4y &= {}^{-}4 \\ \hline y &= 1 \end{aligned}$$

Substituting 1 for y in either equation gives $x = 4$. Thus $x = 4$ and $y = 1$, or $(4, 1)$ is the unique solution of the system.

Second method. In slope-intercept form, the first equation is $y = \frac{2}{3}x - \frac{5}{3}$. The second equation is $y = \frac{1}{2}x - 1$. The slopes of the corresponding lines are $\frac{2}{3}$ and $\frac{1}{2}$, respectively. Consequently, the lines are distinct and are not parallel and, therefore, intersect in a single point whose coordinates are the unique solution to the original system.

b. *First method.* Multiply the first equation by 12 and rewrite the second equation as $^-4x + 3y = {}^-12$. Then, adding the resulting equations gives the following:

$$\begin{aligned} 4x - 3y &= 12 \\ {}^-4x + 3y &= {}^-12 \\ \hline 0 &= 0 \end{aligned}$$

Hence, the two equations represent the same line, and the original system has infinitely many solutions (all the points on the line).

Second method. In slope-intercept form, both equations are

$$y = \frac{4}{3}x - 4.$$

Thus, the two lines are identical, so the system has infinitely many solutions.

c. *First method.* To eliminate y, multiply the first equation by 4 and the second by 3; then, add the resulting equations.

$$\begin{aligned} 24x - 36y &= 20 \\ {}^-24x + 36y &= 21 \\ \hline 0 &= 41 \end{aligned}$$

This is a contradiction (no pair of numbers x and y satisfy $0 \cdot x + 0 \cdot y = 41$), so this equation has no solutions, and, consequently, the original system has no solutions.

Second method. In slope-intercept form, the first equation is $y = \frac{2}{3}x - \frac{5}{9}$. The second equation is $y = \frac{2}{3}x + \frac{7}{12}$. The corresponding lines have the same slope, $\frac{2}{3}$, but different y-intercepts.

Consequently, the lines are distinct and parallel, and the original system has no solution.

NOW TRY THIS 13

Find all solutions (if any) of each of the following systems.

a. $\begin{aligned} x - y &= 1 \\ 2x - y &= 5 \end{aligned}$ **b.** $\begin{aligned} 2x - y &= 1 \\ 2y - 4x &= 3 \end{aligned}$ **c.** $\begin{aligned} 2x - y &= 1 \\ 2y - 4x &= {}^-2 \end{aligned}$

Substitution and elimination methods may be used to solve systems with 3 or more variables. Consider the problem in Example 28. We use the strategy of writing an equation.

Example 28

At a local farmer's market, three purchases were made for the following prices:

- cantaloupe and a vase for $8
- watermelon and an identical vase for $9
- same-priced watermelon and a same-priced cantaloupe for $7

What is the cost of each object?

Solution If the cost of the cantaloupe is c dollars, the cost of the vase is v dollars, and the cost of the watermelon is w dollars, we have the following system of 3 equations with 3 unknowns.

$$c + v = 8$$
$$w + v = 9$$
$$w + c = 7$$

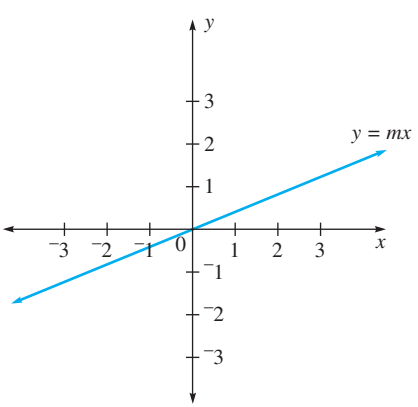
We can eliminate the unknown w by subtracting the third equation from the second: $(v + w) - (c + w) = 9 - 7$ or equivalently $v - c = 2$.

Now we have a system of two equations with two unknowns: $c + v = 8$ and $v - c = 2$. Adding the equations we get $2v = 10$ or $v = 5$. Substituting $v = 5$ into the first equation, we get $c + 5 = 8$ or $c = 3$. Substituting $v = 5$ into the second equation, we get $w + 5 = 9$ or $w = 4$. Thus, the vase costs \$5, a cantaloupe \$3, and a watermelon \$4.

Assessment 8-5A

1. The graph of $y = mx$ is given in the following figure. Sketch the graphs for each of the following on the same figure. Explain your answers.

a. $y = mx + 3$
b. $y = mx - 3$
c. $y = 3mx$
d. $y = {}^{-}3mx$

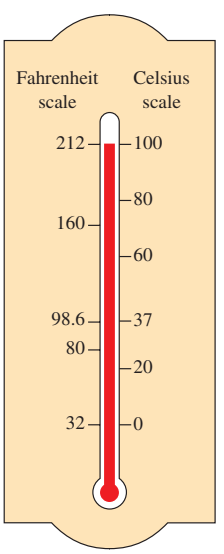


2. Sketch the graphs for each of the following equations:

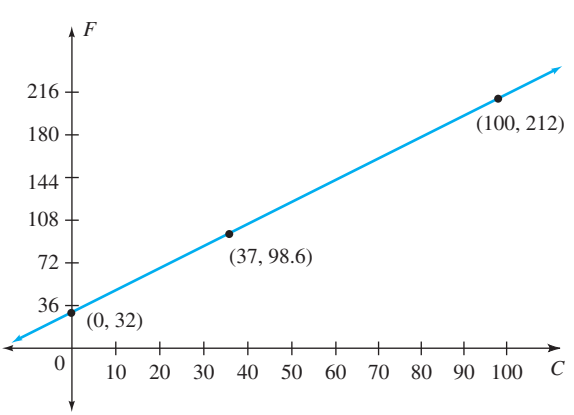
a. $y = \frac{{}^{-}3}{4}x + 3$
b. $y = {}^{-}3$
c. $y = 15x - 30$

3. Find the x-intercept and y-intercept for the equations in exercise 2, if they exist.

4. In the following figure, part (i) shows a dual-scale thermometer and part (ii) shows the corresponding points plotted on a graph.



(i)



(ii)

a. Use the points $(0, 32)$ and $(100, 212)$ to develop a formula for conversion from degrees Celsius (C) to degrees Fahrenheit (F).
b. Use your answer in (a) to find a formula for converting from degrees F to degrees C.
c. Is there a point where degree Celsius and degree Fahrenheit are the same? If so, find it.

5. Write each of the following equations in slope-intercept form and identify the slope and y-intercept.
a. $3y - x = 0$ **b.** $x + y = 3$ **c.** $x = 3y$

6. For each of the following, write the equation of the line determined by the given pair of points in slope-intercept form or in the form $x = a$ or $y = b$.
a. $(^-4, 3)$ and $(1, ^-2)$
b. $(0, 0)$ and $(2, 1)$
c. $\left(0, \frac{^-1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$
d. $(0, ^-3)$ and $(^-1, ^-3)$

7. Find the coordinates of two other **collinear** points (on the same line) with each of the following pairs of given points.
a. $P(2, 2)$, $Q(4, 2)$ **b.** $P(0, 0)$, $Q(0, 1)$

8. For each of the following, give as much information as possible about x and y.
a. The ordered pairs $(^-2, 0)$, $(^-2, 1)$, and (x, y) represent collinear points.
b. The ordered pair (x, y) is in the fourth quadrant.

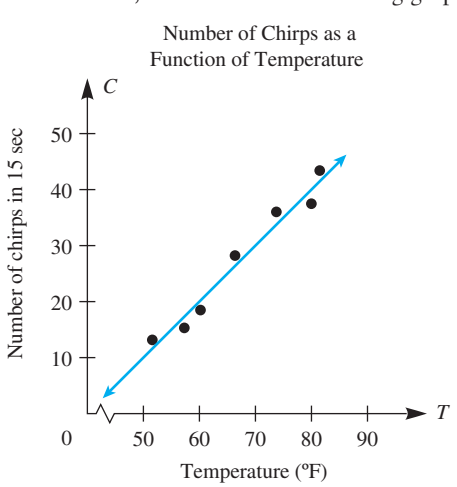
9. Consider lines through $P(2, 4)$ that form 90° angles with the x- and y-axes, respectively. Find the area and the perimeter of the rectangle formed by these lines and the axes.

10. Find the equations for each of the following lines.
a. The line containing $P(3, 0)$ and perpendicular to the x-axis
b. The line containing $P(^-4, 5)$ and parallel to the x-axis

11. For each of the following, find the slope, if it exists, of the line determined by the given pair of points.
a. $(4, 3)$ and $(^-5, 0)$ **b.** $(\sqrt{5}, 2)$ and $(1, 2)$
c. (a, a) and (b, b) **d.** $(\sqrt{5}, 2)$ and $(\sqrt{5}, ^-2)$

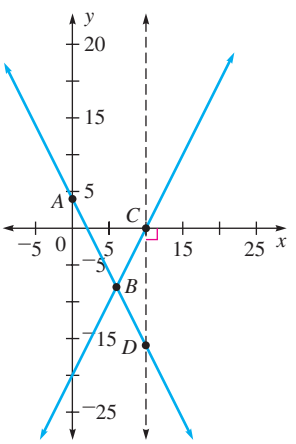
12. Write the equation of each line in exercise 11.

13. Wildlife experts found that the number of chirps a cricket makes in a 15-sec interval is related to the temperature T in degrees Fahrenheit, as shown in the following graph:



a. If C is the number of chirps in 15 sec, write a formula for C in terms of T (temperature in degrees Fahrenheit) that seems to fit the data best.
b. Use the equation in (a) to predict the number of chirps in 15 sec when the temperature is 90°.
c. If N is the number of chirps per minute, write a formula for N in terms of T.

14. The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ shown in the figure are graphs of $y = 2x - 20$ and $y = 4 - 2x$. Through the point C, a vertical line (parallel to the y-axis) was drawn intersecting $\overleftrightarrow{AB}$ at D.
a. Which graph corresponds to which line? Justify your answer.
b. Find the coordinates of point D.



15. What do the graphs of the following equations have in common?
a. $y = m(x + 3)$ for different real number values of m?
b. $y = m(1 - x)$ for different real number values of m?
c. $\frac{x}{a} + \frac{y}{2a} = 1$ for different non-zero values of a?
d. $x + y = a$ for different values of a?

16. Suppose a car is traveling at a constant speed of 60 mph.
a. Draw a graph showing the relation between the distance traveled and the amount of time that it takes to travel the distance.
b. What is the slope of the line that you drew in part (a)?

17. Solve each of the following systems, if possible. Indicate whether the system has a unique solution, infinitely many solutions, or no solution.
a. $y = 3x - 1$
$y = x + 3$
b. $3x + 4y = ^-17$
$2x + 3y = ^-13$
c. $2x - 3y = ^-1$
$\frac{^-2}{3}x + y = \frac{1}{3}$
d. $y = 1 - x$
$y = x - 1$

18. The owner of a 5000-gal oil truck loads the truck with gasoline and kerosene. The profit on each gallon of gasoline is 13¢ and on each gallon of kerosene it is 12¢. How many gallons of each fuel did the owner load if the profit was $640.00?

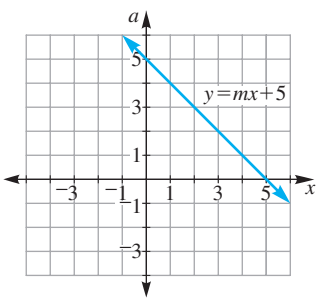
19. Josephine's bank contains 27 coins. If all the coins are either dimes or quarters and the value of the coins is $5.25, how many of each kind of coin are there?

Assessment 8-5B

1. The graph of $y = mx + 5$ is given in the following figure. Sketch the graphs for each of the following equations on the same figure. Explain your answers.

a. $y = mx$ **b.** $y = mx - 5$

c. $y = {}^{-}mx$ **d.** $y = 2mx$



2. Sketch the graphs for each of the following equations:

a. $x = {}^{-}2$

b. $y = 3x - 1$

c. $y = \frac{1}{20}x$

3. Find the x-intercept and y-intercept for the equations in exercise 2, if they exist.

4. On a ruler that measures both metric and English units, we see that 1 m = 39.37 in. Use this information to write an equation to convert x m to y in.

5. Write each of the following equations in slope-intercept form and identify the slope and y-intercept.

a. $\frac{x}{3} + \frac{y}{4} = 1$

b. $3x - 4y + 7 = 0$

c. $x - y = 4(x - y)$

6. For each of the following, write the equation of the line determined by the given pair of points in slope-intercept form or in the form $x = a$ or $y = b$.

a. $(0, 1)$ and $(2, 1)$

b. $(2, 1)$ and $(2, {}^{-}1)$

c. $({}^{-}a, 0)$ and $(a, 0), a \neq 0$

d. $(0, {}^{-}\sqrt{3})$ and $({}^{-}1, {}^{-}\sqrt{3})$

7. Find the coordinates of two other collinear points (on the same line) with each of the following pairs of given points:

a. $P({}^{-}1, 0), Q({}^{-}1, 2)$ **b.** $P(0, 0), Q(1, 1)$

8. Give as much information as possible about x and y in the following.

a. The ordered pairs $({}^{-}2, 1)$, $(0, 1)$, and (x, y) represent collinear points.

b. (x, y), $(0, 0)$ and $({}^{-}1, {}^{-}1)$ are collinear

9. A rectangle has two vertices on the x-axis, two vertices on the y-axis, and one vertex at the point with coordinates $(4, 6)$.

a. Make a sketch showing that there is a rectangle with these characteristics.

b. Write the coordinates of each of the other three vertices.

c. Write the equations of the diagonals of the rectangle.

d. Find the coordinates of the point of intersection of the diagonals.

10. Find the equation for each of the following lines.

a. The line containing $P(0, {}^{-}2)$ and parallel to the x-axis

b. The line containing $P({}^{-}4, 5)$ and parallel to the y-axis

11. For each of the following, find the slope, if it exists, of the line determined by the given pair of points:

a. $({}^{-}4, 1)$ and $(5, 2)$

b. $({}^{-}3, 81)$ and $({}^{-}3, 198)$

c. $(1.0001, 12)$ and $(1, 10)$

d. $(2, \sqrt{5})$ and $({}^{-}2, \sqrt{5})$

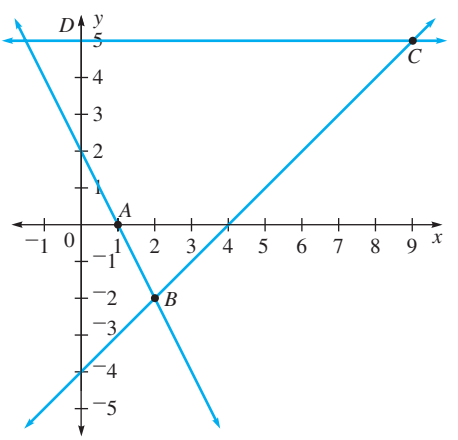
12. Write the equation of each line in exercise 11.

13. At the end of 10 months, the balance of an account earning annual simple interest is \$2100.

a. If, at the end of 18 mo, the balance is \$2180, how much money was originally in the account?

b. What is the annual rate of interest?

14. The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ shown in the figure are graphs of $y = 2 - 2x$ and $y = x - 4$, respectively. The line $\overleftrightarrow{CD}$ goes through $D(0, 5)$ and is parallel to the x-axis. Find the coordinates of A, B, and C.



15. What do the graphs of the following equations under the given conditions in each part of the problem have in common?

a. $y = m(x - 3)$, for different real number values of m?

b. $y = m(3 - x)$, for different real number values of m?

c. $x - y = a$, for different real number values of a?

d. $y = m(x - 1) + 1$, for different real number values of m?

16. Suppose a car is traveling at a constant speed of 45 mph.

a. Draw a graph showing the relation between the distance traveled and the amount of time that it takes to travel the distance.

b. What is the slope of the line you drew in part (a)?

17. Solve each of the following systems, if possible. Indicate whether the system has a unique solution, infinitely many solutions, or no solution.

a. $2x - 6y = 7$
$3x - 9y = 10$

b. $4x - 6y = 1$
$6x - 9y = 1.5$

c. $y = 2 - x$
$y = x - 2$

d. $3x - 2y = {}^{-}1$
$\frac{{}^{-}3}{2}x + y = \frac{1}{2}$

18. The vertices of a triangle are given by $(0, 0)$, $(10, 0)$, and $(6, 8)$. Show that the segments connecting $(5, 0)$ and $(6, 8)$, $(10, 0)$ and $(3, 4)$, and $(0, 0)$ and $(8, 4)$ intersect at a common point.

Mathematical Connections 8-5

Reasoning

1. In this chapter, an arithmetic sequence was associated with a linear graph. Explain whether a geometric sequence should be associated with a linear graph.
2. Explain why two lines with the same slope and different y intercepts, are parallel.
3. Given the equations $y = mx + b$ and $y = nx + a$ answer the following.
 a. Assuming that the lines intersect in a single point, find the coordinates of this point in terms of m, n, and a and b.
 b. Under what condition (in terms of m, n, a and b) do the lines intersect in a single point? Find the answer using part (a) and interpret it using the graphs of the lines.
4. Given the lines in problem 3, once x is found, y can be found by substituting the value of x into either equation. Explain why it does not matter which one.
5. Draw graphs for the following. Justify your solution.
 a. $(x + y)^2 = 4$
 b. $|x + y| = 1$
 c. $|x| = 1$ and y is any real number
 d. $|y| = 1$ and x is any real number
6. Explain why there exist lines with irrational numbers as slopes.

Open-Ended

7. Look for situations in books or the internet that can be modeled by equations whose graphs are straight lines.
8. a. Write equations of two lines that intersect but when graphed look parallel.
 b. At what point do those two lines intersect?

Cooperative Learning

9. Play the following game between your group and another group. Each group makes up four linear equations that have a common property and presents the equations to the other group. For example, one group could present the equations $2x - y = 0$, $4x - 2y = 3$, $y - 2x = 3$, and $3y - 6x = 5$. If the second group discovers a common property that the equations share, such as the graphs of the equations are four parallel lines, they get 1 point. Each group takes a specified number of turns.
10. Consider equations of the form $ax + by = c$, where not both a and b equal 0. Each group should choose three such equations so that a, b, and c are consecutive terms in arithmetic sequence and answer the following.
 a. In your group, choose all pairs of your equations (there are 3 pairs) and solve each pair simultaneously. What do the solutions have in common?
 b. Compare the answers from different groups.
 c. Explain why what you discovered in parts (a) and (b) is always true. Compare the arguments from different groups and choose the best argument. What makes it best?

Connecting Mathematics to the Classroom

11. A student would like to know why it is impossible to find the slope of a vertical line. How do you respond?
12. Jonah tried to solve the equation $^-5x + y = 20$ by adding 5 to both sides. He wrote $5 - 5x + y = 5 + 20$ or $0 \cdot x + y = 25$ and finally $y = 25$. How would you help Jonah?
13. Jill would like to know why two lines with an undefined slope are parallel. How would you respond?
14. Janis noticed that in the simultaneous equations

$$2x - 3y - 5 = 0$$
$$^-4x + 6y + 10 = 0,$$

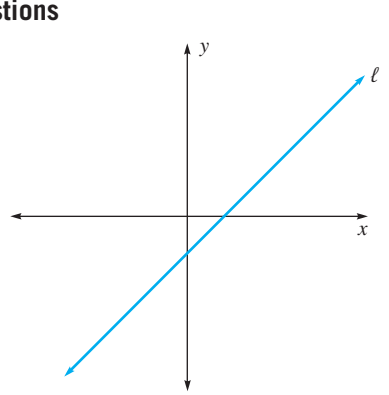
$\frac{2}{^-4} = \frac{^-3}{6} = \frac{^-5}{10}$ and the system has infinitely-many solutions. She would like to know whether, if a similar situation occurs in other systems of two equations in two unknowns, the corresponding system will have infinitely many solutions. How do you respond?

Review Problems

15. Write the following with algebraic expressions or equations.
 a. The cube root of a number is 3 less than its square.
 b. The sum of the squares of two numbers is 36.
 c. $\sqrt{0.\overline{9}}$ has 1 as its principal square root.
16. Give an approximation of $\sqrt{6}$ correct to hundredths.
17. Solve the following for x.
 a. $x\sqrt{2} = ^-3\sqrt{2}x + 2$
 b. $x^2 - 81 = 0$
 c. $3x < ^-\sqrt{7}$
18. Find $f(x)$ for each given value of x when $f(x) = x\sqrt{7} - \sqrt{7}$.
 a. 3
 b. $\sqrt{7}$
 c. $^-4$
19. If $f(x) = 12$ when $f(x) = 3x - \sqrt{2}$, find x.
20. What are the domains for each of the following functions?
 a. $f(x) = \sqrt{x + 1}$
 b. $f(x) = \sqrt{^-x}$

National Assessments

National Assessment of Educational Progress (NAEP) Questions

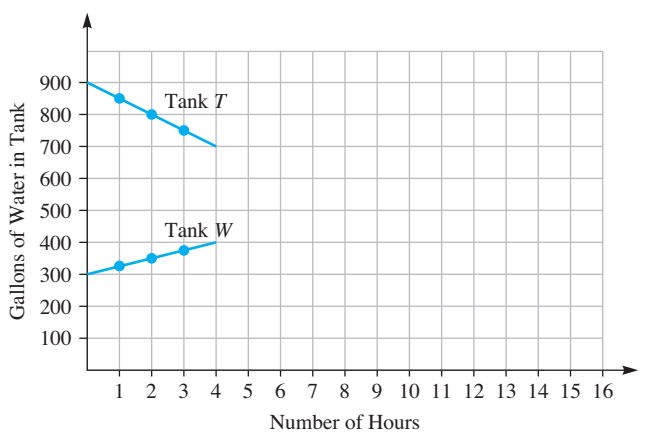


Which of the following statements is true about line ℓ shown above?

A. Line ℓ has a positive slope and a positive y-intercept.
B. Line ℓ has a positive slope and a negative y-intercept.
C. Line ℓ has zero slope and a negative y-intercept.
D. Line ℓ has a negative slope and a positive y-intercept.
E. Line ℓ has a negative slope and a negative y-intercept.

NAEP, Grade 8, 2013

Two large storage tanks, T and W, contain water. T starts losing water at the same time additional water starts flowing into W. The graph below shows the amount of water in each tank over a period of hours.



Assume that the rates of water loss and water gain continue as shown. At what number of hours will the amount of water in T be equal to the amount of water in W?

Show or explain how you found your answer.

NAEP, Grade 8, 2013

Hint for Solving the Preliminary Problem

Use Figure 40 to write a proportion and slove for $\frac{l}{w}$

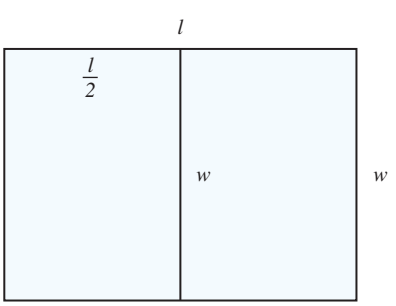


Figure 40

Chapter 8 Summary

KEY CONCEPTS	DEFINITIONS, DESCRIPTIONS, AND THEOREMS
Section 8-1	
Irrational number (p. 388)	Any number that is not rational.
Real number (p. 389)	Any rational or irrational number. Every real number can be represented as a decimal.
Square root (p. 389)	If a is any nonnegative number, a *square root of* a is a number x such that $x^2 = a$. For example, 9 has square roots 3 and $^-3$. If a is any nonnegative number, the *principal square root of* a (denoted $\sqrt{a}$) is the nonnegative number x such that $x^2 = a$. For example, $\sqrt{9} = 3$.
Principal *n*th root (p. 390)	If n is even and b is any nonnegative real number, then the *principal* nth *root* of b (denoted as $\sqrt[n]{b}$) is the positive solution of $x^n = b$. If n is odd and b is any real number, the nth *root* of b (or $\sqrt[n]{b}$) is the unique solution of $x^n = b$.
Pythagorean Theorem (p. 392)	If a and b are the lengths of the shorter sides (legs) of a right triangle and c is the length of the hypotenuse (the side opposite the right angle), then $a^2 + b^2 = c^2$.
Fraction (p. 394)	Any real number of the form $\frac{a}{b}$, where a and b are real numbers with $b \neq 0$.
Addition and multiplication properties for real numbers (p. 394)	• *Closure*—for all real numbers a and b, $a + b$ and ab are unique real numbers. • *Commutative*—for all real numbers a and b, $a + b = b + a$ and $ab = ba$. • *Associative*—for all real numbers a, b, and c, $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$. • *Identity*—for all real numbers a; 0 is the (unique) *additive identity* such that $a + 0 = 0 + a$, 1 is the (unique) *multiplicative identity* such that $a \cdot 1 = 1 \cdot a = a$. • *Inverse*—for every real number a, there is an unique real number $-a$, the *additive inverse of* a, such that $a + {}^-a = 0 = {}^-a + a$. For every nonzero real number a, there is an unique real number $\frac{1}{a}$, the *multiplicative inverse of* a, such that $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$. • *Multiplicative Property of* 0—for all real number a, $a \cdot 0 = 0 \cdot a = 0$. • *Distributive property of multiplication over addition*—for all real numbers a, b, and c, $a(b + c) = ab + ac$.
Density property for real numbers (p. 394)	For all real numbers a and b with $a < b$, there exists a real number c such that $a < c < b$.
Rational exponents (p. 395)	For any positive integer n and any integer m, $x^{1/n} = \sqrt[n]{x}$, if $\sqrt[n]{x}$ is a real number, and $x^{m/n} = \sqrt[n]{x^m}$, if $\text{GCD}(m, n) = 1$ and $\sqrt[n]{x^m}$ is a real number.

Section 8-1

Properties of exponents (p. 395)

Let r and s be any rational numbers, x and y be any real numbers, and n be any nonzero integer. (Assume the following expressions are meaningful.)

- $x^{-r} = \frac{1}{x^r}$, $x^{-1/n} = \frac{1}{x^{1/n}} = \frac{1}{\sqrt[n]{x}}$
- $(xy)^r = x^r y^r$, $(xy)^{1/n} = x^{1/n}y^{1/n}$ and $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$
- $\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}$, $\left(\frac{x}{y}\right)^{1/n} = \frac{x^{1/n}}{y^{1/n}}$ and $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$
- $(x^r)^s = x^{rs}$, $\left(x^{1/n}\right)^p = x^{p/n}$ and $\left(\sqrt[n]{x}\right)^p = \sqrt[n]{x^p}$ where p is an integer.

Section 8-2

Constant (p. 401)

A fixed number or other fixed mathematical object

Variable (p. 401)

A quantity that varies.

A variable may

(1) represent a missing element or an unknown,

(2) stand for more than one thing,

(3) be used in generalizations of patterns, and

(4) be an element of a set, or a set itself.

***n*th term of an arithmetic sequence** (p. 404)

$a_n = a_1 + (n - 1)d$, where a_1 is the first term and d is the difference.

***n*th term of an geometric sequence** (p. 405)

$a_n = a_1 r^{n-1}$, where a_1 is the first term and r is the ratio.

Fibonacci sequence (p. 406)

A sequence in which $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Section 8-3

Properties of equality and equations (p. 412)

- *Addition*—for all real numbers a, b, and c, if $a = b$, then $a + c = b + c$.
- *Multiplication*—for all real numbers a, b, and c, if $a = b$, then $ac = bc$.
- *Cancellation*—For any real numbers a, b, and c,
 if $a + c = b + c$, then $a = b$.
 if $c \neq 0$ and $ac = bc$, then $a = b$.
- *Substitution*—equality is not affected if we substitute a number for its equal.

Section 8-4

Function (p. 423)

A correspondence from set A to set B in which each element of A is paired with one, and only one, element of B.

Let f be a function from A to B. If $x \in A$, then $f(x)$, the value of the function f at x (or the *image of* x) is the element in B that corresponds to x.

Set A is the *domain* of the function, and set B is the *codomain* (or the set of all possible outputs) of the function. The *range* of the function is the set of all outputs of the function.

Section 8-4

Function models (p. 424)	Functions may be modeled in various ways: • as a rule between two sets • as a machine relating inputs and outputs • as an equation relating two variables • as a table relating two variables • as an arrow diagram showing the correspondence between elements of two sets • as a set of ordered pairs • as a graph in two dimensions
Sum of the first *n* terms, *S*(*n*), of an arithmetic sequence (p. 428)	$S(n) = \left(\frac{a_1 + a_n}{2}\right)n$ or $S(n) = a_1 n + \frac{n(n-1)d}{2}$, where a_1 is the first term, a_n is the last term, d is the difference, and n is the number of terms.
Sum of the first *n* terms, *S*(*n*), of a geometric sequence (p. 429)	$S(n) = \frac{a_1(1 - r^n)}{1 - r}$, where a_1 is the first term, r is the ratio ($r \neq 1$), and n is the number of terms.
Composition of functions (p. 431)	Given the functions f and g, the composition of f with g is the function $g \circ f$ such that $(g \circ f)(x) = g(f(x))$.
Relation (p. 433)	A *relation from set A to set B* is any set of ordered pairs; that is, a subset of $A \times B$. A *relation* is a *function* from set A to set B if for every $a \in A$, there is a unique $b \in B$ such that (a, b) is in the relation.
Properties of relations (p. 433)	*Reflexive*—a relation R on a set X is *reflexive* if, and only if, for every element $a \in X$, a is related to a; that is, for every $a \in X$, $(a, a) \in R$. *Symmetric*—a relation R on a set X is *symmetric* if, and only if, for all elements a and b in X, whenever a is related to b, then b also is related to a; that is, if $(a, b) \in R$, then $(b, a) \in R$. *Transitive*— a relation R on a set X is *transitive* if, and only if, for all elements a, b, and c in X, whenever a is related to b and b is related to c, then a is related to c; that is, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$.
Equivalence relation (p. 434)	Any relation that has the reflexive, symmetric, and transitive properties.

Section 8-5

Cartesian coordinate system (p. 442)	A system constructed by placing two number lines perpendicular to each other. The intersection point of the two lines is the *origin*. The two perpendicular lines are commonly called the *x-axis*, and the *y-axis*.
Slope (p. 445, 449)	The measure of steepness (denoted by m) of a line. Give two points $A(x_1, y_1)$ and $B(x_2, y_2)$ with $x_1 \neq x_2$, the *slope* m of the line through A and B is $$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$
***x*-intercept and *y*-intercept** (p.446)	The value of x at the point of intersection of a line with the x-axis is the *x-intercept*. The value of y at the point of intersection of a line with the y-axis is the *y-intercept*.
Equation of a line (p. 447)	Every nonvertical line has an equation of the form $y = mx + b$ (*slope-intercept form*), where m is the *slope* and b is the *y-intercept*. A vertical line has undefined slope and the equation $x = a$. A horizontal line has 0 slope and the equation $y = b$.

Section 8-5	
Solving a system of equations (p. 451)	A system of two equations in two unknowns may be solved using any of the following methods: • *Graphing* • *Substitution* • *Elimination*
Number of solutions of a system of two equations in two variables (p. 454)	The system has *a unique solution* if, and only if, the graphs of the equations intersect in a single point. The system has *no solution* if, and only if, the equations represent distinct parallel lines. The system has *infinitely many solutions* if, and only if, the equations represent the same line.

Chapter 8 Review

1. Classify each of the following as rational or irrational (assume the patterns shown continue):
- **a.** 2.19119911999119999119 . . .
- **b.** $\frac{1}{\sqrt{2}}$
- **c.** $\frac{4}{9}$
- **d.** 0.0011001100110011 . . .
- **e.** 0.001100011000011 . . .

2. Write each of the following in the form $a\sqrt{b}$ or $a\sqrt[n]{b}$, where a and b are positive integers and b has the least value possible:
- **a.** $\sqrt{484}$
- **b.** $\sqrt{288}$
- **c.** $\sqrt{180}$
- **d.** $\sqrt[3]{162}$

3. Answer each of the following and explain your answers.
- **a.** Is the set of irrational numbers closed under addition?
- **b.** Is the set of irrational numbers closed under subtraction?
- **c.** Is the set of irrational numbers closed under multiplication?
- **d.** Is the set of irrational numbers closed under division?

4. Find an approximation for $\sqrt{23}$ correct to three decimal places without using the y^x or the $\sqrt{\ }$ keys on a calculator.

5. Approximate $\sqrt[3]{2}$ to two decimal places by using the squeezing method.

6. Each of the following is a geometric sequence. Find the missing terms.
- **a.** 5, ____, 10
- **b.** 1, ____, ____, ____, 1/4

7. In a geometric sequence, if the nth term is $\sqrt{7}(^-1)^n$, what is the 10th term and what is the ratio?

8. If the first two terms of an arithmetic sequence are 1 and $\sqrt{2}$, what is an algebraic expression for the nth term?

9. There are 13 times as many students as professors at a college. Use S for the number of students and P for the number of professors to represent the given information.

10. Write a sentence that gives the same information as the following equation: $A = 103B$, where A is the number of girls in a neighborhood and B is the number of boys.

11. Write an equation to find the number of feet given the number of yards (let f be the number of feet and y be the number of yards).

12. The sum of n whole numbers is S. If each number is multiplied by 10 and then decreased by 10, what is the sum of the new numbers in terms of n and S?

13. I am thinking of a whole number. If I divide it by 13, then multiply the answer by 12, then subtract 20, and then add 89, I end up with 93. What was my original number?

14. **a.** Think of a number.
Add 17.
Double the result.
Subtract 4.
Double the result.
Add 20.
Divide by 4.
Subtract 20.
Your answer will be your original number. Explain how this "trick" works.

b. Fill in two more steps that will take you back to your original number.
Think of a number.
Add 18.
Multiply by 4.
Subtract 7.
.
.
.

15. Find all the values of x that satisfy the following equations write your answers in simplified form:
- **a.** $4(7x - 21) = 14(7x - 21)$
- **b.** $3(\sqrt{x} - 1) = \sqrt{9x} + 5$
- **c.** $2(3x + 5) = 6x + 11$
- **d.** $2(x + \sqrt{3}) = 3(x - \sqrt{3})$

16. Mike has 3 times as many baseball cards as Jordan, who has twice as many cards as Paige. Together, the three children have 999 cards. Find how many cards each child has.

17. Jeannie has 10 books overdue at the library. She remembers she checked out 2 science books two weeks before she checked out 8 children's books. The daily fine per book is \$0.20. If her total fine was \$11.60, how long was each book overdue?

18. Three children deliver all the newspapers in a small town. Jacobo delivers twice as many papers as Dahlia, who delivers 100 more papers than Rashid. If altogether 500 papers are delivered, how many papers does each child deliver?

19. Which of the following sets of ordered pairs are functions from the set of first components to the set of second components?

a. $\{(a,b),(c,d),(e,a),(f,g)\}$
b. $\{(a,b),(a,c),(b,b),(b,c)\}$
c. $\{(a,b),(b,a)\}$
d. $\{(x,y)\,|\,x \text{ is any real number}, y = 2\}$

20. Given the following function rules and the domains, find the associated ranges:

a. $f(x) = x + 3$; domain $= \{0,1,2,3\}$
b. $f(x) = 3x - 1$; domain is R (all real numbers)
c. $f(x) = x^2$; domain is R (all real numbers)
d. $f(x) = x^2 + 3x + 5$; domain $= \{0,1,2\}$

21. Which of the following correspondences from A to B describe a function? If a correspondence is a function, find its range. Justify your answers.

a. A is the set of college students, and B is the set of majors. To each college student corresponds his or her major.
b. A is the set of books in the library, and B is the set N of natural numbers. To each book corresponds the number of pages in the book.
c. $A = \{(a,b)\,|\,a \in N \text{ and } b \in N\}$, and $B = N$. To each element (a, b) of A corresponds the number $4a + 2b$.
d. $A = N$ and $B = N$. If x is even, then $f(x) = 0$, and if x is odd, then $f(x) = 1$.
e. $A = N$ and $B = N$. To each natural number corresponds the sum of its digits.

22. A health club charges an initiation fee of \$200, which gives 1 month of free membership, and then charges \$55 per month.

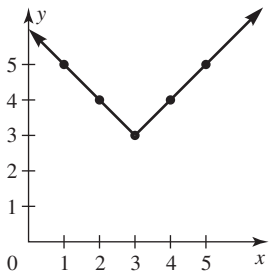
a. If $C(x)$ is the total cost of membership in the club for x months, express $C(x)$ in terms of x.
b. Graph $C(x)$ for the first 12 months.
c. Use the graph in (b) to find when the total cost of membership in the club will exceed \$600.
d. When will the total cost of membership exceed \$6000?

23. For each of the following functions, find possible inputs x, if $f(x)$ is as given.

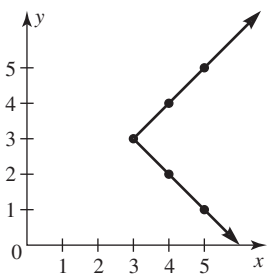
a. $f(x) = 4x - 5; f(x) = 15$
b. $f(x) = x^2 - 1; f(x) = 2$
c. $f(x) = \sqrt{-x}; f(x) = 3$
d. $f(x) = \sqrt{x}; f(x) = -3$
e. $f(x) = \dfrac{x+1}{x+2}; f(x) = -1$
f. $f(x) = \dfrac{x+1}{x+2}; f(x) = 1$

24. Which of the following graphs represent functions? Tell why.

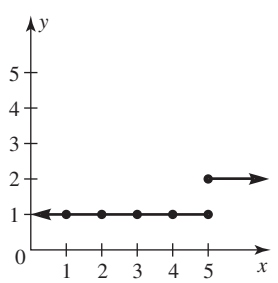
a.



b.



c.



25. a. Jilly is building towers with cubes, placing one cube on top of another and painting the tower (including the top and the bottom, but not the faces touching each other). Find the number of square faces that Jilly needs to paint for towers made of 1, 2, 3, 4, 5, and 6 cubes by filling in the following table:

# of Cubes	# of Squares to Paint
1	6
2	10
3	
4	
5	
6	

b. Graph the information you found in part (a) where the number of cubes in a tower is on the horizontal x-axis and the number of squares to be painted is on the vertical axis.
c. If x is the number of cubes in a tower and y is the corresponding number of squares to be painted, write an equation that gives y as a function of x.
d. Is the graph describing the number of squares as a function of the number of cubes used a straight line?

26. Graph each of the following equations:

a. $y = {}^{-}2x + 5$

b. $-\frac{x}{3} + \frac{y}{5} = 1$

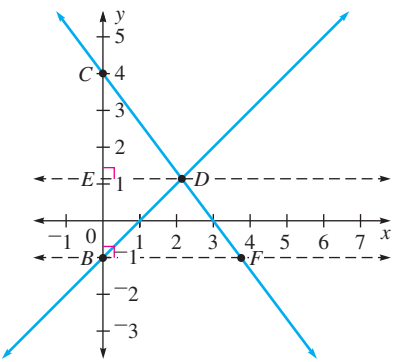
c. $y = {}^{-}x\sqrt{2}$

27. The following are graphs of $x - y - 1 = 0$ and of $4x + 3y - 12 = 0$. Answer the following.

a. Which equation is the graph of $\overleftrightarrow{CD}$?

b. If line $\overleftrightarrow{ED}$ is parallel to the x-axis, find the coordinates of E.

c. If line $\overleftrightarrow{BF}$ is parallel to the x-axis, find the length of segment $\overline{BF}$.



28. Solve the following equations if possible(s).

a. ${}^{-}3x + 12 = 23$

b. $\sqrt{2}x - 5 = \sqrt{8}x + 1$

c. $x^3 = {}^{-}2$

d. $4x^2 - 33 = 3$

e. $(x - 1)^3 = 2$

f. $(2x + 1)^4 = 5$

g. $(2x - 1)(x - 1) = 0$

29. If we take the third root of every term of a geometric sequence is the resulting sequence also geometric? If so, what is its ratio? Justify your answer.

Answers to Problems

Answers to all exercises not shown as annotations.

CHAPTER 1

Assessment 1-1A

2. Building a staircase as seen in (a) gives a visual graphic of the sum $1 + 2 + \ldots + n$. Copying the staircase as in (b) and placing it as shown demonstrates that an array that is n units high and $n + 1$ units long is produced. There are $n(n + 1)$ units in (b), which is twice the number desired. So the sum $1 + 2 + \ldots + n$ must be $n(n + 1)/2$. Gauss's sum when $n = 100$ would be $100(100 + 1)/2$ or 5050. **7.** E is greater by 49. **8.** Dandy, Cory, Alababa, Bubba **10.** $a = 42; b = 32; c = 37; d = 2$ **13.** Choose Box B and pull out a fruit. If you pull an apple then Box B must contain only apples (why?). Because Box A is mislabeled it can't be "Oranges," but it can't be "Apples" either (why?). So, it is "Oranges and Apples." Thus, Box C is "Oranges." If an orange is pulled the reasoning is similar.

Assessment 1-1B

2. a. There are 50 sums of 101 as shown by the arcs, so the total is $50 \cdot 101$ or 5050. **b.** A diagram similar to the one in part 2 (a) would illustrate how the numbers can be paired to form 100 sums of 202. Since the last term is odd, the middle term, 101, is left unpaired. So, the sum of the first 201 natural numbers is $100 \cdot 202 + 101 = 20{,}301$. **4.** Answers vary. For example: $1 + 10, 2 + 4 + 5, 2 + 9, 3 + 8, 4 + 7, 5 + 6, 1 + 1 + 9,$ $2 + 1 + 8, 3 + 1 + 7, 4 + 1 + 6, 5 + 1 + 5, 4 + 4 + 3$ **5.** If there are at least two colors of socks in the drawer then of any three chosen, two must be the same color. **7.** Q is greater by 97. **10.** Answers vary. For example:

17	7	9
3	11	19
13	15	5

11. Put 4 marbles in each tray of the scale. Take the heavier 4 and place 2 in each tray. Take the heavier 2 and place 1 in each tray. The heavier marble will be obvious. **13. a.** Two possibilities: (1) 3 quarters, 4 dimes, and 4 pennies; (2) 1 quarter, 9 dimes, and 4 pennies. **b.** A solution consisting of the least number of coins requires the most use of the quarters. Since 4 quarters cannot be used, use 3. Now, four dimes can be used and no more.

Mathematical Connections 1-1

Reasoning

1. There are two magic squares; strategies vary.

26	21	28
27	25	23
22	29	24

27	20	25
22	24	26
23	28	21

2. Each domino must cover 1 black and 1 red square. Hence, 31 dominoes would cover 31 red and 31 black squares. This is impossible because the board has 30 red and 32 black squares.

3. a. 45 **b.** Answers vary. **c.** $\dfrac{n(n-1)}{2}$

4. a. The first player selects one object from the larger pile, leaving the second player facing a (1, 1)-configuration, which is a loss for him or her. In the (1,100)-game, the first player can take 99 objects from the larger pile to win. **b.** If $a > 1$, the first player can win by taking $a - 1$ objects from the larger pile. **c.** The first player will lose any (a, a)-game. This is because the second player can play by mirroring the first player's move in the other pile. Either the second player has won at this point (by taking all of the remaining objects in one pile) or the first player faces a (b, b)-configuration, where $b < a$. Eventually, the first player will face the (1, 1)-game and lose. **d.** From part (c), we know that the first player to face a (k, k)-configuration will lose the game. The first player of the game therefore can win any (a, b)-game where $a \neq b$, by selecting as many objects as it is necessary to even up the piles ($a - b$ or $b - a$ objects). **5. a.** 7 **b.** $2^{64} - 1$ seconds, or approximately 585 billion years. The number of moves is $2^n - 1$ for n coins. This can be solved using the strategy of examining a simpler problem. If there is one coin, 1 move is necessary. If there are two coins, 3 moves are necessary. For three coins, the number of moves is 7. For four coins, the number of moves is 15.

Open-Ended

6. Answers vary. For example, any of the investigation problems in this section are examples of problems that could be approached with the strategy discussed. **7.** Answers vary. For example, if the average distance between adjacent people was 1.8 m, then it would take approximately 22,000,000 people.

Cooperative Learning

8. a. Play second and make sure that the sum showing when you hand the calculator to your opponent is a multiple of 3. **b.** Play first and press 4. After that, make sure that each time you hand the calculator to your opponent it displays 1 less than a multiple of 5. **c.** Play first and press 3. After that, make sure that each time you hand the calculator to your opponent it displays 3 more than a multiple of 10. **d.** Play second. Make sure the calculator displays a multiple of 3 each time you hand it to your opponent. **e.** Play second. Make sure the calculator displays a multiple of 4 each time you hand it to your opponent. **f.** Play first and subtract 3. Then make sure the calculator displays a multiple of 10 each time.

Connecting Mathematics to the Classroom

9. Answers vary. For example, it is in the last step where students examine whether the answer is reasonable and whether it checks given the original conditions in the problem. Many times,

students discover wrong answers at this point since they may have never bothered to check whether the answer they obtained makes sense. It is also at this step that students reflect on the mathematics used and determine whether there might be different ways of solving the problem. Also at this stage students reflect on any connections to other problems or generalizations. **10.** Answers vary. For example, intelligent guesses will help students systematically zero in on the exact answer because successive guesses can be based on the results given by the previous guesses. This is not possible with random guessing. **11.** Answers vary. For example, if these nine numbers are to be used in a magic square, then the sum in each of the three columns must be the same and must be a natural number. The natural number must be 1/3 of the sum of all nine numbers. However, $1 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 53$ and $53/3 = 17\frac{2}{3}$, which is not a natural number. Therefore, these numbers cannot be used for a magic square.

Assessment 1-2A

2. e. arithmetic; $193 + 10 \cdot 2^{30}, 193 + 11 \cdot 2^{30}, 193 + 12 \cdot 2^{30}$
3. a. 397; $4n - 3$ **b.** 5020; $50n + 20$ **c.** 3^{99}; 3^{n-1}
d. 100^{199}; 10^{2n-1} **e.** $193 + 106 \cdot 2^{30}$; $193 + (n + 6) \cdot 2^{30}$ **10. c.** 15, 17, 21, 27, 35, 45 **14. a.** Answers vary. For example, if $n = 5$, then $\frac{5 + 5}{5} \neq 5 + 1$. **b.** Answers vary. For example, if $n = 2$, then $(2 + 4)^2 \neq 2^2 + 4^2$. **20. a.** 5 (the fifth Fibonacci term) **b.** 8 (the sixth Fibonacci term) **c.** 13 (the seventh Fibonacci term) **d.** $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$ **e.** The sum of the first $n - 2$ terms of the Fibonacci sequence is one less than the nth term.

Assessment 1-2B

1. c.  **2. a.** Arithmetic; 18, 22, 26 **b.** Arithmetic; 39, 52, 65 **c.** Geometric; $4^4, 4^5, 4^6$ **d.** Geometric; $2^{14}, 2^{18}, 2^{22}$ **e.** Arithmetic; $100 + 10 \cdot 2^{50}, 100 + 12 \cdot 2^{50}, 100 + 14 \cdot 2^{50}$
3. a. 398; $4n - 2$ **b.** $99 \cdot 13$; $(n - 1)^{13}$ **c.** 4^{100}; 4^n
d. 2^{398}; 2^{4n-2} **e.** $100 + 202 \cdot 2^{50}$; $100 + 2(n + 1)2^{50}$
5. a. The sum of the first n odd natural numbers is n^2.
10. a. 3, 6, 12, 21, 33, 48 **c.** 4, 7, 13, 22, 34, 49 **13. a.** 4, 9, 14, 19, 24 **b.** 4, 10, 16, 22, 28 **c.** 6, 11, 16, 21, 26 **d.** 0, 3, 8, 15, 24 **14. a.** Answers vary. For example, let $n = 6$; then $\frac{3 + 6}{3} = \frac{9}{3} = 3 \neq 6$. **b.** Answers vary. For example, let $n = 4$; then $(4 - 2)^2 = 2^2 = 4 \neq 4^2 - 4$. **c.** 2, 1, 7 **20. a.** $F_1 + F_3 = 1 + 2 = 3$ and $F_4 = 3$ **b.** $F_1 + F_3 + F_5 = 8 = F_6$ **c.** $F_1 + F_3 + F_5 + F_7 = 21 = F_8$
d. $F_1 + F_3 + F_5 + \ldots + F_{2k-1} = F_{2k}$

Mathematical Connections 1-2

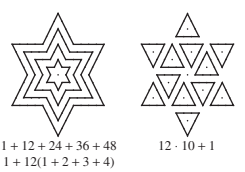
Reasoning

1. a. Yes. The difference between terms in the new sequence is the same as in the old sequence because a fixed number was added to each number in the sequence. **b.** Yes. If the fixed number is k, the difference between terms of the second sequence is k times the difference between terms of the first sequence. **c.** Yes. The difference of the new sequence is the sum of the original differences.

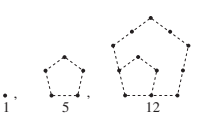
2. Because the population is increasing, the ratio of the geometric sequence is greater than 1. Because the terms of such a geometric sequence increase faster than the terms of any arithmetic sequence, it is possible to show that given any increasing geometric sequence and any arithmetic sequence, for some n the nth term of the geometric sequence will be greater than the corresponding term of the arithmetic sequence. **3. a.** Abby studied for 13.5 hours. Dan studied for 11.25 hours. **b.** Week 17
4. Since a, b, and c form an arithmetic sequence, $b - a = c - b$. Hence, $2b = a + c$ or $b = \frac{a + c}{2}$. **5.** For any x, the sequence $x, 2x, 3x$ are the first three terms in a Fibonacci-type sequence and in an arithmetic sequence. **6. a.** n^2 **b.** If in the nth figure the four corner white tiles are not counted then we see that there are $4n$ white tiles around the $n \times n$ shaded square. Adding the corner tiles, we get the total $4n + 4$ white tiles. Another way is to count all the tiles and subtract the number of the shaded ones, that is, $(n + 2)^2 - n^2$, which also equals $4n + 4$.

Open-Ended

7. Answers vary. For example, two more patterns follow:



8. Answers vary. For example, the following is an array of pentagonal numbers:



The pattern for the number of dots is 1, 5, 12, 22, 35, 51, . . . The number of additional dots for each pentagon forms an arithmetic sequence with first term 1 and second difference 3. For the 100th pentagonal number, we would have $1 + 4 + 7 + 10 + 13 + 16 + \ldots + 298$ dots, where $298 = 1 + (100 - 1)3$. Therefore, the 100th pentagonal number is $(100/2)(1 + 298) = 14{,}950$. **9.** Answers will vary. **10.** Answers vary.

Cooperative Learning

11. a. Answers vary. For example, each number other than 1 is the sum of the two numbers to the top right and top left. There is right-left symmetry. The natural numbers show up along two diagonals. **b.** $1, 2, 4, 8, 16, \ldots$, or $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$
c. $2^{15} = 32{,}768$ **d.** 2^{n-1} **12. a.** 81 **b.** 40 **c.** 3^{n-1}

Connecting Mathematics to the Classroom

13. Answers vary. For example, Joey could be told that he has noticed a pattern that works for many numbers but he must be careful in generalizing his observation to all numbers. He should be encouraged to look at more numbers to see if he can find a *counterexample* to his claim. If he starts investigating, he will find that 14 ends in 4 but is not divisible by 4. 34 is another number that ends in 4 but is not divisible by 4. These counterexamples prove that Joey's conjecture is false. **14.** Al and Betty should be told that both answers could be correct as long as each person's rule works for all the given terms. Al is thinking of a geometric sequence in which each term is multiplied by 2 to get the next term. Betty is thinking of a different sequence. She starts with

2, then adds 2 to get the 2nd term, then adds 4 to get the next term, then 6, then 8, then 10, and so on. Betty's first difference is the arithmetic sequence 2, 4, 6, . . . **15.** The sequence is both arithmetic and geometric (add 0 or multiply by 1 to obtain the terms of the sequence). The only other sequences that are both arithmetic and geometric are of the form $c, c, c, c, \ldots$ for some real number c. **16.** The student is wrong. For example, if we follow the student's procedure for the sequence 35, 40, 45, we get $\frac{45 - 35}{5} = 2$, rather than 3. The student would be correct if she said to add 1 to the quotient.

Review Problems

17. 90 **18.** 7 **19.** We need one 12-person tent and a combination of tents to hold 14 people. There are 10 ways: 662, 653, 6332, 62222, 5522, 5333, 53222, 33332, 332222, and 2222222.

Chapter 1 Review

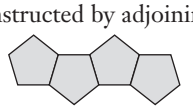
3. The question "What makes you come so soon?" is asked when the scholar arrives two hours later than usual. This makes no sense *unless* the "ten o'clock" is at night. **5. a.** $3n + 2$ if the sequence is arithmetic with difference of 3. **b.** 3^n if the sequence is geometric with ratio 3. **c.** $2^{n+1} - 1$ **6.** 1, 4, 7, 10, 13 **7. a.** 10,100 **b.** 10,201 **8. b.** False; for example, 19 is odd and it ends in 9. **c.** True; the sum of any two even numbers, $2m$ and $2n$, is even because $2m + 2n = 2(m + n)$

9. Answers vary.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

21. $n - m + 1$, answers vary.

22. Answers vary. For example, fill the 4-cup pot and empty it into the 7-cup pot. Repeat. There is now 1 cup in the 4-cup pot. Empty the 7-cup pot and pour the 1 cup into the 7-cup pot. Fill the 4-cup pot and empty it into the 7-cup pot. It will now contain 5 cups. **24. a.** A possible pattern is that each successive figure is constructed by adjoining another pentagon to the previous figure.



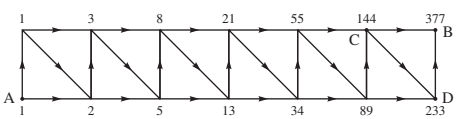
25. a. The circled terms will constitute an arithmetic sequence because the common difference will be twice the difference in the original sequence. **b.** The new sequence will be a geometric sequence because the ratio will be the square of the ratio of the original sequence. **26.** If the sequence is $a_1, a_2, a_3, a_4, \ldots$ to find the first term we substitute $n = 1$ and get $1^2 - 1$ or 0, so $a_1 = 0$. For $n = 2$, we get $2^2 - 2$ or 2. Thus, $a_1 + a_2 = 2$; hence $a_2 = 2$. For $n = 3, a_1 + a_2 + a_3 = 3^2 - 3 = 6$. Substituting for a_1 and a_2, we get $a_3 = 6 - 2 = 4$. For $n = 4, a_1 + a_2 + a_3 + a_4 = 4^2 - 4 = 12$. Substituting for a_1, a_2, and a_3, we get $0 + 2 + 4 + a_4 = 12$. Hence, $a_4 = 6$.

27. a. 7, 20 **b.** 13, 150 **c.** $\frac{a - b}{2}, \frac{a + b}{2}$

Answers to Now Try This

1. In Figure 1, we have $1 + 100 = 101$. To obtain each successive pair from the previous one, we add 1 and subtract 1 and hence the sum does not change. Gauss's approach for the sum of squares does not work because the sums change; for example, $1^2 + 100^2 \neq 2^2 + 99^2$. **2.** First approach:
$25 + 26 + 27 + \ldots + 120 = (1 + 2 + 3 + \ldots + 25 + 26 + 27 + \ldots + 120) - (1 + 2 + 3 + \ldots + 24) =$
$\frac{(1 + 120) \cdot 120}{2} - \frac{(1 + 24) \cdot 24}{2} = 121 \cdot 60 - 25 \cdot 12 = 6960$
Second approach:
There are $120 - 24$ or 96 terms in the sum. Hence, using Gauss's approach, the sum equals $\frac{(25 + 120) \cdot 96}{2}$ or 6960.

3. The first person played with 15 people; the second played with 14 because his game with the first person was already counted and so one. Thus, the total number of games played is $15 + 14 + 13 + \ldots + 1$, which using Gauss's approach is $\left(\frac{15 + 1}{2}\right) \cdot 15$ or 120. **5. a.** Answers vary. For example, the next three terms could be △, △, ○.
b. The pattern could be one circle, two triangles, one circle, two triangles, and so on. **6. a.** Inductive reasoning **b.** The next several numbers also work. **c.** If $x = 11$, then $11^2 + 11 + 11$ is not prime because it is divisible by 11. **7.** The following diagram shows the number of paths from A to every vertex and finally to B.



The pattern follows the Fibonacci sequence. Notice that to get to point B, we need to pass either through point C or through point D. To every path through C, there is only one way to get to point B without passing through D, that is, 144 possible paths. Thus, the number of paths to B is 144 + 233 or 377. This reasoning applies to every vertex other than the first two vertices.
9. $10 \cdot 11$ or 110; $n(n + 1)$

Answer to the Preliminary Problem

Jill takes $1 + 2 + 3 + \ldots 10$, or 55 coins from the 10 boxes. If all the coins were genuine, the 55 coins would weigh $55 \cdot 2$ oz. or 110 oz. If the first box contained all the fake coins, the weight of all the coins would be $110 - 1$, or 109 oz. If the second box contained all the fake coins, the weight of all the coins would be $110 - 2$ or 108 oz. Thus, if all the 55 coins weigh 104 oz., Jill would write $104 = 110 - 6$ and she would know that the sixth box contains the fake coins. In general, if the coins weigh $110 - n$ oz, then the nth box contains the counterfeit coins.

CHAPTER 2

Assessment 2-1A

2. Answers vary. **a.** There exists a natural number n such that $n + 8 = 11$. **b.** There exists a natural number n such that $n^2 = 4$. **c.** For all natural numbers $n, n + 3 = 3 + n$. **d.** For all natural numbers $n, 5n + 4n = 9n$. **3.** Answers vary. **a.** For every natural number $n, n + 8 = 11$. **b.** Every natural number n satisfies $n^2 = 4$. **c.** There is no natural number n such that $n + 3 = 3 + n$. **d.** There is no natural number n such that $5n + 4n = 9n$.

6. a.

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

b.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

7. a.

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

b. $(p \rightarrow q) \equiv (\sim p \vee q)$ **c.** If $2 + 3 = 5$, then $4 + 6 = 10$.
10. Answers vary. **a.** $2 + 3 \neq 5$ or $4 + 7 \neq 10$ **b.** The president of the United States in 2013 was Barack Obama. **c.** With every seat filled, the Supreme Court of the United States does not have 12 justices. **d.** A triangle has three sides of the same length and the triangle does not have two sides of the same length. **e.** A triangle has two sides of the same length and the triangle does not have three sides of the same length.
12. Dr. No is a male spy who is neither poor nor tall.
14. a. Converse: If a triangle has no two sides of the same length, then it is a scalene triangle. Inverse: If a triangle is not scalene, then it has two sides of the same length. Contrapositive: If a triangle has two sides of the same length, then the triangle is not scalene. **b.** Converse: If an angle is a right angle, then it is not acute. Inverse: If an angle is acute, then it is not a right angle. Contrapositive: If an angle is not a right angle, then it is an acute angle. **c.** Converse: If Mary is not a citizen of Cuba, then she is a U.S. citizen. Inverse: If Mary is not a U.S. citizen, then she is a citizen of Cuba. Contrapositive: If Mary is a citizen of Cuba, then she is not a citizen of the United States. **d.** Converse: If a number is not a natural number, then it is a whole number. Inverse: If a number is not a whole number, then it is a natural number. Contrapositive: If a number is a natural number, then it is not a whole number.
15. The statements are negations of each other.

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$p \rightarrow q$	$\sim(p \rightarrow q)$
T	T	F	F	T	T	F
T	F	T	T	F	F	T
F	T	F	F	T	T	F
F	F	T	F	T	T	F

16. Answers vary. If a number is not a multiple of 4, then it is not a multiple of 8.
18.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

19. It is possible that some teachers with IQs of 145 or more are in Mensa if there is one person in the overlap of the circles representing IQs of 145 or more and those in Mensa because all considered are teachers. **20. a.** Some Beta Club students make A's. **b.** If I study for the final, then I will look for a teaching job. **c.** There exist triangles that are isosceles. **21. a.** If a figure is a square, then it is a rectangle. **b.** If a number is an integer, then it is a rational number. **c.** If a polygon has exactly three sides, then it is a triangle.

Assessment 2-1B

2. Answers vary. **a.** For all $n, n + 0 = n$. **b.** No natural number n exists such that $n + 1 = n + 2$. **c.** There exists a natural number n such that $3(n + 2) = 12$. **d.** There exists a natural number n such that $n^3 = 8$. **3.** Answers vary. **a.** There is no natural number n such that $n + 0 = n$. **b.** There exists at least one natural number n such that $n + 1 = n + 2$. **c.** For all natural numbers $n, 3(n + 2) = 12$. **d.** For all natural numbers $n, n^3 = 8$. **4. b.** All cats have nine lives. **c.** There exists a square that is not a rectangle. **9.** Answers vary. **a.** $4 + 6 \neq 10$ and $2 + 3 \neq 5$ **b.** A National Football League team must have 11 or fewer players on the field a game is in progress. **c.** The first president of the United States was not George Washington. **d.** A quadrilateral has three sides of the same length and the quadrilateral does not have four sides of the same length. **e.** A rectangle has four sides of the same length and it does not have three sides of the same length. **11.** Ms. Makeover is a married blond with straight hair. **13. a.** Converse: If $x^2 = 9$, then $x = 3$. Inverse: If $x \neq 3$, then $x^2 \neq 9$. Contrapositive: If $x^2 \neq 9$, then $x \neq 3$. **b.** Converse: If classes are canceled, then it snowed. Inverse: If it does not snow, then classes are not canceled. Contrapositive: If classes are not canceled, then it did not snow. **15. c.** Seven is a natural number but not because of the argument here; the argument is not valid. **16. a.** Helen is in Kappa Mu Epsilon. **17. a.** If a number is a natural number, then it is a real number. **b.** If a figure is a circle, then it is a closed figure. **18. b.** I am not going and she is not going.

Mathematical Connections 2-1

Reasoning

1. The truth value of neither questions nor commands can be determined. As a result, they are not considered mathematical statements. **2.** Answers vary. There are several ways to negate such a statement but one has to make sure that "all" are addressed. One form of the negation is "All A's are not B's." **3. a.** For a disjunction to be true, at least one of the statements forming the disjunction must be true. **b.** An implication is always true unless the conclusion is false and the hypothesis is true.
4. a. If there is no nail, then there is no shoe.
If there is no shoe, then there is no horse.
If there is no horse, there is no rider.
If there is no rider, then the battle is lost.
If the battle is lost, the war is lost.
Therefore, if there is no nail, then the war is lost.
b. Yes, this follows from the repeated use of the chain rule.

5. When a comma or a semicolon is used in an e-mail address (depending on the server), the logical meaning is "and" so that all addresses will receive the e-mail. **6.** When both a statement and its converse are true, then the other forms of the statement (inverse and contrapositive) are true. The statement and its contrapositive have the same truth value, as do the converse and its contrapositive (the inverse of the original statement).

Open-Ended

7. Answers vary. An example could be seen when the statements p is "I bought a car" and q is "I take you on a date." As long as the promise is kept, the implication "If I bought a car, I will take you on a date" is considered true.

Case	p	q	Translation of symbols	Result
1	T	T	I bought a car; I take you on a date.	Promise Kept
2	T	F	I bought a car; I did not take you on a date.	Promise Broken
3	F	T	I did not buy a car; I take you on a date.	Promise Kept
4	F	F	I did not buy a car; I did not take you on a date.	Promise Kept

8. Yes. Consider the truth table below: A tautology $(p \leftrightarrow q)$ is true as long as the truth values for each of p and q are the same. Consider the following table that considers all the cases.

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$
T	T	F	F	T
F	F	T	T	T

Cooperative Learning

9. Answers vary. Statements (a) and (b) are false. Statement (c) causes the problem. If statement (c) is also false, then that makes statement (c) a true assertion, which is a contradiction. On the other hand, if statement (c) is true, then it wrongly asserts that it is false. **10.** Answers vary.

Connecting Mathematics to the Classroom

11. Answers vary. For example, a truth table could be used to show the possibilities. That approach may not satisfy the student, even though it is the mathematical way to do it. An example might satisfy the student, though it would not be a complete mathematical explanation. For example, consider the following:

All whole numbers are natural numbers [*This is a false hypothesis.*]
7 is a whole number.
Conclusion: 7 is a natural number.

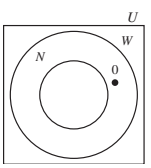
The conclusion is true, and the argument seems valid.

12. Answers vary. For example, Mr. Johnson would have lied had he allowed no one to go to recess. He did not do that by restricting one child from going.

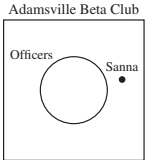
Assessment 2-2A

1. Answers vary. **a.** $\{a, s, e, m, n, t\}$ **b.** $\{20, 21, 22, 23, 24, \ldots\}$ **2. a.** $P = \{p, q, r, s\}$ **4. b.** $n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1 = n!$ The "first" element can be paired with n choices, leaving $n-1$ elements for the "second" pairing and so forth. The Fundamental Counting Theorem says that the choices can be multiplied to find the total number of correspondences. **8.** $\overline{A}$ is the set of all college students with at least one grade that is not an A, that is, those college students who do not have a straight-A average. **9. b.** There is no limit on how many elements set C could have. **10. b.** Answers vary. For example, the sets are equal. **16. b.** No. A may equal B. **d.** No. Consider $A = \{1\}$ and $B = \{1, 2\}$. **17. a.** Let $A = \{1, 2, 3, \ldots, 100\}$ and $B = \{1, 2, 3\}$. Then $n(A) = 100$ and $n(B) = 3$. $B \subset A$. **b.** $n(\varnothing) = 0$. Let $A = \{1, 2, 3\}$, which implies that $n(A) = 3$. $\varnothing \subset A$, which implies that there is at least one more element in A than in $\varnothing$. Thus $0 < 3$. **19.** Answers vary. The set of natural numbers and the set of whole numbers **20.** There is a one-to-one correspondence between the two sets such that each even natural number $2n$ is paired with $2n-1$.

21.



22.

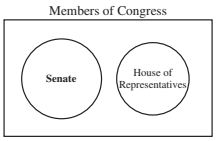


23. Answers vary. The only members of the Adamsville Beta Club are the officers.

Assessment 2-2B

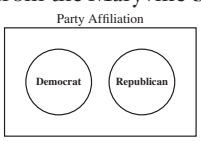
1. Answers vary. **a.** $\{a, l, g, e, b, r\}$ **b.** $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **2. a.** $Q = \{q, r, s\}$ **4. b.** $(n-1)(n-2)(n-3)\cdot\ldots\cdot 1$. There are $n-1$ choices with which to pair the "first" element, $n-2$ choices with which to pair the "second" element, and so forth. The Fundamental Counting Principle then says that the product of the choices gives the total number of ways that the one-to-one correspondence can happen. **8.** $\overline{G}$ is the set of all women who did not graduate from Georgia State University. **9. a.** The empty set is a subset of every set; if $A = \varnothing$, then $n(A) = 0$. **b.** Yes. B can be the empty set if, and only if, $A = \varnothing = B$. **15. a.** No; for example, $A = \{1\}$ and $B = \{1, 2\}$; $A \subseteq B$, but $A \neq B$. **b.** No; $A \subset B$ implies B has at least one element that A does not have. **c.** No; they must be equivalent but they do not have to be equal, for example, $A = \{1\}$ and $B = \{2\}$. **16. a.** $0 = n(\varnothing)$; let $A = \{1, 2\}$; then $n(A) = 2$. Now $\varnothing \subset A$, because every element of $\varnothing$ (of which there are none) is an element of A and there is at least one element in A that is not in $\varnothing$. **b.** Let $A = \{1, 2, 3, \ldots, 99\}$ and $B = \{1, 2, 3, \ldots, 100\}$. Since $A \subset B, n(A) < n(B)$. So $99 < 100$. **19.** For every natural number n, let it be paired with $n-1$. The pairing is a one-to-one correspondence between the set of natural numbers and the set of whole numbers.

20.



21. Answers vary. The only swimmers in the 100-m butterfly race are from the Maryville Swim Team.

22.



23. Answers vary. For example, A is not equivalent to B.

24. a. Answers vary. Let A be the set of all natural numbers not equal to 1 with N as the universal set. Then $\overline{A} = \{1\}$ is finite. **b.** Answers vary. Let A be the set of the even natural numbers with N as the universal set. $\overline{A}$ is the set of odd natural numbers, which is infinite.

Mathematical Connections 2-2

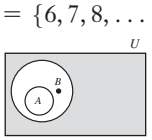
Reasoning

1. A set is well defined if any object can be classified as belonging to the set or not belonging to the set. For example, the set of U.S. presidents is well defined, but the set of rich U.S. presidents is not well defined because "rich" is a matter of opinion. **2. a.** This set is not well defined, since we do not know what is meant by "wealthy." **b.** This set is not well defined, since we do not know what is meant by "great." **c.** This set is well defined. Given a number, we can easily tell if it is a natural number greater than 100. **d.** This set is well defined. We could list them all. **e.** This set is well-defined; it is the empty set. **3.** Suppose set A is not empty. Then A contains at least one element. There is no element in $\varnothing$ to be paired with the element in set A. Thus there can be no one-to-one correspondence between $\varnothing$ and any set with an element. **4.** $\{\varnothing\}$ has exactly one element, namely $\varnothing$. Therefore $\varnothing \in \{\varnothing\}$. $\varnothing$ is a subset of every set. **5.** To show that $A \not\subseteq B$, we must be able to find at least one element of set A that does not belong to set B. **6.** If two sets are not finite, then the cardinal number of one set can be equal to the cardinal number of the other set, even though one is a proper subset of the other. In this case, the sets are infinite. "Less than" is only defined using finite sets. We note that infinities can be compared but not at this level for the most part. **7.** If A and B are finite sets, we say that $n(A) \leq n(B)$ in case A is a (not necessarily proper) subset of B.

Open-Ended

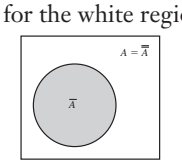
8. Answers vary. **a.** Let $A = \{1, 2, 3, 4, 5\}$; $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$; and $U = \{1, 2, 3, \ldots, 20\}$, $\overline{A} = \{6, 7, 8, \ldots, 20\}$ and $\overline{B} = \{10, 11, 12, \ldots, 20\}$ **b.** $\overline{B} \subset \overline{A}$

c.



$\overline{B}$ contains only the shaded region. $\overline{A}$ contains all the area outside the circle labeled A. The dot in set B is in $\overline{A}$ but not in $\overline{B}$.

9. The simplest symbol for the white region is A.

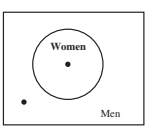


10. We can use a one-to-one correspondence between the given set and a finite set with n elements as shown below:



The pattern is an "add 2" pattern with the first term 3. A formula for the nth term of such a pattern is $2n + 1$. Thus, $2n + 1 = 1009$, so $n = 504$, and there are 504 elements in the given set.

11. Answers vary. The Venn diagram below depicts one way to show the information.



If $m = n$(Men) and $w = n$(Women), then $m \geq 1, n \geq 1$, $m + n = 100$. Other number sentences may be written as well but vary on symbolism and first identified unknowns.

Cooperative Learning

12. Answers vary on how the work is done, but the number of arrangements is 24.

Connecting Mathematics to the Classroom

13. If $A \subseteq B$, then every element in A is an element of B. Since $B \subseteq C$, every element in B is an element of C. Thus every element of A is an element in C and $A \subseteq C$. **14.** Answers vary. Exactly the assumptions that the student is making are unknown, but possibly the student thinks that either A or B must be the universal set. **15.** As a teacher, a response should be that yes, infinite sets can be compared as illustrated by a mathematician, Georg Cantor, around the turn of the twentieth century. It was groundbreaking work at the time. Perhaps the easiest example is that the number of natural numbers and the number of even natural numbers are the same.

Review Problems

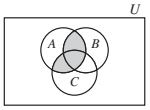
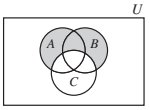
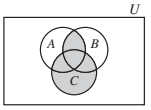
16. a. False **b.** False **c.** True **d.** True **e.** False **17. a.** False **b.** True **c.** False **d.** True **e.** False

18.

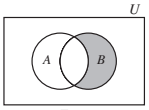
p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Assessment 2-3A

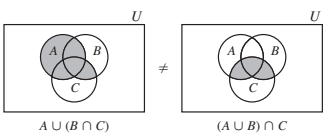
3. b. False. Let $A = \{a, b, c\}$ and $B = \{a, b\}$. Then $A - B = \{c\}$ but $B - A = \varnothing$. **c.** False. Let $U = \{a, b, c\}, A = \{a\}$, and $B = \{b\}$. Then $A \cap B = \varnothing$ and $\overline{A \cap B} = U$. $\overline{A} = \{b, c\}$, $\overline{B} = \{a, c\}$, and $\overline{A} \cap \overline{B} = \{c\}$. Thus $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$. **d.** False. Let $A = \{a, b\}$; $B = \{b\}$. $A \cup B = \{a, b\}$, $(A \cup B) - A = \varnothing \neq B$. **e.** False. Let $A = \{1, 2, 3\}, B = \{3, 4, 5\}$. Then $(A - B) \cup A = \{1, 2, 3\}$, but $(A - B) \cup (B - A) = \{1, 2\} \cup \{4, 5\} = \{1, 2, 4, 5\}$.

5. a.  **b.**  **c.** 

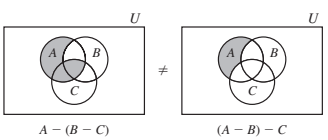
8. Yes; by definition, $A - B$ is the set of all elements in A that are not in B. If $A - B$ is the empty set, then this means that there are no elements in A that are not in B, which makes A a subset of B.

10. a.  **b.** Answers vary. $B - (A \cap B)$

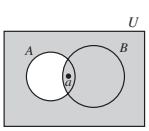
11. a. False $A \cup (B \cap C) \neq (A \cup B) \cap C$



b. False $A - (B - C) \neq (A - B) - C$

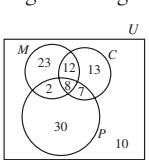


12. a. Yes. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. $A \cap B = \{3\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$. In this example $A \cap B \subseteq A \cup B$, and it is true in general. **b.** Yes. Consider the following Venn diagram: $\overline{A - B}$ is shaded; $a \in \overline{A - B}$ and in A.



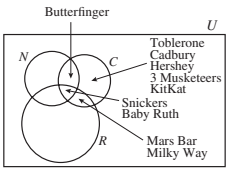
13. a. i. 5 **ii.** 2 **iii.** 2 **iv.** 3 **b. i.** $n + m$ **ii.** m **iii.** m **iv.** N

15. a. 319.2 million **b.** 148.0 million **16. a.** The set of college basketball players more than 200 cm tall **b.** The set of humans who are not college students or who are college students less than or equal to 200 cm tall **c.** The set of humans who are college basketball players or who are college students taller than 200 cm **d.** The set of all humans who are not college basketball players and who are not college students taller than 200 cm **e.** The set of all college students taller than 200 cm who are not basketball players **f.** The set of all college basketball players less than or equal to 200 cm tall **21.** Suppose M is the set of students taking mathematics, C is the set of students taking chemistry, and P is the set of students taking physics. The given information determines the numbers in the Venn diagram's regions below.



The total of all students is seen to be 105, which contradicts John's data that reported 100 students. Thus, on the basis of this report, John should not be hired for the job.

22. Steelers versus Jets, Bills versus Redskins, Cowboys versus Giants, Vikings versus Packers

23. a. 

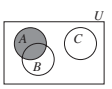
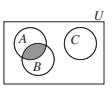
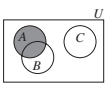
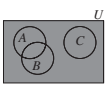
b. iii. Chocolate; Nuts/Peanut butter **iv.** Answers vary.

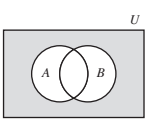
25. The families are the following: Tom, Jane, and Mary; and Abby, Harry, and Dick. **27. b.** $C = \{1, 2\}, D = \{1, 2, 3\}$

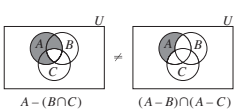
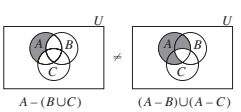
Assessment 2-3B

2. a. No. Let $A = \{a\}$ and $B = \{a, b\}$. $A - B = \varnothing \neq B - A$. **c.** No. Let $B = \{a\}$ and $A = \{a, b\}$. $B - A = \varnothing \neq B \cup A$.

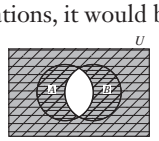
3. a. False. Let $A = \{1, 2\}, B = \{2\}$. Then $A - B = \{1\}$, but $A - \varnothing = \{1, 2\}$. **b.** False. Let $U = \{1, 2, 3\}, A = \{1, 2\}$, $B = \{2, 3\}$. Then $A \cup B = \{1, 2, 3\}$ and $\overline{A} = \{3\}$ and $\overline{B} = \{1\}$, making $\overline{A} \cup \overline{B} = \{1, 3\}$ and $\overline{A} \cup \overline{B} = \varnothing$. **c.** False. Let $A = \{1, 2\}, B = \{2, 3\}$, and $C = \{1, 2, 4\}$. Now $A \cap (B \cup C) = \{1, 2\}$ but $(A \cap B) \cup C = \{1, 2, 4\}$. **d.** False. Let $A = \{1, 2\}, B = \{2, 3\}$. Now $(A - B) \cap A = \{1\}$ but $A = \{1, 2\}$. **e.** False. Let $A = \{1, 2, 3\}, B = \{1, 2, 4\}$, and $C = \{1, 2, 3, 4\}$. Now $A - (B \cap C) = \{3\}$ but $(A - B) \cap (A - C) = \varnothing$.

5. a.  **b.**  **c.**  **d.** 

9. a.  **b.** Answers vary. For example, $\overline{A} \cap \overline{B}$.

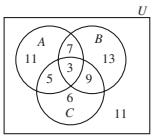
10. a.  **b.** 

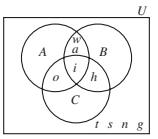
11. The set of persons fitting the description with blond hair, green eyes, and a beard is a subset of persons fitting the description with blond hair and green eyes. One expects the subset to have no more than the same number of elements. With a general populations, it would be expected to have fewer elements.

13. a.  $\overline{A \cap B}$ is gray. $\overline{A}$ has diagonal lines. $\overline{B}$ has horizontal lines. $\overline{A} \cup \overline{B}$ is gray. $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

b. Answers vary. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, and $B = \{3, 4, 5, 6\}$. Now $A \cap B = \{3, 4, 5\}$, $\overline{A} = \{6, 7, 8, 9, 10\}$ and $\overline{B} = \{1, 2, 7, 8, 9, 10\}$. Then we have $\overline{A \cap B} = \{1, 2, 6, 7, 8, 9, 10\}$ and $\overline{A} \cup \overline{B} = \{1, 2, 6, 7, 8, 9, 10\}$.

14. a. The set of all Paxson eighth graders who are members of the band but not the choir **b.** The set of all Paxson eighth graders who are members of both the band and the choir **c.** The set of all Paxson eighth graders who are members of the choir but not the band **d.** The set of all Paxson eighth graders who are members of neither the band nor the choir.

15. 

16. 

17. d. These are students at Hellgate who took algebra but neither biology nor chemistry. **e.** These are students at Hellgate who took both biology and chemistry but not algebra. **f.** Answers vary. $(C \cap B) - A$ or $(C \cap B) \cap \overline{A}$ **g.** Answers vary. $C \cap \overline{B}$ or $C - B$ **h.** Answers vary. $C - (A \cup B)$ or $C \cap \overline{A \cup B}$ **18.** The editor is correct. Sorting the data using a Venn diagram, the total number of people interviewed is 550, not 600. **21. a.** About 146.4 million **b.** About 171.2 million **23. a.** False because these are ordered pairs and order is important. **b.** False because the left side is an ordered pair and the right side is a set.

Mathematical Connections 2-3

Reasoning

1. a. This is not true. If $a \in A \cup B$, this means that a is an element of either A or B. There is nothing in this definition to make a an element of both A and B, which would have to happen for $a \in A \cap B$. **b.** No. For example, let $A = \{1, 2, 3, 4\}, B = \{4, 5, 6\}$, and $C\{3, 5, 6\}$. $A \cup B = A \cup C = \{1, 2, 3, 4, 5, 6\}$ but $B \neq C$. **c.** No. For example, let $A = \{1, 2, 3, 4\}, B = \{3, 5, 6, 7\}$, and $C = \{3, 5, 6\}$. $A \cap B = A \cap C = \{3\}$ but $B \neq C$. **2.** Answers vary. The complements of two sets are equal, if, and only if, the sets are equal. **3.** No. Suppose $A = \{1\}$ and $B = \{2\}$. Then $A \times B = \{(1, 2)\} \neq \{(2, 1)\} = B \times A$. **4.** No; suppose $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7\}$. Then $n(A - B) = n(\{1, 2, 3,4, 5\}) = 5$. However, $n(A) - n(B) = 5 - 2 = 3$ and $5 \neq 3$. **5.** This could happen if there were 110 students taking algebra but not biology and there were 90 students taking biology but not algebra.

Open-Ended

6. Answers vary. Two typical search items would be *mathematics* and *biology*. When those are entered into a search engine, they are treated as *mathematics and biology*. This is comparable to the intersection of two sets when the items found must be in both categories. **7.** Answers vary.

Cooperative Learning

8. a. $A \cup B$ is the set of all blocks that are green, large, or both green and large. $B \cup A$ is the set of all blocks that are large, green, or both large and green. The two sets are equal. **b.** $\overline{A \cap B}$ is the set of all blocks that are not both large and green. $\overline{A} \cap \overline{B}$ is the set of all the small blocks that are not green. The sets are not equal. **c.** $\overline{A \cap B}$ is the set of all blocks that are not both green and large. $\overline{A} \cup \overline{B}$ is the set of all blocks that are not green or are not large. The sets are equal. **d.** $A - B$ is the set of green blocks that are not large. $A \cap \overline{B}$ is the set of green blocks that are not large. The sets are equal. **9.** Answers vary. Typically in an exercise like this, students are looking at an intersection of sets trying to find the one individual with all the given characteristics.

Connecting Mathematics to the Classroom

10. Answers vary. The student is incorrectly thinking that all the elements in each set must be listed in the union of the two sets. However, mathematical convention lists each element exactly once in a set. **11.** The Cartesian product consists of all pairs in which the first component comes from the first set and the second component comes from the second set. In a one-to-one correspondence, each element of the first set appears exactly once, as does each element of the second set. **12.** Answers vary. Addition is not an appropriate operation on sets. However, if the sets are disjoint, finite sets, then $n(A) + n(B) = n(A \cup B)$. **13.** Answers vary. To use set notation to explain the relation $\leq$, one should use the cardinal number of two sets, one that is a subset, not necessarily proper, of the other.

Review Problems

14. The contrapositive of the contrapositive of a statement is the statement itself. **15. a.** Mary will change the lunch menu. **b.** Samuel stays after school. **c.** The lake is not frozen. **16.** Answers vary. The region paired with the set interpretation of $p - q$ could be where p is true and q is false. The region paired with the set interpretation of $q - p$ could be where q is true but p is false. The region paired with the set interpretation of p *intersection* q could be the region where both p and q are true. Finally the region paired with the set interpretation of the intersection of the complements of p and q could represent the region where both are false. All cases are covered. **17. a.** Answers vary. For example, $\{x \mid 3 < x < 10 \text{ and } x \in N\}$ **b.** $\{15, 30, 45\}$ **18. a.** 6 **b.** 6 **19. a.** 8 **b.** 8 **c.** 12 **d.** 4 **e.** 16, 16 **20.** Answers vary. **21.** 60

Chapter 2 Review

1. Answers vary. Statement: $2 + 5 = 7$. Not a statement: He has red hair. **2.** In (i), every student made an A, B, or C on the final exam, whereas in (ii), at least one student made a C or better. **4. c.** Bach wrote some non-classical music. **5.** Converse: If someone will read a tweet, the whole world is tweeting. Inverse: If the whole world is not tweeting, no one will read a tweet. Contrapositive: If no one will read a tweet, the whole world is not tweeting.

6.

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow q$	$\sim q \rightarrow p$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

7. a.

p	q	$\sim q$	$p \vee \sim q$	$p \vee q$	$(p \vee \sim q) \wedge (p \vee q)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	F	T	F
F	F	T	T	F	F

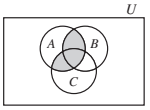
b.

p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge \sim q$	$[(p \vee \sim q) \wedge \sim q] \rightarrow p$
T	T	F	T	F	T
T	F	T	T	T	T
F	T	F	F	F	T
F	F	T	T	T	F

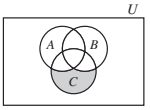
8. a. Alfinia loves Mom and apple pie. **b.** The Washington Monument will eventually crack. **c.** Josef passed the math for elementary teachers course.

9. Answers vary. Pass → Dance; Dance → ~Sit; Sit; Conclusion: ~Pass; Valid: Transitivity and Modus Tollens **10.** Valid: Modus Tollens. **12.** $\varnothing$, $\{m\}$, $\{a\}$, $\{t\}$, $\{h\}$, $\{m, a\}$, $\{m, t\}$, $\{m, h\}$, $\{a, t\}$,$\{a, h\}$, $\{t, h\}$, $\{m, a, t\}$, $\{m, a, h\}$, $\{m, t, h\}$, $\{a, t, h\}$, $\{m, a, t, h\}$

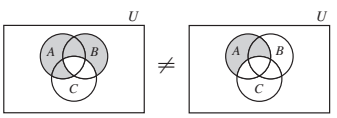
14. a.



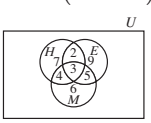
b.



16. a. Answers will vary, for example, $t \leftrightarrow e$, $h \leftrightarrow n$, $e \leftrightarrow d$.
17. It is not true that $A \cap (B \cup C) = (A \cap B) \cup C$ for all A, B, and C, as shown in the following diagrams.



18. a. False, $U = \{1, 2, 3, 4\}, A = \{1, 2\}, B = \{1, 2, 3\}$, $\overline{A} = \{3, 4\}, \overline{B} = \{4\}, \overline{A} \not\subseteq \overline{B}$. **19. a.** False, $U = \{1, 2, 3, 4\}$, $A = \{1, 2, 3\}, B = \{2, 3, 4\}, C = \{2, 3\}, A - B = A - C$, $B \neq C$. **b.** True **21. a.** False. Consider the sets $\{a\}$ and $\{2\}$. **b.** False. It is not a proper subset of itself. **c.** False. Consider the sets $\{t, h, e\}$ and $\{e, n, d\}$. They have the same number of elements, but they are not equal. **d.** False. This can be placed into one-to-one correspondence with the set of natural numbers. **e.** False. The set $\{5, 10, 15, 20, \ldots\}$ is a proper subset of the natural numbers and is equivalent to the natural numbers, since there is a one-to-one correspondence between the two sets. **f.** False. Let $B = \{1, 2, 3\}$ and A be the set of natural numbers. **h.** False. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. **22. a.** Because $A \cup B$ is the union of disjoint sets, $A - B$, $B - A$, and $A \cap B$, the equation is true. **b.** True, because $A - B$ and B are disjoint as well as $B - A$ and A; also, $A \cup B = (A - B) \cup B = (B - A) \cup A$.
26.



27. Answers vary. **a.** $B \cup (C \cap A)$ **b.** $B - C$ or $B \cap \overline{C}$
29. a. Let $A = \{1, 2, 3, \ldots, 13\}$ and $B = \{1, 2, 3\}$; then B is a proper subset of A. Therefore, B has fewer elements than A, so $n(B)$ is less than $n(A)$; thus, $3 < 13$. **b.** Let $A = \{1, 2, 3, \ldots, 12\}$ and $B = \{1, 2, 3, \ldots, 9\}$. B is a proper subset of A, so A has more elements than B; thus $n(A)$ is greater than $n(B)$ and $12 > 9$.

Answers to Now Try This

1. a.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

b. Answers vary, For example suppose p is the statement "I go to the movies" and q is the statement "I drive a car." De Morgan's second law of logic might be interpreted as "It is not the case that I go to the movies or I drive a car" has the same meaning as "I do not go to the movies and I do not drive a car." **2.** A biconditional is true when both statements have the same truth value.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

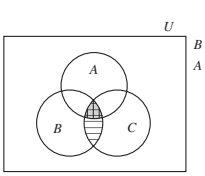
3. a. and **b.** Represent the correspondence as

$$\begin{array}{l} 1 \leftrightarrow \text{A} \\ 2 \leftrightarrow \text{B} \\ 3 \leftrightarrow \text{C} \\ 4 \leftrightarrow \text{D} \end{array} \quad \text{as} \quad \begin{array}{l} 1\ 2\ 3\ 4 \\ \text{ABCD} \end{array}$$

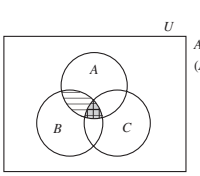
The 24 one-to-one correspondences are

1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
A B C D	B A C D	C A B D	D A B C
A B D C	B A D C	C A D B	D A C B
A C B D	B C A D	C B A D	D B A C
A C D B	B C D A	C B D A	D B C A
A D B C	B D A C	C D A B	D C A B
A D C B	B D C A	C D B A	D C B A

c. We notice that $24 = 4 \cdot 3 \cdot 2 = 4 \cdot 3 \cdot 2 \cdot 1$. We also notice that we had four choices for people to swim in lane 1. After making a choice, we see that we had three choices for the swimmer in lane 2, leaving us with two choices for lane 3 and, finally, one choice for lane 4. Extrapolating from this, we conjecture that there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ distinct one-to-one correspondences between a pair of five-element sets. **5.** Suppose there is a greatest natural number β. If this is the case, then there must be a finite set with β elements in it. Now consider this finite set with one additional element inserted in it. Now there is a set with $\beta + 1$ elements and $\beta + 1$ is greater than β. This is a contradiction and there can be no greatest natural number and hence the set of natural numbers is not finite. **6. a.** 31 **b.** $2^n - 1$, the number of subsets that a set has is 2^n as discussed in the section. Thus the number of proper subsets is 1 less. **7.** It is always true that $A \cap (B \cap C) = (A \cap B) \cap C$. The following figure gives Venn diagrams of each side of this equation. Because the Venn diagrams result in the same set, the equation is always true.

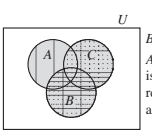


$B \cap C \equiv$
$A \cap (B \cap C)$ ###

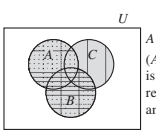


$A \cap B \equiv$
$(A \cap B) \cap C$ ###.

Similarly, it is always true that $A \cup (B \cup C) = (A \cup B) \cup C$. The following Venn diagrams justify this statement.



$B \cup C$ ###
$A \cup (B \cup C)$
is the shaded
region containing
any segments.



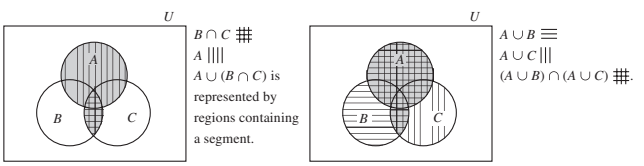
$A \cup B$ ###
$(A \cup B) \cup C$
is the shaded
region containing
any segments.

However, in general, $A - (B - C) \neq (A - B) - C$. To see this, consider the following counterexample:

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{1, 2, 3\}$$
$$C = \{3, 4\}$$

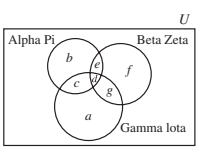
Then $A - (B - C) = A - \{1, 2\} = \{3, 4, 5\}$, but $(A - B) - C = \{4, 5\} - C = \{5\}$. Thus for the preceding choice of A, B, and C, we have that $A - (B - C) \neq (A - B) - C$.

8. Yes, see the Venn diagram for an argument of set equality. The property is the Distributive Property of Set Union over Set Intersection.



Answer to the Preliminary Problem

The information given in the problem can be placed in the following Venn diagram where we know that 5 individuals were not invited. The rest of the numbers of elements in the regions are identified with variables.



Using the problem information we have the following:

1. $c = 9 - d$
2. $g = 5 - d$
3. $e = 10 - d$
4. $a + c + d + g = 12$ or $a + 9 - d + d + 5 - d = 12$ or $a + 14 - d = 12$, which implies $a = d - 2$
5. $d + e + f + g = 21$ or $d + 10 - d + 5 - d + f = 21$ or $15 - d + f = 21$ or $f - d = 6$, which implies $f = d + 6$
6. $b + b + a + e = 20$ or $b + 9 - d + d + 10 - d = 20$ or $b + 19 - d = 20$ or $b - d = 1$, which implies $b = 1 + d$

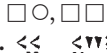
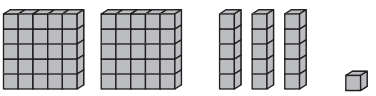
Additionally we have $a + b + c + d + e + f + g + 5 = 37$. Combining all information into the equation 6, we have the following:

$$d - 2 + 1 + d + 9 - d + d + 10 - d + d + 6 + 5 - d + 5 = 37$$
$$d + 34 = 37$$
$$d = 3$$

Thus 3 coeds were invited to join all three groups, and $10 - 3 = 7$ were invited to join Alpha Pi and Beta Zeta but not Gamma Iota.

CHAPTER 3

Assessment 3-1A

1. a. Fifty-six million, two hundred eighty-three thousand, nine hundred fourteen. **b.** Five billion, three hundred sixty-five million, two hundred ninety-five thousand, two hundred thirty-four.
5. a. □, □ Δ, Δ □ ○ **b.** □□, Δ □ ○, □ □ Δ Δ
9. a. MCML; MCMXLVIII. **b.**  ;
c. ; **d.** ; 
15. a. $(1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111)_{\text{two}}$
b. $(1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33)_{\text{four}}$
17. $2032_{\text{four}} = (2 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 2 \cdot 1)_{\text{four}}$
$= 2 \cdot 4^3 + 0 \cdot 4^2 + 3 \cdot 4 + 2 \cdot 1 = 142$
22. 
23. a. 4 quarters, 0 nickels, and 3 pennies. **b.** 73¢, in the fewest number of coins is 2 quarters, 4 nickels, and 3 pennies, so $73 = 243_{\text{five}}$. **28.** $a \cdot 6^3 + b \cdot 6^2 + c \cdot 6^1 + d \cdot 1$ **32.** 1 prize of \$625, 2 prizes of \$125, 1 prize of \$25 **36.** Answers vary; some possibilities might be: **a.** Subtract a number with a 2 in the thousands place and a 2 in the tens place, or 2020. **b.** Subtract a number with a 5 in the tens place, or 50.

Assessment 3-1B

1. a. Two hundred fifty-three million, two hundred forty-three thousand, eight hundred ten **b.** Sixty-two billion, three hundred fifty-five million, two hundred eighty-eight thousand, four hundred thirty-two. **8. a.** $\overline{\text{M}}$DCXXIV. DC is greater than CD. **b.** 30,456. The 3 in 30,456 represents 30,000 while the 3 in 3456 represents only 3000. **c.** . The space indicates that is multiplied by 60; the < without the space represents just 10. **d.** represents 300. The other figure represents 211. **e.** , represents $2 \cdot 20$. The other figure represents 13.
12. a. ; LXXVIII **b.** ; **c.** ;
13. $47; 1 \cdot 27 + 2 \cdot 9 + 1 \cdot 2 = 47$
17. $(2 \cdot 10^3 + 0 \cdot 10^2 + 2 \cdot 10^1 + 2 \cdot 1)_{\text{three}}$
$= 2 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3 + 2$
$= 62$
23. 1 gross, 11 dozen, and 1 unit **24. a.** $EET9E_{\text{twelve}}$: EE_{twelve} is greater than $E0_{\text{twelve}}$. **b.** 1011011_{two}; any seven-digit number is greater than any six-digit number. **c.** 51000_{six}; 1000_{six} is greater than 0555_{six}. **36.** Answers vary.

Mathematical Connections 3-1

Reasoning

1. Answers vary. A number is an abstraction that, whereas a numeral is a written symbol such as 2 or 3 represent the abstract concept of a number. **2.** Answers vary. Place value is based on powers of the base. In the Hindu-Arabic system place value is based on powers of 10. Place value assigns a value to a digit based on its placement in a numeral. To find the value of a numeral we multiply the place value of a digit by its face value. For example in the numeral 3524, the 3 has place value thousands (10^3) and face value 3. **3.** Answers vary. Zero is a place holder in the Hindu-Arabic system. It is used to differentiate between numbers like 54 and 504. If zero were nothing, then we could eliminate it without changing our number system. **4. a.** Answers vary. The Egyptian system has no zero symbol and no place value. It is cumbersome to compute with this system and hard to work with large numbers. **b.** The Babylonian system has too few symbols to be useful. Also the use of the spacing to indicate place value makes the system unusable unless the numbers can be read in context. **c.** The Roman system is fairly complicated. It has no place value. It has no zero symbol. Large numbers are difficult to write. Computation is not easy. **5. a.** Answers vary. We name

our numbers in groups of three digits. Thus the grouping in threes is natural. For example, 323,243. It helps with readability. It has been proposed with the metric system to drop the commas and simply use spaces instead. **b.** Answers vary. **6. (a)** If each symbol in the Roman numeral has a value that is greater than or equal to the symbol to its immediate right, then the value of the numeral is found by adding the values of the symbols. For example, CXXIII has value $100 + 10 + 10 + 1 + 1 + 1 = 123$. **(b)** The only symbols whose value can be subtracted from the value of the symbol to the right are I, X, and C. The value of the symbol to be subtracted must be no less than one-tenth of the value of the symbol to its right, for example, CMIX has value $(1000 - 100) + (10 - 1) = 909$. **(c)** If a bar is placed above a number, its value is multiplied by 1000. For example, $\overline{\text{V}}$ represents 5000.

Open-Ended

7. Answers vary. **8.** Four weights (1, 2, 4, and 8) are needed to check weights from 1 to 15; five weights (1, 2, 4, 8, and 16) are needed to check weights from 1 to 31; six weights (1, 2, 4, 8, 16, and 32) are needed to check weights 1 to 63, and so on.

Cooperative Learning

9. a. and **b.** Answers vary. Most students will probably emulate one of the systems studied in the section.

Connecting Mathematics to the Classroom

10. Answers vary. For example, years 1 to 100 were the first century, years 101 to 200 were the second century, and so on. So years 2001 to 2099 are the 21st century. **11.** Answers vary. The technique that Allie used works with an additive system such as the Egyptian systerm, it works with smaller numbers in the Babylonian system, but it will not work in general with a system that has place value. This would be comparable to adding $41 + 41$ in the Hindu-Arabic system and getting 4411. Consider problems like $97 + 89$ and see what Allie would do. Convert the Babylonian numerals to Hindu-Arabic numerals and compare answers. **12.** Answers vary. The two 3s have the same face value but have different place values. **13.** A base-ten system uses place value and face value. The Roman system uses the additive, subtractive, and multiplicative properties and does not use place value. **14.** In theory, it is correct. However, Romans reserved the bar for numbers greater than 4000. Because M is a special symbol for 1000, it is preferable to write MI for 1001 rather than $\overline{\text{I}}\text{I}$. Because the Romans did not write $\overline{\text{I}}$, by definition it is not correct.

Assessment 3-2A

1. 13 and 4 **4.** Let $A = \{1, 2\}$, $B = \{2, 3\}$, so $A \cup B = \{1, 2, 3\}$. Then $n(A) = 2, n(B) = 2$. But $n(A) + n(B) = 4 \neq 3 = n(A \cup B)$. **7.** Answers vary. A possible answer is 4, 6, 8, 5, 7, 9 **8.** 0, 1 **9. a.** 1, commutative property of addition **b.** 7, commutative property of addition **c.** 0, additive identity **d.** 7, associative and commutative properties of addition **10. b.** Associative property of addition **c.** Commutative property of addition **f.** Associative property of addition **g.** Closure property of addition **11.** No. If $k = 0$ and $a = b = 0$, then $a = b + k$ which implies $a < b$, which would imply that $0 < 0$, which is a contradiction. **12. i.** For any whole numbers a and b, $a < b$ if, and only if, there exists a natural number k such that $b - k = a$ (or equivalently, if, and only if, $b - a$ is a natural number). **ii.** For any whole numbers a and b, $a > b$ if, and only if, there exists a natural number k such that $a - k = b$ (or equivalently, if, and only if, $a - b$ is a natural number).

14.

8	3	4
1	5	9
6	7	2

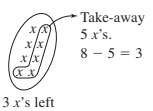
or

8	1	6
3	5	7
4	9	2

17. a. Kent is shortest; Vera is tallest.

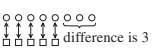
b. Answers vary, as long as the players' heights increase in the order Kent, Mischa, Sally, Vera

19. a. Take-away:

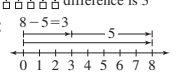


b. Missing addend: $\square + 5 = 8$ implies $\square = 3$

c. Comparison:



d. Number line:

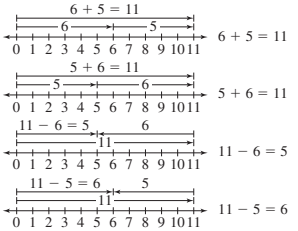


20. Answers vary.

21. a. i. 1 **ii.** 1 **iii.** 1 **b.** Subtract 10 and add 1.

22. a. $5 + 6 = 11; 11 - 6 = 5; 11 - 5 = 6$

b.



24. a. $x + (y + z) = (x + y) + z$ (associative property of addition)
$= z + (x + y)$ (commutative property of addition)

b. $x + (y + z) = (y + z) + x$ (commutative property of addition)
$= y + (z + x)$ (associative property of addition)
$= y + (x + z)$ (commutative property of addition)

26. $a - (b + c)$ or $a - b - c$ **27.** Yes in both cases. A multiple of 5 plus a multiple of 5 is always a multiple of 5.

Assessment 3-2B

7. a. Answers vary. For example, $A = \{2, 5, 7, 8, 9, 10, 11, 12, 13, 14, \ldots\}$ **b.** The set must contain the rest of the whole numbers greater than 1. **8.** All even numbers greater than or equal to 2. **9. a.** 4, commutative property of addition **b.** 5, commutative property of addition **c.** 0, identity property of addition **d.** 4, associative property of addition **10. c.** Commutative property of addition **d.** Associative property of addition

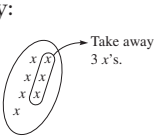
12. $a \geq b$ if, and only if, $a - b$ is a whole number. Alternative: $a \geq b$ if, and only if, $a - b = k$, for some whole number k.

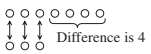
14.

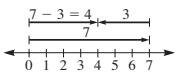
6	7	2
1	5	9
8	3	4

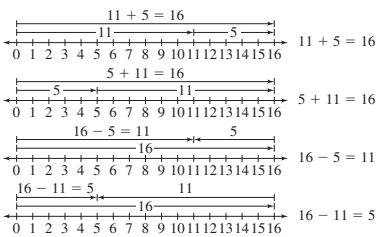
16. b. If the domino is horizontal and the sum of the dots in the right square plus the sum of the dots in the left square is known, and the domino is then turned 180°, the sum is the same.

17. *a*, *b*, *c*, *e*, *d* **19. a.** Take-away:



b. Missing addend: $\square + 3 = 7$ implies $\square = 4$.
c. Comparison: 

d. Number line: 

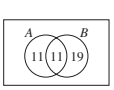
21. a. $11 + 7 = 7 + 11 = 7 + (7 + 4) = (7 + 7) + 4 = 14 + 4 = 18$ **b.** $24 + 28 = 24 + (24 + 4) = (24 + 24) + 4 = 48 + 4 = 52$ **c.** $35 + 37 = 35 + (35 + 2) = (35 + 35) + 2 = 70 + 2 = 72$ **22. a.** $5 + 11 = 16$; $16 - 5 = 11$; $16 - 11 = 5$
b. 

23. a. $a \leq b$ **b.** $a \geq 3$ and $b \geq a - 3$
24. a. $a + (b + c) = (a + b) + c$ (associative property of addition)
$= c + (a + b)$ (commutative property of addition)
b. $a + (b + c) = a + (c + b)$ (commutative property of addition)
$= (c + b) + a$ (commutative property of addition)
27. Yes, a multiple of 7 plus a multiple of 7 is always a multiple of 7.

Mathematical Connections 3-2

Reasoning

1. No; sets A (students taking algebra) and B (students taking biology) do not have to be disjoint. For example, suppose there are 11 students taking both algebra and biology. This implies the following Venn diagram:



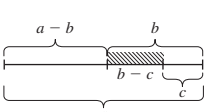
From this Venn diagram we see that $n(A \cup B) = 41$ and there are not necessarily 52 students taking algebra or biology. **2.** The student is using the associative property to think of $9 + (1 + 6)$ as $(9 + 1) + 6$, which is $10 + 6$. She then uses place value to write this as 16. **3.** Answers vary. For example, **(i)** if $a > b$ then $a = b + k$ for some counting number k. Then $a + c = b + c + k$ and therefore $a + c > b + c$. **(ii)** If Adam has a dollars and Bob has b dollars and Adam has more money than Bob and each gets c additional dollars, Adam will still have more than Bob.
4. a. Suppose we have a blocks, and we want to take away b blocks and then take away c more blocks. This is the same as starting with a blocks, taking away c blocks, and then taking away b blocks. **b.** Suppose you have a dollars in your checking account and you spent b dollars in one grocery store and c dollars in another grocery store. Then your balance can be calculated as $a - b - c$ or by finding how much was spent on groceries altogether and subtracting that amount from a; that is, $a - (b + c)$. **c.** Suppose you have two checking accounts with a dollars in one and b dollars in the other. If you spend c dollars your balance in the combined account is $(a + b) - c$. If $b \geq c$ and you write a check of c dollars from the second account, your total balance will be $a + (b - c)$. However, the total balance will be as before. **d.** Notice that $a + b - c = (a + b) - c$ and $a - c + b = (a - c) + b$. The explanation is similar to the one given in part (c) except that the check is written now from the first account. **5.** Answers vary. It is very useful to have students learn more than one method to model addition and subtraction if the methods are to generalize to sets of numbers other than whole numbers. The missing addend approach is useful for all sets of numbers in solving subtraction problems. The method of counting to compute additions is not effective with sets of fractions or real numbers. **6.** Answers vary. For example, you need estimations or mental math to tell if a calculator is correct. You need basic facts to make estimations. **7. a.** When you put 9 and 4 next to each other, it is equal to the same length as 13. **b.** If you put 9 and 4 together on top of 4 and 9 together, they both equal 13. **c.** Take the length 9 away from 13 and the length left is equal to 4. **d.** Take the length 4 away from 13 and the length left is equal to 9. **8.** Answers vary. Subtraction and addition are inverses of each other; that is, they "undo" each other. For example, if we start with 8 objects and subtract 3, then add 3, we are right back where we started. **9.** Yes, the nonempty set $\{0\}$ is closed under subtraction since $0 - 0 = 0$. It is the only such subset.
10.

1	2	3
8	9	4
7	6	5

Explantions vary.

Open-Ended

11. Answers vary. Colored rods may be used to teach addition to students using the principle of counting on. Choose the rod representing the greater number and make a train of rods using unit rods until the second number is added. The total train represents the sum. **12.** Answers vary. For example, let $A = \{a, b\}$ and $B = \{a, b, c, d\}$. Then $4 - 2 = n(B - A) = n(\{c, d\}) = 2$. **13. a.** Answers vary. For example, on Friday 25 tickets to the play were sold. On Saturday, 8 more tickets were sold. How many tickets were sold altogether? **b.** Answers vary. For example, Tom traveled 25 mi on the first day of his trip. On the second day he traveled 8 miles. How many miles did he travel in the first two days of his trip? **c.** Answers vary. For example, the following diagram can be used:



Cooperative Learning

14. a. The table shows that if you add any two single-digit whole numbers, your answer is also a whole number. **b.** The table shows that if you add any two single-digit whole numbers, then the order is not important; that is, if $a \in W$ and $b \in W$, $a + b = b + a$. Each row of answers has a corresponding column of identical answers. **c.** The first row and first column show that if you add any number to the identity, 0, you get the identical number. **d.** The properties reduce the number of facts to be remembered. For example, the 19 facts in the first row and column can be learned by just knowing

that 0 is the additive identity. The commutative property also reduces the number of facts; for example, if you know $9 + 2$, then you know $2 + 9$. **15. a.** Yes, if any two elements of $\{a, b, c, d\}$ are chosen and the operation Δ is performed, the result belongs to the original set. **b.** Yes, for any elements x and y belonging to $\{a, b, c, d\}$, we have $x \,\Delta\, y = y \,\Delta\, x$. **c.** Yes, a **d.** For any elements x, y, z belonging to $\{a, b, c, d\}$, we have $x \,\Delta\, (y \,\Delta\, z) = (x \,\Delta\, y) \,\Delta\, z$. **16.** Answers vary.

Connecting Mathematics to the Classroom

17. It should be noted that 0 would be the identity for subtraction if, and only if, $0 - a = a$ and $a - 0 = a$ for all a were true which it is not. **18.** Answers vary. For example, the take-away model or the missing-addend model do not work for all types of subtraction problems. The model that is appropriate in this case is the *comparison* model. The pencils Karly has can be matched to some of the pencils that Sam has and then the answer is the number of Sam's pencils remaining. **19.** Answers vary. For example, the examples that the student is showing are all true, but producing examples that work is not proof that the claim is true in general. All that is needed is to produce one *counterexample* to show that the claim is not true. If we consider $5 - 8$, we can see that there is no whole number that satisfies this subtraction. If the set were closed under subtraction, then if any two elements were picked and one subtracted from the other, the answer would have to be a whole number. In the case of $5 - 8$, this does not happen. **20.** Answers vary. For example, John is correct in that he can get the same answer by adding up or adding down. He is using the fact that if a, b, and c are whole numbers, then $(a + b) + c = (c + b) + a$. The commutative and associative properties can be used to justify the steps given below.

$$\begin{aligned} a + b + c &= (a + b) + c = c + (a + b) \\ &= c + (b + a) = (c + b) + a \end{aligned}$$

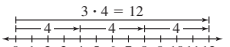
21. Answers vary. You can discuss the difference between 0 and *nothing* emphasizing that 0 is a number that tells "how many." Computations $2 - 1, 2 - 2$ might be addressed first and the need for a single answer to problems. A number line might be used to show that in $2 - 4$ you end up moving 2 units past 0 on the number line in the negative direction, and so there is a need for numbers to represent these points. The missing addend approach might be used. For example, $5 - 2 = x$ if, and only if, $5 = x + 2$. So $2 - 4 = x$ if and only if $2 = 4 + x$. We can see that 0 does not work as an answer. **22.** Answers vary. When you turn the numbers around you are looking at a different problem. For example, if you start out with 4 cookies and give a cookie to each of two friends, this is very different than starting out with 2 cookies and trying to give a cookie to each of 4 friends. **23.** In the first case we have $8 - (3 + 2)$, and in the second case we have $8 - (2 + 3)$. This is the commutative property of addition. The commutative property of subtraction does not hold; for example, $8 - 3 \neq 3 - 8$. **24.** You can show she in not correct by doing a physical example. Suppose Jane and Mark each have 5 pencils. After Jane gave Mark two pencils she now has 3 pencils and Mark has 7 pencils for a difference of 4 pencils. Jane is decreasing her total by 2 pencils and Mark is increasing his total by two pencils so the difference is $2 + 2 = 4$ pencils.

Review Problems

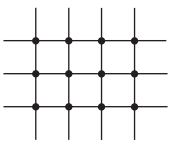
25. $5 \cdot 10^3 + 2 \cdot 10^2 + 8 \cdot 10 + 0 \cdot 1$ **26.** 1410 **27. a.** 1594 **b.** 11 **c.** 23 **28.** 10111_{twelve}

Assessment 3-3A

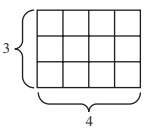
1. a.



b.



c.



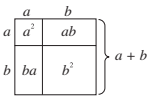
d.
$$\begin{aligned} A &= \{a, b, c\} \\ B &= \{x, y, z, w\} \\ A \times B &= \{(a, x), (a, y), (a, z), (a, w), (b, x), (b, y), \\ &\quad (b, z)\,(b, w), (c, x), (c, y), (c, z), (c, w)\} \\ n(A) &= 3, n(B) = 4 \\ n(A \times B) &= 3 \cdot 4 = 12 \end{aligned}$$

2. a. $\{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z)\}$

6. a. No; $2 + 3 = 5$ and 5 does not belong to the set. **b.** Yes. There will be no numbers in the set that will multiply to give a product of 5. **12. f.** Distributive property of multiplication over addition **14. a.** Distributive property of multiplication over addition **b.** $32 \cdot 12 = 32(10 + 2) = 320 + 64 = 384$

16. a.
$$\begin{aligned} (a + b)^2 &= (a + b)(a + b) = a(a + b) + b(a + b) \\ &= a^2 + ab + ba + b^2 = a^2 + 2ab + b^2 \end{aligned}$$

b.



The area of a square with sides $a + b$ can be expressed as $(a + b) \cdot (a + b)$, and also as the sum of areas of the four regions: two squares, $a \cdot a$ and $b \cdot b$, and two rectangles $a \cdot b$ and $b \cdot a$. Thus:

$$\begin{aligned} (a + b)^2 &= a \cdot a + a \cdot b + b \cdot a + b \cdot b \\ &= a^2 + 2ab + b^2. \end{aligned}$$

17. The question is really to show that the area of the large square minus the area of the small square is the same as the area of the four rectangles. The area of the large square is $(a + b)^2$. The area of the small square is $(a - b)^2$. The difference between the two areas is the four shaded rectangles, each with area ab; the total area of the four is $4ab$. Therefore $(a + b)^2 - (a - b)^2 = 4ab$.

18. a.
$$\begin{aligned} 51^2 &= (50 + 1)(50 + 1) = 50^2 + 2 \cdot 50 \cdot 1 + 1^2 \\ &= 2500 + 100 + 1 = 2601 \end{aligned}$$

b.
$$\begin{aligned} 102^2 &= (100 + 2)(100 + 2) = 100^2 + 2 \cdot 100 \cdot 2 + 2^2 \\ &= 10{,}000 + 400 + 4 = 10{,}404 \end{aligned}$$

19. a.
$$\begin{aligned} (ab)c &= c(ab) \text{ (commutative property of multiplication)} \\ &= (ca)b \text{ (associative property of multiplication)} \end{aligned}$$

b.
$$\begin{aligned} (a + b)c &= c(a + b) \text{ (commutative property of multiplication)} \\ &= c(b + a) \text{ (commutative property of addition)} \end{aligned}$$

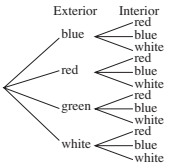
23. $72/8 = 9; 72/9 = 8; 8 \cdot 9 = 72; 9 \cdot 8 = 72$ **24.** The process always results in the original number.

Step 1: "Think of a number." Name this number x.
Step 2: $5x$
Step 3: $5x + 5$
Step 4: $\dfrac{5x + 5}{5} = x + 1$
Step 5: $x + 1 - 1 = x$

25. Answers vary, for example: **a.** There is no associative property of division; e.g., $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$. **b.** There is no distributive property of division; e.g., $8 \div (2 + 2) \neq (8 \div 2) + (8 \div 2)$. **26.** Answers vary. For example, suppose there are two bags of marbles containing a marbles in one and b marbles in the other. It is desired to divide the marbles equally among c boys. Then the number of marbles that each boy receives can be found in two different ways: **(i)** Put all the marbles in one bag. There will be $a + b$ marbles, and each boy will receive $(a + b) \div c$ marbles.

(ii) Divide the marbles in the first bag first and then the second. Each boy would receive $(a \div c) + (b \div c)$ marbles. Another way to show this is to let $a \div c = x$ and $b \div c = y$. Then $a = cx$ and $b = cy$. So $a + b = cx + cy = c(x + y)$. By the definition of division: $(x + y) = (a + b) \div c$. Substituting for x and y: $(a \div c) + (b \div c) = (a + b) \div c$.

28. Answers vary. One possibility is:

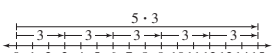


This results in $4 \cdot 3 = 12$ color schemes. **34.** Yes, for example, $3 \# 2 = 3 + 2 + 4 = 9$ and $2 \# 3 = 2 + 3 + 4 = 9$. In general, if, $a, b \in W$ then $a \# b = a + b + 4 = b + a + 4 = b \# a$

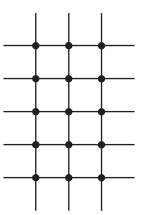
35. a. 625 **b.** n^2 **c.** 32

Assessment 3-3B

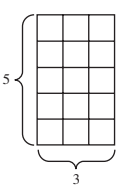
1. a.



b.



c.



d. $A = \{a, b, c, d, e\}$
$B = \{x, y, z\}$
$A \times B = \{(a,x), (a,y), (a,z), (b,x), (b,y), (b,z), (c,x), (c,y), (c,z), (d,x), (d,y), (d,z), (e,x), (e,y), (e,z)\}$
$n(A) = 5, n(B) = 3, n(A \times B) = 5 \cdot 3 = 15$

2. a. $A \times B = \{(a,x), (a,y), (a,z), (b,x), (b,y), (b,z), (c,x), (c,y), (c,z), (d,x), (d,y), (d,z)\}$
b. 12 **c.** $4 \cdot 3 = 12$ **6. a.** No, $1 + 1 = 2$ **b.** Yes, the product of two whole numbers greater than 1 is greater than one and still a whole number. Also, any whole number times 0 is 0 which belongs to this set. **12. e.** Distributive properly of multiplication over addition **f.** Distributive property of multiplication over addition **13. a.** Commutative property of multiplication
b. Distributive property of multiplication over addition
14. a. Distributive property of multiplication over addition.
b. $8 \cdot 34 = 8(30 + 4) = 8 \cdot 30 + 8 \cdot 4 = 240 + 32 = 272$
15. a. $15(10 - 2) = 15 \cdot 10 - 15 \cdot 2 = 150 - 30 = 120$
b. $30(9 - 2) = 30 \cdot 9 - 30 \cdot 2 = 270 - 60 = 210$

16. The area of the shaded region is $a(b - c)$, but it can also be expressed as the area of the large rectangle minus the area of the small rectangle. **17.** The first two figures on the left illustrates a^2, viewed as the area of the large square, minus b^2, viewed as the area of the small square on the bottom right. The figure on the right illustrates how the pieces can be rearranged to form a rectangle with height $a + b$ and length $a - b$.

18. a. $19 \cdot 21 = 21 \cdot 19 = (20 + 1)(20 - 1) = 20^2 - 1 = 400 - 1 = 399$
b. $25 \cdot 15 = (20 + 5)(20 - 5) = 20^2 - 5^2 = 400 - 25 = 375$
c. $99 \cdot 101 = 101 \cdot 99 = (100 + 1)(100 - 1) = 100^2 - 1^2 = 10000 - 1 = 9999$
d. $101^2 - 99^2 = (101 + 99)(101 - 99) = (200)(2) = 400$

19. a. $(ab)c = (ba)c$ (commutative property of multiplication)
$= b(ac)$ associative property of multiplication)
b. $a(b + c) = ab + ac$ (distributive property of multiplication over addition)
$= ac + ab$ (commutative property of addition)

20. a. Answers vary. $y(x - 1)$. **b.** $(x + 1)(y - 1)$
c. $ab^2(ab - 1)$ **23.** $30/6 = 5; 30/5 = 6; 5 \cdot 6 = 30; 6 \cdot 5 = 30$ **24. (a)** The original number is returned. **(b)** Yes, $x \to 2x \to 2x + 2 \to x + 1 \to x$. **25.** Answers vary, for example: **a.** There is no commutative property in division; e.g., $(8 \div 4) \neq (4 \div 8)$. **b.** There is no commutative property in subtraction; e.g., $8 - 4 \neq 4 - 8$. **26.** Write $(a - b) \div c = d$, where d is a whole number. This can be written as a multiplication problem $a - b = cd$. Similarly, $b \div c$ can be written as a multiplication problem $b = ck$. Combining this, we have $a - b = cd \Rightarrow a = cd + b \Rightarrow a = cd + ck \Rightarrow a = c(d + k) \Rightarrow a \div c = d + k$. Recall that $b \div c = k$. So, $a \div c = d + k \Rightarrow a \div c - k = d \Rightarrow a \div c - b \div c = d = (a - b) \div c$.
34. Yes, for example, $3\#5 = 0 = 5\#3$; $3\#4 = 0 = 4\#3$; $4\#3 = 0 = 3\#4$; $2\#4 = 1 = 4\#2$.

Mathematical Connections 3-3

Reasoning

1. Answers vary. For example, a number is odd if, and only if, it has the form $2k + 1$ for some whole number k. Hence the product of two odd numbers can be written as $(2k + 1)(2m + 1) = 2(2km + k + m) + 1$, an odd number. **2.** 0 is not the identity for multiplication because if it were, then any number times 0 must be that number. The result in every case is 0, not the number multiplied by 0. **3.** Answers vary. For example, you might think of $9 \cdot 7$ as $7 \cdot 9$ and see if that helps. You might think of $9 \cdot 7$ as $9 \cdot 6 + 9$ or $54 + 9 = 63$ or $9 \cdot 7 = 9 \cdot 5 + 9 \cdot 2 = 45 + 18 = 63$. Also $9 \cdot 7 = (10 - 1)7 = 70 - 7 = 63$.
4. No, this is not the case when $x = 0$. **5.** Yes, this is the case when x is either 0 or 1. **6.** Let x and y be the whole numbers. Then $x = 0$ and $y = 0$, or $x = 2$ and $y = 2$.
7. Answers vary. Division with remainder 0 and multiplication are related. Division is the inverse of multiplication and vice versa; that is, these operations "undo" each other. For example, $(15 \div 3) \cdot 3 = 15$ and $(15 \cdot 3) \div 3 = 15$. **8.** You could factor out 37 and then perform the easy addition and multiplication; $37 \cdot 52 + 37 \cdot 48 = 37(52 + 48) = 37(100) = 3700$.

Open-Ended

9. Answers vary. **10.** Answers vary. For example, a taxi driver charges \$3 for entering a cab and \$2 per minute for 6 minutes. (The prices here are unrealistic, but this is the type of problem students may suggest.)

Cooperative Learning

11. a. Answers vary. **b.** Answers vary. We know that if $a \div b = c$, then $a = bc$, where a, b, and c are whole numbers and $b \neq 0$. Therefore, to find $35 \div 5$ we look in the table and go down to the 5 row, then over to 35, and then to find the other factor we go up to 7. Therefore, $35 \div 5 = 7$. **c.** The only way for a product of two numbers to be odd is if the two factors are odd. If the odd number is not in the first row nor in the first column, the eight products surrounding it all involve an even factor and hence are even. If the odd number is in the first row or first column the

reasoning is similar. **12. a.** Yes **b.** 18 and 19 **c.** Even **d.** Some reach 1, but it is not known if it terminates for all numbers.

Connecting Mathematics to the Classroom

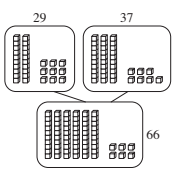
13. The argument is not valid. By definition of division, $0 \div 0 = x$ if, and only if, $0 = 0 \cdot x$ has a unique solution. But the last equation is true for all whole numbers x. Because the equation has no unique solution, $0 \div 0$ is not meaningful. **14.** You could encourage Sue to substitute numbers for a and b to see if her claim is true. For example, if $a = 2$ and $b = 4$, then $3(2 \cdot 4) = 3 \cdot 8 = 24$ and $(3 \cdot 2)(3 \cdot 4) = 6 \cdot 12 = 72$. Therefore, we have a counterexample to show Sue's claim is false. At this point the correct associative and distributive properties could be demonstrated. **15.** This is true as long as $b \neq 0$. By definition of division, $(ab) \div b = x$ if, and only if, $ab = xb$ has a unique solution. If $b \neq 0$, the unique solution is a. **16.** Since $1 \div a$ is not defined in the set of whole numbers, $1 \div a \neq a$ and therefore 1 is not the identity for division. **17.** Have Amy consider whether the sets $\emptyset$, $\{0\}$, $\{1\}$, $\{0, 1\}$ are closed under multiplication. **18.** By the distributive property of multiplication over addition $a(b + c) = ab + ac \neq a + bc$. A counterexample could be used, for example, $3 + 9 \cdot 7 = 3 + 63 = 66$ and $3(9 + 7) = 3 \cdot 16 = 48$.

Review Problems

19. Answers vary; an example would be $\{2, 6, 10, 14, \ldots\}$. **20.** No. For example, $5 - 2 \neq 2 - 5$. **21.** In base two there are only two digits, 0 and 1. The base-ten number 2 is represented by 10_{two}. **22.** 10000_{three}

Assessment 3-4A

2.



3. a.
$$\begin{array}{r} \boxed{7}\boxed{6}\boxed{2} \\ +\ \boxed{8}\boxed{5}\boxed{3} \\ \hline 1\ 6\ 1\ 5 \end{array}$$

b.
$$\begin{array}{r} \boxed{2}\boxed{6}\boxed{7} \\ +\ \boxed{3}\boxed{5}\boxed{8} \\ \hline 6\ 2\ 5 \end{array}$$

4.
$$\begin{array}{r} \boxed{3}\boxed{4}\boxed{2}\boxed{8} \\ +\ \boxed{5}\boxed{6}\boxed{3}\boxed{1} \\ \hline 9\ 0\ 5\ 9 \end{array}$$

5. b. $321 - 38 \Rightarrow 321 + 2 - (38 + 2) \Rightarrow 323 - 40 \Rightarrow 323 + 60 - (40 + 60) \Rightarrow 383 - 100 = 283$

6. a. i.
$$\begin{array}{r} 6\ 8\ 7 \\ +\ 5\ 4\ 9 \\ \hline 1\ 6 \\ 1\ 2 \\ \hline 1\ 1 \\ \hline 1\ 2\ 3\ 6 \end{array}$$

ii.
$$\begin{array}{r} 3\ 5\ 9 \\ +\ 6\ 7\ 3 \\ \hline 1\ 2 \\ 1\ 2 \\ \hline 9 \\ \hline 1\ 0\ 3\ 2 \end{array}$$

b. The algorithm works because the placement of partial sums under their addends accounts for place value. **7.** Answers vary; some possibilities are: **a.** $8 + 5 = 13$ and 13 was written with no regrouping, $2 + 7 = 9$ and 9 was simply placed in front of the 13. **b.** $8 + 5 = 13$, but instead of writing 3 and regrouping with the 1, the 1 was written, and the 3 was regrouped. **c.** Only the absolute difference in the units $(9 - 5 = 4)$, tens $(5 - 0 = 5)$, and the hundreds $(3 - 2 = 1)$ was recorded, without taking into account the signs of the numbers. **d.** Three hundreds was regrouped as 2 hundreds and 10 tens, but 10 tens was not regrouped as $9 \cdot 10 + 15$ in order to obtain $15 - 9 = 6$ in the ones place. **8.** By dinner time Tom had consumed $90 + 120 + 119 + 185 + 110 + 570 = 1194$ calories. Subtracting 1194 from 1500 gives $1500 - 1194 = 306$. Tom may have fish or salad, but not both. (He may have tea with either.)

9. Step 1 → Expanded form;
Step 2 → Commutative and associative properties of addition;
Step 3 → Distributive property of multiplication over addition;
Step 4 → Closure property of addition, one digit addition facts;
Step 5 → Expanded form condensed.

10. a.
$$\begin{aligned} 66 + 23 &= (6 \cdot 10 + 6) + (2 \cdot 10 + 3) \\ &= (6 \cdot 10 + 2 \cdot 10) + (6 + 3) \\ &= (6 + 2) \cdot 10 + (6 + 3) \\ &= 8 \cdot 10 + 9 \\ &= 89 \end{aligned}$$

b.
$$\begin{aligned} 124 + 235 &= (1 \cdot 100 + 2 \cdot 10 + 4) \\ &\quad + (2 \cdot 100 + 3 \cdot 10 + 5) \\ &= (1 \cdot 100 + 2 \cdot 100) \\ &\quad + (2 \cdot 10 + 3 \cdot 10) + (4 + 5) \\ &= (1 + 2) \cdot 100 + (2 + 3) \cdot 10 \\ &\quad + (4 + 5) \\ &= 3 \cdot 100 + 5 \cdot 10 + 9 \\ &= 359 \end{aligned}$$

11. a.
$$\begin{array}{r} 4\ 3\ 5\ 8 \\ +\ 3\ 8\ 6\ 4 \\ \hline \not{0}\not{7}\ \not{1}\not{1}\ \not{1}\not{1}\ \not{1}\not{2} \\ 8\ 2\ 2\ 2 \end{array}$$

b.
$$\begin{array}{r} 4\ 9\ 2\ 3 \\ +\ 9\ 8\ 9\ 7 \\ \hline \not{1}\not{3}\ \not{1}\not{7}\ \not{1}\not{1}\ \not{1}\not{0} \\ 1\ 4\ 8\ 2\ 0 \end{array}$$

13.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

a. 474_{eight}

b. 667_{eight}

16. a.
$$\begin{array}{r} {\scriptstyle 1}\quad {\scriptstyle 1} \\ 4\ 3\ 2 \\ \not{9}_{4}\ \not{7}_{1}\ 6 \\ +\ {}^{1}1\ 4\ 1\ \not{8} \\ \hline 2\ 8\ 2\ 6 \end{array}$$

b.
$$\begin{array}{r} {\scriptstyle 3}\quad {\scriptstyle 3} \\ \not{3}_{1}\ 2_{\text{five}} \\ 1\ \not{3}_{0}\ _{\text{five}} \\ 2\ 2\ _{\text{five}} \\ \not{4}_{3}\ \not{3}_{0}\ _{\text{five}} \\ 2_{0}\ 3\ _{\text{five}} \\ +\ 1\ 2_{0}\ _{\text{five}} \\ \hline 3\ 1\ 0\ _{\text{five}} \end{array}$$

Answer to the Preliminary Problem

The table below shows the differences in the scores of each pair of students.

Ordered Pair (a, b)	$a - b$
(20, 17)	$20 - 17 = 3$
(15, 18)	$15 - 18 = {}^{-}3$
(21, 19)	$21 - 19 = 2$
(17, 22)	$17 - 22 = {}^{-}5$
(18, 18)	$18 - 18 = 0$
(19, 15)	$19 - 15 = 4$
(16, 14)	$16 - 14 = 2$
(20, 18)	$20 - 18 = 2$
(21, 20)	$21 - 20 = 1$
(18, 25)	$18 - 25 = {}^{-}7$

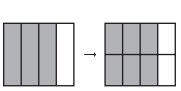
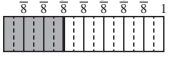
There are fewer negative differences than positive differences so that would be a case for saying the experiment worked. However, the sum of all the differences is

$$3 + {}^{-}3 + 2 + {}^{-}5 + 0 + 4 + 2 + 2 + 1 + {}^{-}7 = {}^{-}1.$$

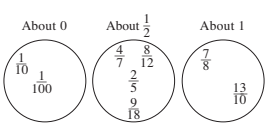
Because the sum of all the differences is negative, it would be very difficult to make a case for the experiment being a success.

CHAPTER 6

Assessment 6-1A

1. Answers vary. **a.** The solution to $8x = 7$ is $\frac{7}{8}$. **b.** Jane ate $\frac{7}{8}$ of the pizza. **c.** The ratio of boys to girls is 7 to 8. **2. a.** $\frac{1}{6}$ **b.** $\frac{1}{4}$ **c.** $\frac{2}{6} = \frac{1}{3}$ **d.** $\frac{7}{12}$ **3.** $\frac{5}{8}$ **4. a.** $\frac{2}{3}$ **b.** $\frac{4}{6} = \frac{2}{3}$ **c.** $\frac{6}{9} = \frac{2}{3}$ **d.** $\frac{8}{12} = \frac{2}{3}$. The diagrams illustrate the Fundamental Law of Fractions. **5. a.** No, the parts do not have equal areas. The shaded part could be $\frac{1}{4}$ of the circle, but we can't tell from the figure. **b.** Yes **c.** Yes **6.** Answers vary. **a.**  $\frac{2}{8}$ **b.**  $\frac{3}{9}$ **c.**  $\frac{3}{6}$ **7. a.** $\frac{9}{24} = \frac{3}{8}$ **b.** $\frac{12}{24} = \frac{1}{2}$ **c.** $\frac{4}{24} = \frac{1}{6}$ **d.** $\frac{8}{24} = \frac{1}{3}$ **8. a.** $\frac{4}{43}$ **b.** $\frac{17}{43}$ **c.** 1 **d.** $\frac{39}{43}$ **9.** Answers vary. **a.** $\frac{4}{18}, \frac{6}{27}, \frac{8}{36}$ **b.** $\frac{{}^{-}4}{10}, \frac{2}{{}^{-}5}, \frac{{}^{-}10}{25}$ **c.** $\frac{0}{1}, \frac{0}{2}, \frac{0}{4}$ **d.** $\frac{2a}{4}, \frac{3a}{6}, \frac{4a}{8}$ **10. a.** $\frac{52}{31}$ **b.** $\frac{3}{5}$ **c.** $\frac{{}^{-}5}{7}$ **11. a.** Undefined **b.** Undefined **c.** 0 **d.** Cannot be simplified **e.** Cannot be simplified **12. a.** $\frac{a-b}{3}$ **b.** $\frac{2x}{9y}$ **13. a.** Equal **b.** Equal **14. a.** Not equal **b.** Not equal **15.** Answers vary.  **16.** $\frac{36}{48}$ **17. a.**  **b.** 14 gallons **18. a.** $\frac{32}{3}$ **b.** ${}^{-}36$ **19. a.** $>$ **b.** $<$ **20. a.** $\frac{11}{13}, \frac{11}{16}, \frac{11}{22}$ **b.** $\frac{{}^{-}1}{5}, \frac{{}^{-}19}{36}, \frac{{}^{-}17}{30}$ **21.** Answers vary. **a.** $\frac{10}{21}, \frac{11}{21}$ **b.** $\frac{{}^{-}22}{27}, \frac{{}^{-}23}{27}$ **22. a.** $\frac{6}{16} = \frac{3}{8}; \frac{6}{32{,}000} = \frac{3}{16{,}000}$ **b.** $\frac{10}{100} = \frac{1}{10}$ **c.** $\frac{15}{60} = \frac{1}{4}$ **d.** $\frac{8}{24} = \frac{1}{3}$ **23.** False **24.** $\frac{1}{6}$ **25.** Answers vary. **a.** 2, 3, 4 **b.** 3, 5, 7, 9 **26.** n cannot be 0; otherwise there would $\frac{0}{0}$ in the theorem. **27.** $n = \frac{n}{1}$ for all integers n, so every integer can be written as a fraction. **28.** Drawing is only a representation, not to scale.  **29.** 2 **30.** $\frac{3}{5}$

Assessment 6-2A

1. a. $\frac{7}{6} = 1\frac{1}{6}$ **b.** $\frac{{}^{-}4}{12} = \frac{{}^{-}1}{3}$ **c.** $\frac{5y - 3x}{xy}$ **d.** $\frac{-3y + 5x + 14y^2}{2x^2y^2}$ **e.** $\frac{71}{24} = 2\frac{23}{24}$ **f.** $\frac{{}^{-}23}{3} = -7\frac{2}{3}$ **g.** $8\frac{4}{12} = 8\frac{1}{3}$ **2. a.** $18\frac{2}{3}$ **b.** ${}^{-}2\frac{93}{100}$ **3.** Answers vary. **a.** $\frac{27}{4}$ **b.** $\frac{{}^{-}29}{8}$ **4.** Answers vary. **a.** $\frac{1}{3}$, high **b.** $\frac{1}{6}$, low **c.** $\frac{3}{4}$, low **d.** $\frac{1}{2}$, high **5. a.** Beavers **b.** Ducks **c.** Bears **6.** 

7. a. $\frac{1}{2}$, high **b.** 0, low **c.** $\frac{3}{4}$, high **d.** 1, high **8. a.** 2 **b.** $\frac{3}{4}$ **9. a.** $\frac{1}{4}$ **b.** 0 **10. a.** A **b.** H **c.** T **d.** H **11.** Answers vary. Approximately $4 \cdot 3 = 12$ **12.** $\frac{1}{4}$ **13.** $6\frac{7}{12}$ yd **14.** $2\frac{5}{6}$ yd **15.** Answers vary. **a.** $\frac{1}{2} + \frac{3}{4} \in Q$ **b.** $\frac{1}{2} + \frac{3}{4} = \frac{3}{4} + \frac{1}{2}$ **c.** $\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4} = \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$ **16.** $1, \frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \frac{10}{6}, \frac{11}{6}, 2$ **17. a. i.** $\frac{1}{4} + \frac{1}{3 \cdot 4} = \frac{1}{4} + \frac{1}{12} = \frac{16}{48} = \frac{1}{3}$ **ii.** $\frac{1}{5} + \frac{1}{4 \cdot 5} = \frac{1}{5} + \frac{1}{20} = \frac{25}{100} = \frac{1}{4}$ **iii.** $\frac{1}{6} + \frac{1}{5 \cdot 6} = \frac{1}{6} + \frac{1}{30} = \frac{36}{180} = \frac{1}{5}$ **b.** $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$ **18. a.** $\frac{5}{6}$ **b.** $\frac{21}{6} = 3\frac{1}{2} = \frac{7}{2}$ **19.** $\frac{1}{2}$ mi **20.** No, you need $\frac{1}{4}$ cup more milk.

13.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	10
2	2	3	4	5	10	11
3	3	4	5	10	11	12
4	4	5	10	11	12	13
5	5	10	11	12	13	14

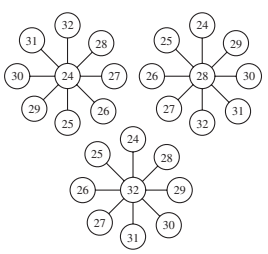
a. 43_{six} **b.** 154_{six}

16. a.
$$\begin{array}{r} {}^{1}\,{}^{2} \\ 5\ 3\ 7 \\ 3\ 1\ 5 \\ +\ {}^{1}2\ 3\ 4\ 3 \\ \hline 3\ 2\ 0\ 0 \end{array}$$

b.
$$\begin{array}{r} {}^{2}4_{0}\ 1_{six} \\ 3\ 2_{six} \\ 2\ 2_{six} \\ 4_{3}\ 3_{2_{six}} \\ 2\ 2_{six} \\ +\ 5_{4}\ 4_{2_{six}} \\ \hline 3\ 4\ 2_{six} \end{array}$$

17. There is no numeral 6 in base six. $23_{six} + 43_{six} = 110_{six}$.

20. a.



23. Step 1 → Associative property of addition;
Step 2 → Commutative property of addition;
Step 3 → Associative property of addition;
Step 4 → Closure property of whole-number addition.

24. a. $997 - 30 = 967$; $967 - 2 = 965$ **b.** $560 + 100 = 660$; $660 + 30 = 690$; $690 + 6 = 696$

25. Answers vary.
a. $75 - 38 \Rightarrow (75 + 2) - (38 + 2) = 77 - 40 = 37$
b. $57 + 35 \Rightarrow (57 + 3) + (35 - 3) = 60 + 32 = 92$
c. $137 - 29 \Rightarrow (137 + 1) - (29 + 1) = 138 - 30 = 108$
d. $78 + 49 \Rightarrow (78 + 2) + (49 - 2) = 80 + 47 = 127$

26. a. $63 + 7 = 70$; $70 + 4 = 74$
Therefore, $7 + 4 = 11$ is the difference.
b. $57 + 3 = 60$; $60 + 10 = 70$; $70 + 3 = 73$
Therefore, $3 + 10 + 3 = 16$ is the difference.

29. a. $2 + 5 + 4 + 0 = 11 \Rightarrow 11{,}000$
$11{,}000 + 1700\ (\text{adjust}) = 12{,}700$
b. $3 + 5 + 6 + 2 = 16 \Rightarrow 1600$
$1600 + 200\ (\text{adjust}) = 1800$

30. a. (i) No, the numbers are not clustered. **(ii)** Yes, the numbers are clustered around 2000, so $5 \cdot 2{,}000 = 10{,}000$. **b.** Estimates vary. **31.** Estimates may vary: **a.** The range is $100 + 700 = 800$ to $200 + 800 = 1000$. Then $800 < (123 + 780) < 1000$. **b.** The range is $400 + 200 = 600$ to $500 + 300 = 800$. Then $600 < (482 + 246) < 800$. **32.** Answers vary. For example, $65 + 45$ is about 100 and $190 + 212$ is about 400, so $100 + 400 + 420 = 920$. Then 1237 is about 1220 and $1220 - 920 = 300$. Therefore an estimate is about 300.

33. a. Yes. Rounding, Josh plans to write checks for about \$40, \$30, \$60, and \$250, or about \$380. Since the check amounts were rounded up, he will have enough. **b.** Yes, $\$981 + \$1140$ is greater than $\$900 + \$1100 = \$2000$. **c.** Alberto. He received 10 more votes than Juan from the first district, but only 1 less from the second.

34. Answers vary. Possibilities include:

Rounding

$$\begin{array}{r} 71{,}000 \\ 65{,}000 \\ 68{,}000 \\ 73{,}000 \\ 85{,}000 \\ +\ 70{,}000 \\ \hline 432{,}000 \end{array}$$

Trading off

Observe that the high attendance on Friday can be distributed among the other day's attendance for approximately 72,000 visitors per day. $72{,}000 \times 6 = 432{,}000$.

Mathematical Connections 3-4

Reasoning

1. Answers vary. This approach emphasizes the meaning of place value of the digits and may be easier for young children than the standard algorithm. It can also serve as a transition to the standard algorithm. **2.** Answers vary. As in problem 1, this approach may serve as a transition from a concrete base-ten blocks approach to the standard algorithm. **3.** Answers vary. For example, the "scratch marks" represent the normal "carries" and is just a way to keep track of them. **4. a.** You first subtract 200, leaving 385. Then subtract 100, leaving 285. At this point, we have subtracted $100 - 77 = 23$ too much, so we add $23 + 285$ to obtain 308.

4. b.
$$\begin{aligned} 787 - 398 &= 787 - 300 - 98 \\ &= 487 - 98 \\ &= 487 - 100 + 2 \\ &= 387 + 2 \\ &= 389 \end{aligned}$$

5. Answers vary. The terms *regroup* and *trade* match more closely the physical actions done with manipulatives. **6.** Answers vary. The front-end estimation is almost always less than the exact sum because each of the estimates is usually less than the actual values being added. The only case where this is not true is when each of the front-end numbers that is used is followed by zeroes. In this case, the estimate would equal the exact sum. **7.** Answers vary. For example, students might suggest that mental computation mathematics and estimation are necessary every day in order to determine quickly whether their bills at various businesses are being computed correctly. Mental math and estimation help students to know whether the answers that appear on the calculator are reasonable. The *Common Core Standards* makes several points about the importance of mental math and estimation. **8.** The estimate is always too high. If we increase x by an amount p and decrease y by an amount q, then the estimate is $(x + p) - (y - q)$. This can be rewritten as $(x - y) + (p + q)$. This shows that the estimate is always $(p + q)$ greater than the difference $(x - y)$.

Open-Ended

9. Students are encouraged to seek mathematics education articles to answer this question. **10.** Answers vary. For example, an estimate is sufficient, when you are determining the amount of a tip for a waiter in a restaurant. **11.** Answers vary. For example, **a.** an estimate for $230 + 520 + 640 + 170 + 160$ using front-end estimation is $1500 + 200$ (adjustment) $= 1700$. This is the same result as the one obtained by rounding and adding. **b.** An estimate for $210 + 520 + 640 + 130 + 100$ is 1600 using front-end estimation and 1500 using rounding.

Cooperative Learning

12. Answers vary. For example, the topic of constructivism might be discussed along with whether a student will understand and remember an algorithm that he or she has developed. There is also merit in finally developing an algorithm that is efficient; students will welcome it more after they have struggled developing their own. Some discussion might involve research that claims that providing opportunities for students to develop, use, and discuss invented algorithms helps to enhance number and operation sense. Other discussions might include the argument that exposing students to various algorithms only confuses them and students only need the standard algorithms. There are excellent articles on this topic in NCTM's 1998 *Yearbook*, *The Teaching and Learning of Algorithms in School Mathematics*. **13.** Answers vary. **14.** Play the game.

Connecting Mathematics to the Classroom

15. Answers vary. For example, you could talk about how adding 1 and subtracting 1 has the effect of adding 0 to the problem, which produces an equivalent problem. The reason for changing the original problem to the new problem is that the numbers are easier to compute. You could tell her that her technique works well and discuss how it works. **16.** Answers vary. For example, you might ask her to explain her algorithm with different problems, making sure to choose computations where regrouping is necessary. She should understand that what she is really doing is the following: $97 - 28 = (90 + 7) - (20 + 8) = (90 - 20) + (7 - 8) = 70 + (-1) = 69$. **17.** Answers vary. Joe is wrong. The teacher could ask Joe to perform the computation with base-ten blocks. **18.** Answers vary. We could point out that Jill's $4 - 2$ is actually $400 - 200 = 200$, and $200 + 3 = 203$ not 23. **19.** Betsy is confused on what you should add to check a subtraction. She should try smaller numbers to get a feeling for what needs to be done, for example, $9 - 5 = 4$. To check we add $4 + 5$ to see that we get back to 9. Number lines or colored rods could be used to show that $9 - 5 = 4$ and that $4 + 5$ gets her back to the length of 9. **20.** Even though the answer is correct, the use of the equal sign is not correct, and this should not be allowed. The student wrote $7 + 2 = 9 + 3$, which is obviously incorrect even though the student knew that what he wanted to do was add 3 to the sum of $7 + 2$. The correct use of the equal sign must be emphasized. **21.** The first thing Richie should do in rounding is to determine between which two consecutive tens the number lies. In this case they are 2560 and 2570. Next he should determine to which number 2568 is closer. In this case 2568 is closer to 2570, and so 2570 is the proper rounding. It should be pointed out that Richie's answer of 2578 is not between 2560 and 2570. **22.** When rounding to the nearest hundred Ashley should determine between which two consecutive hundreds the number lies. In this case they are 300 and 400. Because 345 is closer to 300, the proper rounding is to 300. She rounded to tens first and then to hundreds. **23.** In this case one thousand is exchanged for 10 hundreds, leaving 4 thousands. Then one hundred is exchanged for 10 tens, leaving 9 hundreds. Then one hundred is exchanged for 10 tens, leaving 9 hundreds. Then one ten is exchanged for 10 ones, leaving 9 tens and a total of 14 ones. With these exchanges Pauli can perform the indicated subtraction. **24.** Molly has a good idea and almost has it right. Point out that she needs to add 2 at the end instead of subtract 2. The reasons for this should be explained. For example,

$$\begin{aligned} 261 - 48 &= 261 - (50 - 2) \\ &= 261 - 50 + 2 \\ &= (261 - 50) + 2 \\ &= 211 + 2 \\ &= 213. \end{aligned}$$

25. Answers vary. For example, the rounding strategy may not always be the easiest or fastest way to do an estimation. The clustering strategy is much faster if all the numbers are roughly grouped around a particular value. Depending upon the values of the numbers, strategies like using compatible numbers make the estimation process much easier. Front-end estimation is quick and easy and in many cases can lead to a better estimate. **26.** By using the calculator first, she is not learning the estimation skills that are so useful in making decisions. One important use of estimation is to determine whether an answer is reasonable. For example, if the problem is $492 \cdot 63$, by estimating, the student would know that the answer should be approximately 30,000. If she gets an answer such as 17,712 ($492 \cdot 36$) or 59,346 ($942 \cdot 63$), she would know that there was an error in computation. Maybe she transposed two digits.

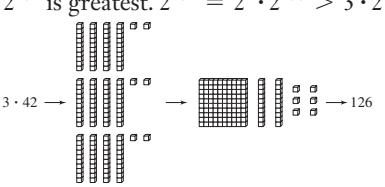
Review Problems

27. 1410 **28. a.** 1594 **b.** 11 **c.** 23 **29.** No. For example $2 + 3 = 5 \notin \{1, 2, 3\}$ **30.** For example, $(2 + 3) + 4 = 2 + (3 + 4)$ **31. a.** 10 **b.** 14 **c.** All whole numbers

Assessment 3-5A

2. a. 2^{100} is greater. $2^{80} + 2^{80} = 2^{80}(1 + 1) = 2^{80} \cdot 2 = 2^{81} < 2^{100}$
b. 2^{102} is greatest. $2^{102} = 2^2 \cdot 2^{100} > 3 \cdot 2^{100} > 4 \cdot 2^{99} = 2^{101}$

3.



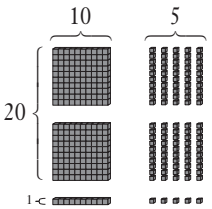
4. a. The following partial products are obtained through the distributive property of multiplication over addition. The parenthetical computations below illustrate the corresponding base-ten-block notions from the model:

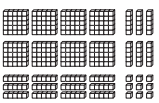
$$\begin{array}{rl} 22 & \\ \times\ 13 & \\ \hline 6 & (3 \times 2) \\ 60 & (3 \times 20) \\ 20 & (10 \times 2) \\ 200 & (10 \times 20) \\ \hline 286 & \end{array}$$

b.

$$\begin{array}{rl} 15 & \\ \times\ 21 & \\ \hline 5 & (1 \times 5) \\ 10 & (1 \times 10) \\ 100 & (20 \times 5) \\ 200 & (20 \times 10) \\ \hline 315 & \end{array}$$

Which is illustrated by:



c.  To find $43_{\text{five}} \cdot 23_{\text{five}}$ count the number of flats, longs, and ones. There are $4 \cdot 2$ flats, $2 \cdot 3 + 3 \cdot 4$ longs, and $3 \cdot 3$ units. 5 units = 1 long, 5 longs = 1 flat, and 5 flats = 1 block; thus $43_{\text{five}} \cdot 23_{\text{five}} = 2$ blocks, 1 flat, 4 longs, and 4 units, or 2144_{five}.

6. a. $728 \times 94 = 68{,}432$:

	7	2	8	
6	6/3	1/8	7/2	9
8	2/8	0/8	3/2	4
	4	3	2	

b. $306 \times 24 = 7344$:

	3	0	6	
	0/6	0/0	1/2	2
7	1/2	0/0	2/4	4
	3	4	4	

7. In millions of liters: 168,000; 40,000; 6400; 48,750; 11,500; 1500; 360 **8. a.** In base two, two is 10_{two}. Consider a three-digit number in base two: $abc_{\text{two}} \cdot 10_{\text{two}} = (a \cdot 10^2_{\text{two}} + b \cdot 10_{\text{two}} + c \cdot 1_{\text{two}}) \cdot 10_{\text{two}} = a \cdot 10^3_{\text{two}} + b \cdot 10^2_{\text{two}} + c \cdot 10_{\text{two}} + 0 \cdot 1_{\text{two}} = abc0_{\text{two}}$. So multiplying abc_{two} by 10_{two} appended a 0 in the "ones" place. **b.** In base two, 4 can be expressed as $10_{\text{two}} \cdot 10_{\text{two}}$. Thus, given what we learned in (a), multiplying by 4 in base two appends a base-two numeral by 00 by appending the original number by 0 twice.

c.
$$\begin{aligned} 110_{\text{two}} \cdot 11_{\text{two}} &= 110_{\text{two}}(10_{\text{two}} + 1_{\text{two}}) \\ &= 110_{\text{two}} \cdot 10_{\text{two}} + 110_{\text{two}} \cdot 1_{\text{two}} \\ &= 1100_{\text{two}} + 110_{\text{two}} \\ &= 10010_{\text{two}} \end{aligned}$$

9.

			Halves		Doubles	
		→	17	×	(63)	
Halve	17		8		126	Double 63
Halve	8		4		252	Double 126
Halve	4		2		504	Double 252
Halve	2	→	1		(1008)	Double 504

The numbers in the doubles column paired with odd numbers in the halves column are 63 and 1008; $63 + 1008 = 17 \times 63 = 1071$.

13. a. Repeated subtraction

$8\overline{)623}$

560 70 eights

63

56 7 eights

7 77 R7

Standard algorithm

$$\begin{array}{r} 77 \text{ R}7 \\ 8\overline{)623} \\ \underline{560} \\ 63 \\ \underline{56} \\ 7 \end{array}$$

b. Repeated subtraction

$36\overline{)298}$

288 8 36's

10 8 R10

Standard algorithm

$$\begin{array}{r} 8 \text{ R}10 \\ 36\overline{)298} \\ \underline{288} \\ 10 \end{array}$$

c. Repeated subtraction

$391\overline{)4001}$

3910 10 391's

91 10 remainder 91

Standard algorithm

$$\begin{array}{r} 10 \text{ R}91 \\ 391\overline{)4001} \\ \underline{3910} \\ 91 \end{array}$$

16. a. Answers vary. One such is $36 \cdot 84 = 3024$ and $63 \cdot 48 = 3024$. **b.** Let the digits be a, b, c, and d. Then $(10a + b) \cdot (10c + d) = (10b + a) \cdot (10d + c) \Rightarrow 100ac + 10bc + 10ad + bd = 100bd + 10bc + 10ad + ac \Rightarrow 99ac = 99bd \Rightarrow ac = bd$. So if $a \cdot c = b \cdot d$ then the products will always be the same when the digits are reversed (e.g., in part (a) above, $a \cdot c = b \cdot d \Rightarrow 3 \cdot 8 = 6 \cdot 4$). **19.** Answers vary. **a.** 5 was multiplied by 6 to obtain 30. The 3 was regrouped, then 3 was multiplied by 2 to obtain 6. The regrouping was added to obtain 9 which was recorded in the tens place. **b.** When 1 was brought down the quotient of 0 was not recorded.

20. $56 \cdot 10$

$= (5 \cdot 10 + 6) \cdot 10$	(expanded form)
$= (5 \cdot 10) \cdot 10 + 6 \cdot 10$	(distributive property of multiplication over addition)
$= 5(10 \cdot 10) + 6 \cdot 10$	(associative property of multiplication)
$= 5 \cdot 10^2 + 6 \cdot 10$	(definition of a^n)
$= 5 \cdot 10^2 + 6 \cdot 10 + 0 \cdot 1$	(additive identity and multiplication property of 0)
$= 560$	(place value)

21. c. Arithmetic, because each subsequent term is the result of adding 4 to the previous term. **23.** $323_{\text{five}} \cdot 42_{\text{five}} = 30221_{\text{five}}$:

	3	2	3	
3	2/2	1/3	2/2	4
0	1/1	0/4	1/1	2
	2	2	1	

24. $32_a = 23_b \Rightarrow 3a + 2 = 2b + 3 \Rightarrow b = \frac{3a - 1}{2}$. The least value of $a\,(a > 1)$ for b to be whole is $a = 3 \Rightarrow a = 3$ and $b = 4$. But 23_{three} is not possible because there is no digit 3 in base three. The least possible solution is $a = 5$ and $b = 7$, or $32_{\text{five}} = 23_{\text{seven}} = 17$.

25. a.
$$\begin{array}{r} 768 \\ \times \quad 8 \\ \hline 6104 \end{array}$$

b.
$$\begin{array}{r} 678 \\ \times \quad 3 \\ \hline 2034 \end{array}$$

28. Estimates vary. The range is from $30 \cdot 20 = 600$ to $40 \cdot 30 = 1200$. Rounding, $38 \approx 35$ or 40 and $23 \approx 20$ or 25. Thus estimates are about $35 \cdot 20 = 700$ seats, or $40 \cdot 25 = 1000$ seats. 700 would be low (rounded down) and 1000 would be high (rounded up). **29.** Answers vary. **a.** Different; one factor is the same in each and the other is 4 times larger. **b.** Same; 22 was divided by 2 to obtain 11 while 32 was multiplied by 2 to obtain 64. The result is to multiply the original computation by $\frac{2}{2} = 1$, which does not change it. **c.** Same; 13 was multiplied by 3 and 33 was divided by 3. Thus the original computation was multiplied by 1. **30.** Answers vary. Note that in parts (b) through (d), a larger divisor produces a lower quotient when the dividend stays the same, and a larger dividend produces a higher quotient when the divisor stays the same. **a.** High; $299 \cdot 300 < 300 \cdot 300$. **b.** Low; $6001 \div 299 > 6000 \div 300$. **c.** Low; $6000 \div 299 > 6000 \div 300$. **d.** Low; $10 \cdot 99 = 990 < 999 \Rightarrow 999 \div 99 > 10$. **31.** Answers vary. One strategy to find $(n5)^2$ is to write $n(n + 1)$ and append 25 (because 5^2 is always 25); e.g., $65^2 = 6 \cdot 7$ and append 25 to give 4225 and

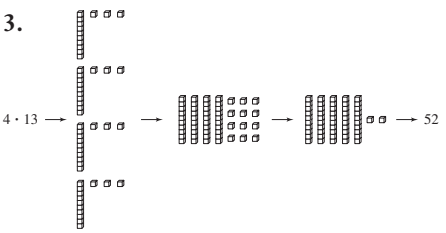
$75^2 = 7 \cdot 8$ and append 25 to give 5625. **32.** Answers vary; for example, $612 \cdot 73$.

Assessment 3-5B

2. a. They are the same because $2^{20} + 2^{20} = 2^{20}(1 + 1) = 2^{20} \cdot 2 = 2^{21}$.

b. $3^{31} < 9 \cdot 3^{30} < 3 \cdot 3^{32}$ because $9 \cdot 3^{30} = 3^2 \cdot 3^{30} = 3^{32}$ and $3 \cdot 3^{32} = 3^{33}$.

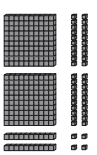
3.



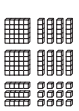
4. a.

$$\begin{array}{rl} 13 & \\ \times\ 12 & \\ \hline 6 & (2 \cdot 3) \\ 20 & (2 \cdot 10) \\ 30 & (10 \cdot 3) \\ \underline{100} & (10 \cdot 10) \\ 156 & \end{array}$$

b.

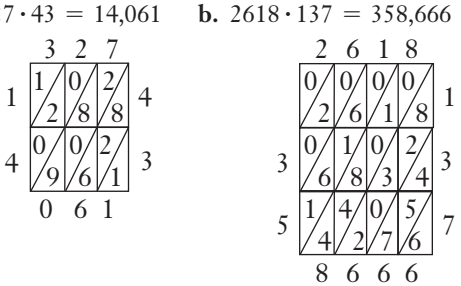


c.



There are $2 \cdot 1$ blocks, $(2 \cdot 4 + 3 \cdot 1)$ longs, and $4 \cdot 3$ units = 2 blocks + (2 blocks + 1 long) + (2 longs + 2 units) = 4 blocks + 3 longs + 2 units = 432_{five}.

6. a. $327 \cdot 43 = 14{,}061$ **b.** $2618 \cdot 137 = 358{,}666$



8. a. Use a three-digit number in base five, abc_{five}: $abc_{\text{five}} \cdot 10_{\text{five}} = (a \cdot 10^2 + b \cdot 10^1 + c \cdot 10^0)_{\text{five}} \cdot 10_{\text{five}} = (a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1)_{\text{five}} = abc0_{\text{five}}$ so multiplying by 10_{five} appended a 0 in the ones place **b.** Multiplying by 100_{five} is the same as multiplying twice by 10_{five} (i.e., $10_{\text{five}} \cdot 10_{\text{five}} = 100_{\text{five}}$). Since a zero is appended each time a number is multiplied by 10_{five}, multiplication by 100_{five} will result in annexation of two zeros (this is true for multiplying by 100 in any base, not just base-five).

c. $14_{\text{five}} \cdot 23_{\text{five}} = (10 + 4)_{\text{five}} \cdot 23_{\text{five}} = (10 \cdot 23 + 4 \cdot 23)_{\text{five}} = (230 + 202)_{\text{five}} = 432_{\text{five}}$ Observe that in base-five notation

$$\begin{aligned} 4_{\text{five}} \cdot 23_{\text{five}} &= [4 \cdot (2 \cdot 10 + 3)]_{\text{five}} \\ &= (130 + 22)_{\text{five}} = 202_{\text{five}}. \end{aligned}$$

9.

			Halves		Doubles	
		→	31	×	(69)	
Halve	31	→	15		(138)	Double 69
Halve	15	→	7		(276)	Double 138
Halve	7	→	3		(552)	Double 276
Halve	3	→	1		(1104)	Double 552

69
138
276
552
1104
2139 which is the product of 31×69.

13. a. Repeated subtraction

$$\begin{array}{r} 7\overline{)3\ 9\ 2} \\ \underline{3\ 5\ 0}\quad 50 \text{ sevens} \\ 4\ 2 \\ \underline{4\ 2}\quad \underline{6} \text{ sevens} \\ 0\quad 56 \text{ R0} \end{array}$$

Standard algorithm

$$\begin{array}{r} 5\ 6 \text{ R0} \\ 7\overline{)3\ 9\ 2} \\ \underline{3\ 5} \\ 4\ 2 \\ \underline{4\ 2} \\ 0 \end{array}$$

b. Repeated subtraction

$$\begin{array}{r} 3\ 7\overline{)9\ 2\ 5} \\ \underline{7\ 4\ 0}\quad 20\ 37\text{'s} \\ 1\ 8\ 5 \\ \underline{1\ 8\ 5}\quad \underline{5}\ 37\text{'s} \\ 0\quad 25 \text{ R0} \end{array}$$

Standard algorithm

$$\begin{array}{r} 2\ 5 \text{ R0} \\ 3\ 7\overline{)9\ 2\ 5} \\ \underline{7\ 4} \\ 1\ 8\ 5 \\ \underline{1\ 8\ 5} \\ 0 \end{array}$$

c. Repeated subtraction

$$\begin{array}{r} 4\ 2\ 3\overline{)5\ 0\ 0\ 2} \\ \underline{-\ 4\ 2\ 3\ 0}\quad 10\quad 423\text{'s} \\ 7\ 7\ 2 \\ \underline{-\ 4\ 2\ 3}\quad \underline{1}\quad \underline{4\ 2\ 3} \\ 3\ 4\ 9\quad 11 \text{ R}349 \end{array}$$

Standard algorithm

$$\begin{array}{r} 1\ 1 \text{ R}349 \\ 4\ 2\ 3\overline{)5\ 0\ 0\ 2} \\ \underline{4\ 2\ 3} \\ 7\ 7\ 2 \\ \underline{4\ 2\ 3} \\ 3\ 4\ 9 \end{array}$$

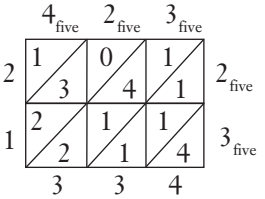
17. 8, 2, and 24 respectively **18.** \$175 **19. a.** Regrouping was not used. Place value was not observed. **b.** 4 was multiplied by 6 to obtain 24. The 2 was regrouped; then 6 and 3 were added instead of multiplied and the 2 was added to obtain 11.

20.

$$\begin{aligned} 35 \cdot 100 &= (3 \cdot 10 + 5) \cdot 100 && \text{(expanded form)} \\ &= (3 \cdot 10 + 5) \cdot 10^2 && \text{(definition of } a^n\text{)} \\ &= (3 \cdot 10) \cdot 10^2 + 5 \cdot 10^2 && \text{(distributive property of multiplication over addition)} \\ &= 3(10 \cdot 10^2) + 5 \cdot 10^2 && \text{(associative property of multiplication)} \\ &= 3 \cdot 10^3 + 5 \cdot 10^2 && (a^n \cdot a^m = a^{n+m}) \\ &= 3 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 0 \cdot 1 && \text{(additive identity and multiplication property of 0)} \\ &= 3500 && \text{(place value)} \end{aligned}$$

21. c. This is an arithmetic sequence with $a_1 = 1$ and $d = 4$.

23. $423_{\text{five}} \cdot 23_{\text{five}} = 21334_{\text{five}}$:



25. a.
$$\begin{array}{r} 762 \\ \times\ 83 \\ \hline 63246 \end{array}$$
b.
$$\begin{array}{r} 378 \\ \times\ 26 \\ \hline 9828 \end{array}$$

27. Answers vary.
$$\begin{aligned} 6 \cdot 65 &= 6(60 + 5) \\ &= 6 \cdot 60 + 6 \cdot 5 = 360 + 30 \\ &= 390 \end{aligned}$$

28. Answers vary; for example $6 \cdot 3540$ is about $6 \cdot 3500 = 3(2 \cdot 3500) = 3 \cdot 7000 = 21{,}000$.

29. Answers vary: **a.** Different. The products are quite different. The number 22 is $\frac{1}{4}$ the number 88. **b.** Same; 93 was divided by 3 to obtain 31 and 15 was multiplied by 3 to obtain 45. **c.** Different; $20 \cdot 17 = 17 \cdot 20$; $12 < 17$ and $18 < 20$. **30. a.** High; $398 \cdot 500 < 400 \cdot 500$. **b.** Low; $8001 \div 398 > 8001 \div 400 > 8000 \div 400$. **c.** Low; $10{,}000 \div 999 > 10{,}000 \div 1000$. **d.** High; $1999 \div 201 < 2000 \div 200$. **31.** Answers vary. One strategy to mentally find $ab \cdot 99$, where ab represents the number $10a + b$, would be to write $(ab - 1)$ as the first two digits and then append the digits obtained from $100 - ab$. For example, to find $12 \cdot 99$ the first two digits are $12 - 1 = 11$ and the second two digits are $100 - 12 = 88$, or $12 \cdot 99 = 1188$. Another method is to show $ab \cdot 99$ as $ab(100 - 1) = 100ab - ab$.

32. Answers vary, for example, $34 \cdot 82$.

Mathematical Connections 3-5

Reasoning

1. Because $345 \cdot 678 = 345 \cdot (6 \cdot 10^2 + 7 \cdot 10 + 8)$, one explanation could be as follows. Using the distributive property of multiplication over addition, first multiply 345 by 6 and the result by 10^2; then multiply 345 by 7 and the result by 10; then multiply 345 by 8. Add all the numbers previously obtained. **2.** Answers vary. If the number is ab, then the value is $10a + b$, the product is $1000a + 100b + 10a + b$, or $abab$. **3.** The result is always 4. Let the original number be x. The calculation appears as follows:

$$[(2x)3 + 24] \div 6 - x = 4$$

4. Answers vary. Some students may find one algorithm easier to understand or to use than others and therefore it will be easier for him or her to remember or to reproduce. **5.** Answers vary. NCTM suggests that the traditional algorithm for division be considered toward the end of grades 3–5 as one efficient way to calculate. Students should be able to justify the procedure. However, NCTM recommends that students consolidate and practice a small number of strategies for division and be able to use them fluently. Long division can be practiced by working on context problems and not necessarily just drill problems. If a student can demonstrate that he or she can handle division problems, there is no need to continue to practice them in excess. Division can then receive less emphasis. **6. a.** We are asking how many 43s are in 5283. To answer this we first answer how many 43s are in 5200. The answer 1 hundred is the result of regrouping five thousand, two hundred as 52 hundreds. Therefore there are one hundred 43s, so we place a 1 in the hundreds place in the quotient. **b.** We are regrouping 9 hundreds and 8 tens as 98 tens. **c.** We are regrouping 12 tens and 3 ones as 123. **d.** The R37 indicates a remainder of 37; that is, there are 122 43s in 5283 with 37 remaining. **7. a.** In this problem there are 100_{five} groups of 31_{five} in 4000_{five} with 400_{five} left over. The 4 represents the 400_{five}.

b.
$$\begin{array}{r} 111\text{R}30_{\text{five}} \\ 31_{\text{five}}\overline{)4021_{\text{five}}} \\ -31 \\ \hline 42 \\ -31 \\ \hline 111 \\ -31 \\ \hline 30 \end{array}$$

Open-Ended

8. Answers vary. The recommendation should be to try to complete a proof that the algorithm would work on a general set of numbers. If the student does not have the capacity to complete that task, one method of trial and error is to try a great number of cases. This is not a proof, and that should be emphasized, but it might produce a counterexample. **9.** Answers vary. She used the distributive property of multiplication over addition:
$49 \cdot 51 + 49 \cdot 49 = 49(51 + 49) = 4900$
$98 \cdot 37 + 2 \cdot 37 = (98 + 2) \cdot 37 = 3700$
$99 \cdot 37 + 37 = (99 + 1) \cdot 37 = 3700$.
Additional examples vary; for example,
$101 \cdot 37 - 37 = 101 \cdot 37 - 1 \cdot 37 = (101 - 1)37 = 3700$.

Cooperative Learning

10. Arguments vary depending upon groups. Some will argue that addition should be followed by subtraction, because they are inverses of each other. Others will argue that addition should be followed by multiplication, because multiplication is repeated addition. This would also postpone subtraction until students are more ready for it. **11. a.** Answers vary; for example, $19{,}875 \cdot 44$ is greater because $19{,}876 \cdot 43 = 19{,}875 \cdot 43 + 43$ and $19{,}875 \cdot 44 = 19{,}875 \cdot 43 + 19{,}875$ and $19{,}875 > 43$. **b.** Answers vary. **12. a.** A COMPUTER **b.** Answers vary. **c. (i)** When the person tells his age by listing cards$_n$ he is giving the base-two representation for his age. The number can then be determined by adding the numbers in the upper left-hand corners of the named cards. In this case, $16 + 4 + 2 = 22$. **(ii)** Card F would have 32 in the upper-left corner. Each of the numbers 1 through 63 could be written in base two, so you can tell where to place them on the cards.

Connecting Mathematics to the Classroom

13. If the repeated subtraction algorithm is done with large multiples of the divisor, the repeated subtraction can be quite efficient. However, if a student uses repeated subtraction by subtracting small multiples of the divisor, the process can be very time consuming. **14.** Evidently the student does not understand the process of long division. The repeated subtraction method should help in understanding the mistake.

$$\begin{array}{rl} 6\overline{)36} & \\ -\ 6 & 1 \text{ six} \\ \hline 30 & \\ -\ 30 & \underline{5 \text{ sixes}} \\ \hline & 6 \text{ sixes} \end{array}$$

Instead of adding 1 and 5, the student wrote 15. **15.** We see on the calculator that 593 divided by 36 is $16.47\overline{2}$. This tells us there are 16 sets of 36 in 593 with some remain-

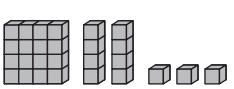
ing. To determine how many remain, we could multiply $16 \cdot 36 = 576$ and subtract 576 from 593 to obtain a remainder of 17. Another way to think about it is to use the division algorithm; that is, $a = bq + r$. Solving for r, we have $r = a - bq = 593 - 36 \cdot 16 = 17$. **16.** If the number has three digits with the units digit 0, we have $ab0 \div 10 = ab$ since $ab \cdot 10 = ab0$. This will not work if the 0 digit is not the units digit. **17. a.** The first equation is true because $39 + 41 = 39 + (1 + 40) = (39 + 1) + 40 = 40 + 40$. Similarly, $39 \cdot 41 = (40 - 1)(40 + 1) = 40^2 - 1^2$, but $40^2 - 1 \neq 40^2$. **b.** Yes, this pattern continues because the numbers being considered are in the form $(a - 1)(a + 1)$, which is equal to $a^2 - 1$. All the answers vary. **18.** Howard may be used to solving problems in the form $30 \div 5 = x$ rather $x \div 5 = 6$; that is, he is used to seeing the blank come last. He needs more work with the division/multiplication relationship; that is, $x \div 5 = 6$ can be thought of as $x = 5 \cdot 6$. **19. a.** The student is ignoring the remainder in each division. For example, the student is thinking of $6 \div 4$ as 1 and not $6 \div 4 = 1\text{R}2$. **b.** The student is ignoring the 0 and is treating it as if it has no value. The student should be asked to estimate the value of $906 \div 5$, which is close to $1000 \div 5$ or 200. The student will then see that the value cannot be 19. **20.** Answers vary. **a.** The student multiplied the ones ($5 \cdot 8 = 40$ or 40 ones) and the 4 over the 6 represents the 4 tens from regrouping 40 as $4 \cdot 10 + 0$. **b.** The student multiplied and added the tens ($5 \cdot 6$ tens + 4 tens = 34 tens). The 3 in the hundreds place represents the regrouping of 34 tens as 3 hundreds, 4 tens. **21. a.** The student multiplied the ones ($4_{\text{five}} \cdot 4_{\text{five}} = 31_{\text{five}}$ or 31_{five} ones) and the 3 over the 3 represents the 3 fives from the regrouping of 31_{five} as $3 \cdot 10_{\text{five}} + 1_{\text{five}}$.

Review Problems

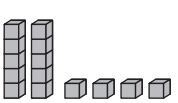
22. For example, $3 + 0 = 0 + 3 = 0$
23. a. $(a + b + 2)x$ **b.** $(3 + x)(a + b)$ **24.** 6979 mi
25. a. $36 = 4 \cdot 9$ **b.** $112 = 2x$ **c.** $48 = x \cdot 6$ or $48 = 6x$
d. $x = 7 \cdot 17$

Chapter 3 Review

7. a.



b.



10. Answers vary. One example would be selling pencils by units, dozens, and gross; i.e., using base twelve. **11.** Answers vary. **a.** The Egyptian system had seven symbols. It was a tally system and a grouping system, and it used the additive property. It did not have a symbol for zero, which was not important inasmuch as they did not use place value. **b.** The Babylonian system used only two symbols. It was a place value system (base 60) and was additive within the positions. It lacked a symbol for zero until around 300 BCE **c.** The Roman system used seven symbols. It was additive, subtractive, and multiplicative. It did not have a symbol for zero. **d.** The Hindu-Arabic system uses ten symbols. It uses place value involving base ten and has a symbol for zero. **13. b.** $E0T018_{\text{twelve}}$ **15. a.** Distributive property of multiplication over addition **b.** Commutative property of addition **c.** Identity property of multiplication **d.** Distributive property of multiplication over addition **e.** Commutative property of multiplication **f.** Associative property of multiplication **16. a.** $13 = 3 + 10$. Since 10 is a natural number, $3 < 13$. **b.** $12 = 3 + 9$. Since 3 is a natural number, $12 > 9$. **18. d.** $(x + 5)(3 + y)$
25. a. Let n be the original number. Then

$$\frac{2[2(n + 17) - 4] + 20}{4} - 20 =$$

$$\frac{4(n + 17) - 8 + 20}{4} - 20 =$$

$$\frac{4n + 80}{4} - 20 = n + 20 - 20 = n.$$

b. Answers vary. For example, if n is the original number: $4(n + 18) - 7 = 4n + 65$. Then two more steps might be: $4n + 65 - 65$ (subtract 65); $\frac{4n}{4}$ (divide by 4). **c.** Answers vary; use the techniques of parts (a) and (b).

26. Scratch:

$$\begin{array}{r} {}^{1}3\ 1\ 6 \\ 7_{1}\ 1\ 2 \\ +\ 9_{1}\ 1 \\ \hline 1\ 1\ 1\ 9 \end{array}$$

Traditional:

$$\begin{array}{r} {}^{1} \\ 3\ 3\ 1\ 6 \\ 7\ 1\ 2 \\ +\ 9\ 1 \\ \hline 1\ 1\ 1\ 9 \end{array}$$

27. Traditional:

$$\begin{array}{r} 6\ 1\ 3 \\ \times\ 9\ 8 \\ \hline 4\ 9\ 0\ 4 \\ 5\ 5\ 1\ 7 \\ \hline 6\ 0\ 0\ 7\ 4 \end{array}$$

Lattice:

	6	1	3	
6	5/4	0/9	2/7	9
0	4/8	0/8	2/4	8
	0	7	4	

28. a. Repeated subtraction:

$$\begin{array}{r} 912)4803 \\ -\ 4560 \\ \hline 243 \end{array} \qquad \begin{array}{l} 5\ 912\text{s} \\ 5\ 912\text{s} \end{array} \Rightarrow 5\ \text{R}243$$

Traditional:

$$\begin{array}{r} \quad 5\ \text{R}243 \\ 912)4803 \\ -\ 4560 \\ \hline 243 \end{array}$$

b. Repeated subtraction:

$$\begin{array}{r} 11)1011 \\ -\ 990 \\ \hline 21 \\ -\ 11 \\ \hline 10 \end{array} \qquad \begin{array}{l} 90\ 11\text{'s} \\ \\ \\ 1\ 11 \\ 91\ 11\text{'s} \end{array} \Rightarrow 91\ \text{R}10$$

Traditional:

$$\begin{array}{r} \quad 91\ \text{RI}0 \\ 11)1011 \\ -\ 990 \\ \hline 21 \\ -\ 11 \\ \hline 10 \end{array}$$

c. Repeated subtraction:

$$\begin{array}{r} 23_{\text{five}})3312_{\text{five}} \\ -\ 2300 \\ \hline 1012 \\ -\ 1010 \\ \hline 2 \end{array} \qquad \begin{array}{l} (100\ 23\text{s})_{\text{five}} \\ \\ (20\ 23\text{s})_{\text{five}} \\ (120\ 23\text{s})_{\text{five}} \end{array}$$

$\Rightarrow 120_{\text{five}}\ \text{R}\ 2_{\text{five}}$

Traditional:

$$\begin{array}{r} \quad 120_{\text{five}}\ \text{R}2_{\text{five}} \\ 23_{\text{five}})3312_{\text{five}} \\ \quad 23 \\ \hline \quad 101 \\ \quad 101 \\ \hline \quad\ \ 02_{\text{five}} \end{array}$$

$\Rightarrow 120_{\text{five}}\ \text{R}\ 2_{\text{five}}$

d. Repeated subtraction:

$$\begin{array}{r} 11_{\text{two}})1011_{\text{two}} \\ 110 \\ \hline 101 \\ 11 \\ \hline 10_{\text{two}} \end{array} \qquad \begin{array}{l} (10\ 11\text{s})_{\text{two}} \\ \\ (1\ 11)_{\text{two}} \\ (11\ 11\text{s})_{\text{two}} \end{array}$$

$\Rightarrow 11_{\text{two}}\ \text{R}\ 10_{\text{two}}$

Traditional:

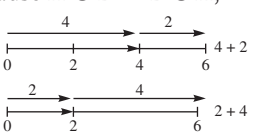
$$\begin{array}{r} \quad 11_{\text{two}}\ \text{R}10_{\text{two}} \\ 11_{\text{two}})1011_{\text{two}} \\ \quad -\ 11 \\ \hline \quad 101 \\ \quad -\ 11 \\ \hline \quad\ \ 10_{\text{two}} \end{array}$$

$\Rightarrow 11_{\text{two}}\ \text{R}\ 10_{\text{two}}$

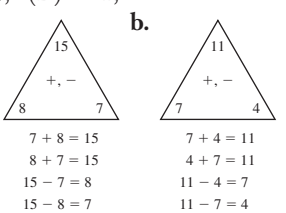
29. a. If $4803 \div 912 = 5$ remainder 243, then $912 \cdot 5 + 243 = 4803$.
b. If $1011 \div 11 = 91$ remainder 10, then $11 \cdot 91 + 10 = 1011$.
c. If $(3312 \div 23)_{\text{five}} = (120 \text{ remainder } 2)_{\text{five}}$, then $(23 \cdot 120)_{\text{five}} + 2_{\text{five}} = 3312_{\text{five}}$ **d.** If $(1011 \div 11)_{\text{two}} = (11 \text{ remainder } 10)_{\text{two}}$, then $(11 \cdot 11)_{\text{two}} + 10_{\text{two}} = 1011_{\text{two}}$.
30. c. $8 \cdot 481 \cdot 73 \cdot 125 = (481 \cdot 73) \cdot 1000 = 35{,}113{,}000$
d. $374 \cdot 200 \cdot 893 \cdot 50 = (374 \cdot 893) \cdot 10{,}000 = 3{,}339{,}820{,}000$
36.
$$\begin{aligned} 44_{\text{five}} \cdot 34_{\text{five}} &= (4 \cdot 10_{\text{five}} + 4) \cdot 34_{\text{five}} \\ &= 4 \cdot 34_{\text{five}} \cdot 10_{\text{five}} + 4 \cdot 34_{\text{five}} \\ &= 3010_{\text{five}} + 301_{\text{five}} \\ &= 3311_{\text{five}} \end{aligned}$$
37.
$$\begin{array}{r} 4_{\text{five}})\overline{4\,3\,4_{\text{five}}} \\ \underline{-4\,0\,0_{\text{five}}} \quad (100 \cdot 4)_{\text{five}} \\ 3\,4_{\text{five}} \\ \underline{-3\,1_{\text{five}}} \quad (4 \cdot 4)_{\text{five}} \\ 3_{\text{five}} \end{array}$$
Thus, $434_{\text{five}} = 104_{\text{five}} \cdot 4_{\text{five}} + 3_{\text{five}}$ **38.** Answers vary. For example: **a.** $(26 + 24) + (37 - 7) = 50 + 30 = 80$
b. $(7 \cdot 9) \cdot (4 \cdot 25) = 63 \cdot 100 = 6300$ **39.** Methods vary. For example: **a.** $63 \cdot 7 = (7 \cdot 60) + (7 \cdot 3) = 420 + 21 = 441$ (front-end multiplying) **b.** $85 - 49 = (85 + 1) - (49 + 1) = 86 - 50 = 36$ (trading off) **c.** $(18 \cdot 5) \cdot 2 = 18 \cdot (5 \cdot 2) = 18 \cdot 10 = 180$ (using compatible numbers) **d.** $2436 \div 6 = (2400 \div 6) + (36 \div 6) = 400 + 6 = 406$ (breaking up the dividend) **40.** Answers vary; for example: **(a)** Front-end: $5 + 3 + 2 + 4 + 9 = 23$; place value 2300; adjustments $40 + 100 + 60 + 0 + 100 = 300$; adjusted sum $= 2300 + 300 = 2600$. **(b)** Rounding: $500 + 400 + 300 + 400 + 1000 = 2600$. **41.** The addends cluster around 2400, so one would estimate the sum to be $4 \cdot 2400 = 9600$.
42. a. $999 \cdot 47 + 47 = 47(999 + 1) = 47 \cdot 1000 = 47{,}000$
b. $43 \cdot 59 + 41 \cdot 43 = 43 \cdot (59 + 41) = 43 \cdot 100 = 4300$
c. $1003 \cdot 79 - 3 \cdot 79 = 79 \cdot (1003 - 3) = 79 \cdot 1000 = 79{,}000$
d. $1001 \cdot 113 - 113 = 113 \cdot (1001 - 1) = 113 \cdot 1000 = 113{,}000$
e. $101 \cdot 35 = (100 + 1)35 = 100 \cdot 35 + 1 \cdot 35 = 3500 + 35 = 3535$ **f.** $98 \cdot 35 = (100 - 2)35 = 100 \cdot 35 - 2 \cdot 35 = 3500 - 70 = 3430$ **43. a.** $(3x^3 + 4x^2 + 7x + 8) + (5x^2 + 2x + 1) = 3x^3 + 9x^3 + 9x + 9$
c. Answers vary; for example,
$$\begin{aligned} 34 \cdot 10^2 &= (3 \cdot 10 + 4) \cdot 10^2 \\ &= 3 \cdot 10^3 + 4 \cdot 10^2 + 0 \cdot 10 + 0 \cdot 1 \\ &= 3400; \end{aligned}$$
$(3x + 4)x^2 = 3x^3 + 4x^2$
44. Suppose $a, b \in B$. Then $a = 5j$ and $b = 5k$, where $j, k \in W$. Therefore $a + b = 5j + 5k = 5(j + k)$, where $(j + k) \in W$. Therefore $a + b \in B$ and B is closed under addition.

Answers to Now Try This

1. a. 3 blocks 12 flats 11 longs 17 units
= 3 blocks; 1 block 2 flats; 10 longs 1 long; 10 units 7 units
= 4 blocks; 2 flats; 1 flat 1 long; 1 long 7 units
= 4 blocks 3 flats 2 longs 7 units
= 4327
b. $3282 = 3 \cdot 10^3 + 2 \cdot 10^2 + 8 \cdot 10 + 2 \cdot 1$
2. a.  **b.** 203,034 **c.** Answers vary. For example, writing large numbers is very cumbersome as the system is additive and does not use place value. Performing operations involving addition, subtraction, multiplication, and division is complicated because of the way that numbers are represented.
3. a.  **b.** $2 \cdot 60^2 + 11 \cdot 60 + 1 \cdot 1 = 7861$
c. Answers vary. The Hindu-Arabic system has a symbol for 0, and this is very important in a system that uses place value. Because it uses base sixty, the Babylonian system requires the use of many symbols to write numbers such as 59.
6. For example, if the sets of elements are $\{a, b, c, d\}$ and $\{a, c, d\}$, then the union of the sets of elements is $\{a, b, c, d\}$. The union has only 4 elements while the original sets have 4 and 3, respectively. The sum of 4 and 3 is 7, not 4, the number of elements in the set union. **7.** Answers vary; for example, students are used to starting with 1 when they count, so they sometimes start with 1 on the number line. It should be stressed that we are working with whole numbers and the first whole number is 0. Next, to represent 3 on a number line an arrow (vector) of length 3 units must be used.
8. a. Closed an even number plus an even number is always even. **b.** Not closed, for example, $1 + 3 = 4$ and $4 \notin F$. **c.** Not closed, for example, $2 + 2 = 4$ and $4 \notin G$.
9. a. Use disjoint sets A and B such that $n(A) = 3$ and $n(B) = 5$. Then $n(A) + n(B) = n(A \cup B)$, $n(B) + n(A) = n(B \cup A)$. Because $A \cup B = B \cup A$, $n(A) + n(B) = n(B) + n(A)$. **b.**



c. Answers vary. **10.** Let $B \subseteq A$. If $n(A) = a$ and $n(B) = b$, then $a - b = n(A - B)$. For example, we can model the subtraction $4 - 1 = 3$ as follows: $A = \{a, b, d, e\}$ and $B = \{a\}$, $A - B = \{b, d, e\}$, $n(A) = 4$, $n(B) = 1$, $n(A - B) = 3$. **11.** Answers vary. **a.** **b.**



$7 + 8 = 15$
$8 + 7 = 15$
$15 - 7 = 8$
$15 - 8 = 7$

$7 + 4 = 11$
$4 + 7 = 11$
$11 - 4 = 7$
$11 - 7 = 4$

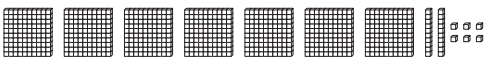
12. a. The set of whole numbers is not closed under subtraction; for example, $2 - 5$ is not a whole number.
b. Subtraction is not associative for whole numbers; for example, $9 - (7 - 2) \neq (9 - 7) - 2$. **c.** Subtraction is not commutative for whole numbers; for example, $3 - 2 = 1$, but $2 - 3$ is not a whole number. **d.** There is no identity for whole-number subtraction; for example, $5 - 0 = 5 \neq 0 - 5$.
14. a. The set of whole numbers is not closed under division; for example, $8 \div 5$ is not a whole number. Likewise, $8 \div 2 \neq 2 \div 8$ and $(8 \div 4) \div 2 \neq 8 \div (4 \div 2)$ show it is neither commutative nor associative. **b.** 1 is not the identity for whole-number division because $n \div 1 = 1$ for all whole numbers n, but $1 \div n \neq n$ except when $n = 1$.
15. (i) The method is valid because subtracting and adding the same number is zero and does not change the original sum, i.e., $x + y - y + z = x + z$, **(ii)** Answers vary. For example,
$$\begin{aligned} 97 + 69 &= (97 + 3) + (69 - 3) \\ &= 100 + 66 \\ &= 166. \end{aligned}$$

16. a.

$+$	0_{two}	1_{two}
0_{two}	0_{two}	1_{two}
1_{two}	1_{two}	10_{two}

b. (i) $1111_{two} + 111_{two} = 10110_{two}$ (ii) $1101_{two} - 111_{two} = 110_{two}$

19. The base-ten representation of 726 is



If 1 flat is traded for 10 longs then 6 equal-sized piles can be formed. Therefore, $726 \div 6 = 121$. Answers vary for preferred technique. **20.** Answers vary. For example, **a.** To estimate $4525 \cdot 9$, we know $4525 \cdot 10 = 45{,}250$, and since we have only 9 sets of 4525, we can take away approximately 5000 from our estimate, and we have 40,250. **b.** To estimate $3625 \div 42$, we know the answer will be close to $3600 \div 40$, or 90.

Answer to Preliminary Problem

There are many ways to solve this problem. Consider rows and columns with only two symbols, for example,

Row 1 $2\square + \Delta = 35$
Column 2 $2\square + 2\Delta = 50$

We see an addition of one Δ adds 15 to the sum. Therefore, $\Delta = 15$. Then substitute $\Delta = 15$ in $2\square + \Delta = 35$ and we have $2\square + 15 = 35$ or $2\square = 20$ and $\square = 10$. Now $\bigcirc + 2\square = 32$ so $\bigcirc + 20 = 32$ and $\bigcirc = 12$. Therefore, $\bigcirc = 12$, $\Delta = 15$, and $\square = 10$. These answers can be checked by substitution in the original table.

CHAPTER 4

Assessment 4-1A

4. c. Yes; $8 \mid 1216$. **7. b.** $19 \mid 3800$ and $19 \mid 19$ **8. a.** True. Because $13 \mid 390{,}000$ and $13 \mid 26$; 13 divides the sum. **b.** True. Because $13 \mid 260{,}000$ and $13 \nmid 33$; 13 does not divide the sum. **c.** False. Because $17 \mid 34{,}000$ and $17 \nmid 15$; 17 does not divide the sum. **d.** True. Because $17 \mid 34{,}000$ and $17 \mid 51$; 17 divides the sum. **e.** False. Because $19 \mid 19{,}000$ and $19 \nmid 31$; 19 does not divide the sum. **f.** True. Because $31 \mid 93$; $31 \mid 93 \cdot 93^{10}$.
15. a. $(7 + 2 + 4 + 2 + 8 + 1) + 5$. Since the sum in the parentheses is divisible by 3, the remainder is found when 5 is divided by 3. The remainder is 2. **b.** A four-digit number in base ten can be written as $(a \cdot 999 + b \cdot 99 + c \cdot 9) + (a + b + c + d)$. Because the number in the first parentheses is divisible by 3, it leaves remainder 0 upon division by 3. So the remainder when $abcd$ is divided by 3 is the same as when $a + b + c + d$ is divided by 3. **c.** Yes, because $a \cdot 999 + b \cdot 99 + c \cdot 9$ also leaves remainder 0 upon division by 9 (see part (b)). **16.** 1, 3, 5, 9, and 15 divide n. 1 divides every number. Because $n = 45 \cdot d, d \in W$, then $n = (3 \cdot 15)d$ which implies n is divisible by 3 and 15. Also, $n = (5 \cdot 9)d$ which implies n is divisible by 5 and 9.
17. The last four digits represent a number divisible by 16.
18. a. $12{,}343 + 4546 + 56 = 16{,}945$; $4 + 1 + 2 = 7$ has a remainder of 7 when divided by 9, as does $1 + 6 + 9 + 4 + 5 = 25$. **b.** $987 + 456 + 8765 = 10{,}208$; $6 + 6 + 8 = 20$ has a remainder of 2 when divided by 9, as does $1 + 0 + 2 + 0 + 8 = 11$. **c.** $1003 - 46 = 957$; $4 - 1 = 3$ has a remainder of 3 when divided by 9, as does $9 + 5 + 7 = 21$. **d.** $345 \cdot 56 = 19{,}320$; $3 \cdot 2 = 6$ has a remainder of 6 when divided by 9, as does $1 + 9 + 3 + 2 + 0 = 15$. **e.** No. For example, $56 \div 4 = 14$; when divided by 9, $1 + 4$ does not have the same remainder as $2 \div 4$. **19. b.** Yes. Because

$$\begin{aligned} abccba &= (a \cdot 10^5 + a) + (b \cdot 10^4 + b \cdot 10) + c \cdot 10^3 + c \cdot 10^2 \\ &= a \cdot 100{,}001 + b \cdot 10{,}001 + c \cdot 1100 \\ &= 11(a \cdot 9091 + b \cdot 910 + c \cdot 100) \end{aligned}$$

20. $a \cdot 10^4 + b \cdot 10^3 + c \cdot 10^2 + d \cdot 10 + e$
$= a(9999 + 1) + b(999 + 1) + c(99 + 1) + d(9 + 1) + e$
$= (9999a + 999b + 99c + 9d) + (a + b + c + d + e)$ Because $(9999a + 999b + 99c + 9d)$ is divisible by 9, it follows that if $9 \mid (a + b + c + d + e)$, then $9 \mid (a \cdot 10^4 + b \cdot 10^3 + c \cdot 10^2 + d \cdot 10 + e)$.

Assessment 4-1B

6. a. True; 5 is a factor of $2 \cdot 3 \cdot 5 \cdot 7$. **b.** False; $5 \mid 2 \cdot 3 \cdot 5 \cdot 7$ and $5 \nmid 1$, so $5 \nmid [(2 \cdot 3 \cdot 5 \cdot 7) + 1]$. **c.** True; $6 \mid (2 \cdot 3) \cdot 2^2 \cdot 3 \cdot 17^4$ because $6 \mid (2 \cdot 3)$. **d.** True; $7 \mid 4200$ but $7 \nmid 22$ so $7 \nmid 4222$.
7. c. Because $2 \cdot 4 \cdot 6 = 2 \cdot 2^2 \cdot 2 \cdot 3 = 2^4 \cdot 6$, $2^4 \mid (2 \cdot 4 \cdot 6 \cdot 8 \cdot 17^{10})$. Because $2^4 \nmid 1$, $2^4 \nmid (2 \cdot 4 \cdot 6 \cdot 8 \cdot 17^{10})$. **d.** $2^4 \mid (10^4 + 6^4)$ because $2^4 \mid (2^4 \cdot 5^4)$ and $2^4 \mid (2^4 \cdot 3^4)$. **8. a.** False. Because $12 \mid 24{,}000$ and $12 \nmid 13$; 12 does not divide the sum. **b.** True. Because $12 \mid 24{,}000$ and $12 \mid 36$; 12 divides the sum. **c.** False. Because $19 \mid 38{,}000$ and $19 \nmid 37$; 19 does not divide the sum. **d.** True. Because $19 \mid 3{,}800{,}000$ and $19 \nmid 18$; 19 does not divide the sum. **e.** False. Because $23 \mid 23{,}000$ and $23 \mid 23$; 23 divides the sum. **f.** True. Because $23 \mid 46$; $23 \mid 46 \cdot 46^9$. **10. a.** No, $12 \mid 24{,}000$ and $12 \nmid 13$, so $12 \nmid 24{,}013$. **b.** Yes, $12 \mid 24{,}000$ and $12 \mid 36$, so $12 \mid 24{,}036$. **c.** Yes, $17 \mid 17{,}000$ and $17 \mid 34$, so $17 \mid 17{,}034$. **d.** Yes, $3 \mid 3(2 \cdot 5 \cdot 7)$. **e.** No, $6 \mid (2 \cdot 3 \cdot 5 \cdot 7)$ and $6 \nmid 1$, so $6 \nmid (2 \cdot 3 \cdot 5 \cdot 7) + 1$. **12. a.** True **b.** False
13. b. 4 TDs with extra points and 4 FGs or 1 TD with extra points and 11 FGs **c.** 5 **14. a.** True **b.** True **c.** False **d.** True **15. f.** The remainder when n is divided by 9 seems to be equal to the remainder when the sum of the digits of n is divided by 9. **16.** 1, 2, 4, 7, 14 divide n. 1 divides every number. Because $n = 28 \cdot d$, for some $d \in W$, then $n = (2 \cdot 14)d$ which implies n is divisible by 2 and 14. And, $n = (4 \cdot 7)d$ which implies n is divisible by 4 and 7.
17. A number is divisible by 25 if, and only if, the last two digits are 00, 25, 50, or 75. **18. a.** $99 + 28 = 227$; $0 + 1 = 1$ has a remainder of 1 when divided by 9, but $2 + 2 + 7 = 11$ has a remainder of 2 upon division by 9. **b.** $11{,}199 - 21 = 11{,}168$; $3 - 3 = 0$ has a remainder of 0 when divided by 9, but $1 + 1 + 1 + 6 + 8 = 17$ has a remainder of 8 upon division by 9. **c.** $99 \cdot 26 = 2575$; $0 \cdot 8 = 0$ has a remainder of 0 when divided by 9, but $2 + 5 + 7 + 5 = 19$ has a remainder of 1 upon division by 9. **19. a. i** $11 \mid [(4 + 5) - (5 + 4)]$, so $11 \mid 4554$
ii. $11 \mid [(9 + 3) - (3 + 9)]$, so $11 \mid 9339$
iii. $11 \mid [(2 + 0) - (0 + 2)]$, so $11 \mid 2002$
iv. $11 \mid [(2 + 2) - (2 + 2)]$, so $11 \mid 2222$ **20.** The proof is analogous to the proof of Theorem 4-8 for four digits. Just replace 3 by 9.

Mathematical Connections 4-1

Reasoning

1. a. We are given that n^2 is odd and must show that n is odd. Assume, to the contrary, that n is even. Then $n = 2k$, for some $k \in W$. But then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. This means that n^2 is even, contradicting the conditions of the problem. Because the assumption that n was even led to a contradiction, n must be odd.

b. We are given that n^2 is even and must show that n is even. Assume, to the contrary, that n is odd. Then $n = 2k + 1$, for some $k \in W$. But then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. This means that n^2 is odd, contradicting the conditions of the problem. Because the assumption that n was odd led to a contradiction, n must be even. **2.** Yes; \$18.95 is divisible by 5¢. **3. a.** No; in some cases the calculator rounds the answers. Try the following on an 8-digit calculator: 79,012,351 ÷ 2,469,136 or 79,012,353 ÷ 2,469,136. One or both will produce the answer 32. **b.** No; $d \nmid n$ because the display implies that n is not a multiple of d. **4. a.** Yes; $4 \mid 52{,}832$ **b.** Yes; $2 \mid 52{,}834$ and $2 \mid 324{,}514$ so $4 \mid (52{,}834 \cdot 324{,}514)$ **5. a.** No; if it is not divisible by 5, then it does not end in 0 or 5 therefore it is not divisible by 10. **b.** Yes; all odd multiples of 5 are not divisible by 10, but are divisible by 5. **6.** In the sum of the digits we can write each digit times 3 then factor out a 3, which makes the sum divisible by 3, and therefore the number is divisible by 3. For example, 777,555,222 has a sum of the digits $7 + 7 + 7 + 5 + 5 + 5 + 2 + 2 + 2 = 3 \cdot 7 + 3 \cdot 5 + 3 \cdot 2 = 3 \cdot (7 + 5 + 2)$ and $3 \mid 3 \cdot (7 + 5 + 2)$ so $3 \mid 777{,}555{,}222$. **7.** 243; yes. Consider any number n of the form $abcabc$. Then we have the following:

$$\begin{aligned} n &= a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + a \cdot 10^2 + b \cdot 10^1 + c \\ &= a \cdot (10^5 + 10^2) + b \cdot (10^4 + 10^1) + c \cdot (10^3 + 1) \\ &= a \cdot (100{,}100) + b \cdot (10{,}010) + c \cdot (1001) \\ &= a \cdot (1001 \cdot 100) + b \cdot (1001 \cdot 10) + c \cdot (1001 \cdot 1) \\ &= (1001)(100a + 10b + c) \\ &= (7 \cdot 11 \cdot 13)(100a + 10b + c) \end{aligned}$$

Therefore, if you divide by 1001, the quotient is abc.
8. 10-digit: 3,816,547,290; 9-digit: 381,654,729 **9.** 4, 6, 8 in any order ($4 + 6 + 8 = 18$ is divisible by 3 and 9 and all digits have to be even to be divisible by 2)

Open-Ended

10. a. No; there is no winning combination. **b.** Answers vary. For example, since many of these multiples of 3 sum to 99 (33 + 66, 45 + 51 + 3, etc.), the company could place at most 1000 cards with the number 1 on the card. This would ensure that there are at most 1000 winners. Other numbers could also be used.
11. a. You can use counters and divide them into two groups. If no counters are left, then the number is even. If one counter is left, then the number is odd. **b.** You can arrange tiles in rows of three and if there are no tiles left over then the number is divisible by 3, but if there is a row with 1 or 2 counters then the number is not divisible by 3. **c.** Similar to (b) but use your tiles in rows of nine. If all counters fit into rows of nine, the number is divisible by 9. Each row of nine can then be divided into 3 groups of 3 tiles, showing that if $9 \mid a$, then $3 \mid a$.

Cooperative Learning

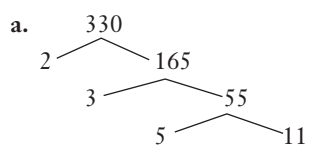
12. The sum of n consecutive numbers is divisible by n if, and only if, n is odd. **13.** Use base-ten blocks and use them similar to divisibility by 2 in the section to show divisibility by 3.

Connecting Mathematics to the Classroom

14. Use counters to show why 10 is even. When we divide 10 counters into two groups there are no counters left over. When a number is divided into two equal groups and nothing is leftover, then the number is even, so 10 is even. **15.** She is right. If each of the last two digits is divisible by 4 then the number formed by the last two digits is divisible by 4 and hence the original number is divisible by 4. **16.** It has been shown that any four-digit number n can be written in the form $n = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d = (a \cdot 999 + b \cdot 99 + c \cdot 9) + (a + b + c + d)$. The test for divisibility by some number g will depend on the sum of the digits $a + b + c + d$ if, and only if, $g \mid (a \cdot 999 + b \cdot 99 + c \cdot 9)$ regardless of the values of a, b, and c. Since the only numbers greater than 1 that divide 9, 99, and 999 are 3 and 9, the test for divisibility by dividing the sum of the digits by the number works only for 3 and 9 and not for 6. **17.** It is true that a number is divisible by 21 if, and only if, it is divisible by 3 and by 7. However, the general statement is false. For example, 12 is divisible by 4 and by 6, but not by $4 \cdot 6$, or 24. One part of the statement is true, that is, "if a number is divisible by $a \cdot b$, then it is divisible by a and b." The statement "if a number is divisible by a and by b, it is divisible by ab" is true if a and b have no primes in common. This follows from the Fundamental Theorem of Arithmetic, which will be introduced in Section 4-2. **18.** You could review division into zero and show that we can divide 2 into zero. Brian might be confusing this with division by zero if he thinks we can't divide zero into two equal piles.

Assessment 4-2A

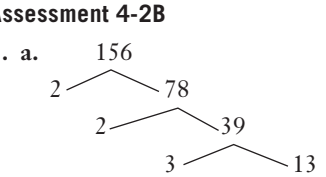
1. a.



2. a. $2^3 \cdot 3^2 \cdot 7$ **b.** $3^2 \cdot 5^2 \cdot 11$ **c.** $2 \cdot 3^2 \cdot 5^4$ **8. a.** The Fundamental Theorem of Arithmetic states that n can be written as a product of primes in one and only one way. Since $2 \mid n$ and $3 \mid n$ and 2 and 3 are both prime, they must be included in the unique factorization: $2 \cdot 3 \cdot p_1 \cdot \ldots \cdot p_m = n$; $(2 \cdot 3)(p_1 \cdot p_2 \cdot \ldots \cdot p_m) = n$; therefore, $6 \mid n$. **b.** Yes. If $a \mid n$, there exists a whole number c such that $ca = n$. If $b \mid n$, there exists a whole number d such that $db = n$. Therefore $(ca)(db) = n^2 \Rightarrow (dc)(ab) = n^2 \Rightarrow ab \mid n^2$.
10. 25 is not a prime factor. $2^3 \cdot 3^2 \cdot 5^6$ is the prime factorization.
11. No, because $8^z = 2^{3z}$, which is its prime factorization. Because of the uniqueness of prime factorization, 2^{3z} cannot have a 5 to a non-zero power in its prime factorization.
12. If $32n = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$, then $n = 2 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^{11} = (3^4 \cdot 5^3 \cdot 7^2 \cdot 11) \cdot (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11^6)$. Thus, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11^6$ is a factor of n. **13.** Yes; because $7^5 \cdot 11^3 = (7^4 \cdot 11^3) \cdot 7$, $7^4 \cdot 11^3$ is a factor $7^5 \cdot 11^3$. **14. a.** 1 by 48, 2 by 24, 3 by 16, 4 by 12 **b.** 1 by 47
15. a. 1, 2, 3, 5, 6, 10, 15, or 30 rows **b.** 1, 2, 4, 7, 14, or 28 rows **c.** 1 or 23 rows **d.** 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, or 120 rows **20.** (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103), (107, 109), (137, 139), (149, 151), (179, 181), (191, 193), (197, 199)

Assessment 4-2B

1. a.



2. b. $2 \cdot 5 \cdot 7 \cdot 43$ **c.** $2 \cdot 5^2 \cdot 191$ **5. g.** Not prime **h.** Not prime
8. If $4 \mid n$, then 2^2 is in the prime factorization of n. If $9 \mid n$, then 3^2 is in the prime factorization of n. Because 2^2 and 3^2

have no primes in common, by the Fundamental Theorem of Arithmetic, $2^2 \cdot 3^2$ must appear in the prime factorization of n. Therefore, $2^2 \cdot 3^2 | n$, or $36 | n$. **10.** 9 is not a prime factor. $2^2 \cdot 3^4 \cdot 5^3$ is the prime factorization. **11.** No, because 5^c has no factors of either 2 or 3. **12.** If $2n = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$, then $n = 2^5 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^6 = (2^4 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^5) \cdot (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)$. Thus, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ is a factor of n. **13.** No, because no whole number times $3^2 \cdot 2^4$ will give $3^3 \cdot 2^2$ **14. a.** 1 by 24, 2 by 12, 3 by 8. **b.** 1 array: 1 by 43. **15. a.** 1 by 15, 3 by 5. **b.** 1 by 24, 2 by 12, 3 by 8, 4 by 6. **c.** 1 by 17. **d.** 1 by 200, 2 by 100, 4 by 50, 5 by 40, 8 by 25, 10 by 20. **19.** 11 red, 2 green, 23 blue **20. c.** 23, 29, 31, 37, 53, 59, 71, 73, 79 **d.** Answers vary. For example, 233. **21. a. i.** Abundant **ii.** Deficient **iii.** Neither

Mathematical Connections 4-2

Reasoning

1. In any set of three consecutive numbers, there is one number that is divisible by 3 and at least one of the other two numbers is divisible by 2. Therefore, the product will be divisible by 2 and by 3 and so it is divisible by 6. **2.** In any set of four consecutive numbers, there are one or two numbers divisible by 3, a number divisible by 4, and a number divisible by 2 but not 4. Therefore, the product of the four consecutive numbers will have 2, 4, and 3 as distinct factors and so the product is divisible by $2 \cdot 3 \cdot 4 = 24$. **3.** The multiples of 4, 8, and 10 were crossed out with the multiples of 2. The multiples of 6 were crossed out when multiples of 2 and 3 were crossed out. The multiples of 9 were crossed out when the multiples of 3 were crossed out. All composite numbers less than 100 must have some factors less than or equal to 10 and these are all accounted for. Therefore, the remaining numbers that have not been crossed out are prime. **4.** Let $a = 2 \cdot 3 \cdot 5 \cdot 7$ and $b = 11 \cdot 13 \cdot 17 \cdot 19$. Then each prime p less than or equal to 19 appears in the prime factorization of a or of b but not in both. If p is in the prime factorization of a, then $p | a$ but $p \nmid b$ and hence $p \nmid (a + b)$. A similar argument holds if p is in the prime factorization of b. **5.** In Theorem 4-16 it states that if n is a whole number greater than 1 and not divisible by any prime p such that $p^2 \leq n$, then n is prime. The possible primes p such that $p^2 \leq 353$ are 2, 3, 5, 7, 11, 13, and 17. Because $2 \nmid 353$, $3 \nmid 353$, $5 \nmid 353$, $7 \nmid 353$, $11 \nmid 353$, $13 \nmid 353$, and $17 \nmid 353$, the number 353 is prime. **6.** Suppose n is composite and d is its least positive divisor other than 1. We need to show that d is prime. If not, then some prime less than d will divide d and hence will divide n which contradicts the fact that d is the least divisor of n. **7. a.** When n is divided by a prime number in the list the remainder is 1. **b.** Since $n > 1$, some prime appears in the prime factorization of n and hence it divides n. **c.** Yes, the argument assures us that for each prime we can find a greater prime. **8.** The students must have had the first 23 prime numbers for the numbers of tiles; that is 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53, 59, 61, 67, 71, 73, 79, 83 tiles. Therefore, the number of tiles is the sum of the first 23 prime numbers, which is 874 tiles. **9. a.** It has factors other than 1 and itself. **b.** Because $(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8) + 2 = 2[(3 \cdot 2 \cdot 5 \cdot 6 \cdot 7 \cdot 8) + 1]$ and $2 | 2[(3 \cdot 2 \cdot 5 \cdot 6 \cdot 7 \cdot 8) + 1]$, $(3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8) + 2$ is composite. **c.** Because $(3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) + 5 = 5[(3 \cdot 1 \cdot 7 \cdot 11 \cdot 13) + 1]$ and $5 | 5[(3 \cdot 1 \cdot 7 \cdot 11 \cdot 13) + 1]$, $(3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) + 5$ is composite. **d.** Because $(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10) + 7 = 7[(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10) + 1]$ and $7 | 7[(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10) + 1]$, $10! + 7$ is composite. **10.** 139 marbles **11.** 3, 3, 8

Open-Ended

12. a. In the interval 0–99. **i.** 25 **ii.** 21 **b.** The unique longest string in 100–199 is 114 through 126, of length 13. **c. i.** 8 **ii.** 7 **d.** Answers vary.

Cooperative Learning

13. The people who won free admission and had their hats on at the end of the contest were 1st, 4th, 9th, 25th, 36th, 49th, 64th, 81st, and 100th in line. A possible strategy would be to start with a simpler problem of 50 people and making a chart to see who will win free admission. **14.** Answers vary.

Connecting Mathematics to the Classroom

15. No. The method works for 3 and 4 because they have no common prime factors, but the method does not work for 2 and 6 because they have a common prime factor. A counterexample of 18 can be used to explain this to the students. **16.** 25 ants can make 2 queues of 12 and have a remainder of 1, or 3 queues of 8 with a remainder of 1, or 4 queues of 6 with a remainder of 1, or 5 queues of 5 with a remainder of zero. Modeling this story with tiles can help students find the remainders and the solution. **17.** As long as you factor to all prime factors the end result will be the same. **18.** Bob has almost the right idea but it needs a little work. It should be pointed out that if a number is not divisible by 2 and 3, then it couldn't be divisible by 6, so there is no need to check 6. Also, if a number is not divisible by 2, then it can't be divisible by 4 or 8, so there is no need to check 4 and 8. Next, if a number is not divisible by 5, then it can't be divisible by 10, so there is no need to check 10. Thus, we have cut Bob's list to 2, 3, and 5, which are all primes. The Sieve of Eratosthenes can be used to motivate these concepts. Next, we need to explore what happens when the number to be checked is large and show that just checking 2, 3, and 5 is not adequate. For example, just checking for divisibility by 2, 3, and 5 is not enough to check whether 169 is prime. We can use the sieve to show that if we are trying to determine if a whole number n is prime, we only have to check primes p such that $p^2 \leq n$. **19.** Not true: one counterexample is 11. **20.** There are infinitely many primes, but there's no pattern that allows us to find the next prime so it is more complicated than knowing there are infinitely many numbers. **21.** 1 is not prime because it does not have two distinct divisors; it only has one.

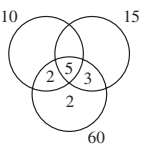
Review Problems

22. a. True **b.** True **c.** True **23. a.** 2, 3, 6 **b.** 2, 3, 5, 6, 9, 10 **24.** If $15 | n$, $(3 \cdot 5) | n$ so $3 | n$.

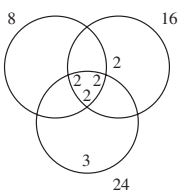
Assessment 4-3A

1. GCD(6, 8) = 2 (1 red); LCM(6, 8) = 24 (4 dark green, 3 brown) **2. a.** GCD(18, 12) = 6, LCM(18, 12) = 36
b. GCD(20, 36) = 4, LCM(20, 36) = 180
c. GCD(8, 24, 64) = 8, LCM(8, 24, 64) = 192
d. GCD(7, 9) = 1, LCM(7, 9) = 63
3. a. GCD(132, 504) = 12, LCM(132, 504) = 5544

b. GCD(65, 1690) = 65, LCM(65, 1690) = 1690
c. GCD(96, 900, 630) = 6, LCM(96, 900, 630) = 50,400
d. GCD(108, 360) = 36, LCM(108,360) = 1080
4. a. GCD(220, 2924) = 4 **b.** GCD(14,595, 10,856) = 1
5. a. LCM(24, 36) = 72 **b.** LCM(72, 90, 96) = 1440
c. LCM(90, 105, 315) = 630
d. LCM(9^{100}, 25^{100}) = $9^{100} \cdot 25^{100}$
8. b. GCD(a, a) = a LCM(a, a) = a
c. GCD(a^2, a) = a, LCM(a^2, a) = a^2
d. GCD(a, b) = a LCM(a, b) = b
11. The only prime factor of 4 is 2 and 97, 219, 988, 751 is odd. Consequently, 1 is their only common divisor and they are relatively prime.
12. a.



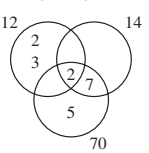
b.



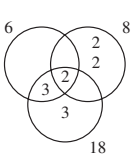
13. Answers vary. For example, (1, 48), (3, 48), (24, 48).
17. 5 packages **18.** They should pass the starting point after LCM(12, 18, 16) = 144 min. **19.** 6 in each bag

Assessment 4-3B

1. GCD(4, 10) = 2 (1 red); LCM(4, 10) = 20 (5 purple, 2 orange) **2. a.** GCD(12, 30) = 6; LCM(12, 30) = 60
b. GCD(18, 58) = 2; LCM(18, 58) = 522
c. GCD(6, 18, 24) = 6; LCM(6, 18, 24) = 72
d. GCD(36, 13) = 1; LCM(36, 13) = 468
3. a. GCD(11, 19) = 1; LCM(11, 19) = 209
b. GCD(140, 320) = 20; LCM(140, 320) = 2240
c. GCD(800, 75, 450) = 25; LCM(800, 75, 450) = 7200
d. GCD(103, 320) = 1; LCM(103, 320) = 32960
11. $11 \nmid 181{,}345{,}913$
12. a.



b.



13. Answers vary. For example, (1, 60), (2, 60), (30, 60).
16. 120th customer **18.** After 450 minutes, or 7.5 hrs at 2:30 P.M.

Mathematical Connections 4-3

Reasoning

1. No, the set of common multiples is infinite, and therefore there could be no GCM. **2.** No. GCD(2, 4, 10) = 2 LCM(2, 4, 10) = 20. GCD times LCM is 40 but $2 \cdot 4 \cdot 10 = 80$. **3.** No. Let $a = 2 \cdot 3$, $b = 3 \cdot 5$, $c = 5 \cdot 7$. Then GCD(a, b, c) = 1, but GCD(a, b) = 3, and GCD(b, c) = 5.
4. Yes. The hypothesis implies that 2 is a common divisor of a, b, and c. If a number greater than 2 divided a, b, and c, then in particular that number would have to divide a and b, which is impossible because GCD(a, b) = 2. Thus, GCD(a, b, c) = 2.
5. Yes. The LCM equals the product of the numbers if, and only if, the numbers have no prime factors in common. Because GCD(a, b) · LCM(a, b) = ab if, and only if, GCD(a, b) = 1; that is a and b have no factors in common. **6.** No. Answers vary: one way is to notice that $\text{LCM}(a, b) = \dfrac{ab}{\text{GCD}(a, b)}$. **7.** True. Answers vary. For example, $m = 6$, and $n = 9$; then GCD(6, 9) = 3 and LCM(6, 9) = 18. GCD(15, 18) = 3.

Open-Ended

8. Answers vary. For example, LCM(8, 12) = 24 < 96; LCM(6, 15) = 30 < 90; LCM(10, 25) = 50 < 250.
9. a. GCD($2a$, $2b$) = 2 if GCD(a, b) = 1
b. GCD($6a$, $6b$) = 6 if GCD(a, b) = 1
c. GCD($91a$, $91b$) = 91 if GCD(a, b) = 1
10. For example, 1500 teeth and 72 teeth; 750 teeth and 36 teeth; 375 teeth and 18 teeth; smallest number of teeth is 6 teeth.

Cooperative Learning

11. a. Answers vary **b.** Answers vary.

Connecting Mathematics to the Classroom

12. Answers vary. Most teachers will use an example such as 6. The divisors are {1, 2, 3, 6}, whereas the multiples of 6 are {6, 12, 18, 24, 30, 36, . . .}. **13.** She's incorrect. GCD(a, 0) = a for $a \neq 0$. **14.** Using the prime factorization method for finding the LCM you can tell Aiko that he can multiply the numbers together if there are no common factors in their prime factorization. If there are common factors, then we use the prime factor with the highest exponent. **15.** There are 3333 multiples of 3 and 2000 multiples of 5, but you need to subtract multiples of 15 which have been double counted: (3333 + 2000) − 666 = 4667.

Review Problems

16. a. 7 **b.** 6 **c.** 9 **17. a.** $2^3 \cdot 3^7$ **b.** $3^2 \cdot 5 \cdot 727$ **c.** $2^{10} \cdot 3^9$
18. Not prime; $3 \mid 2223$ **19.** Answers vary. For example, 2^5.
20. 120 **21.** 53

Chapter 4 Review

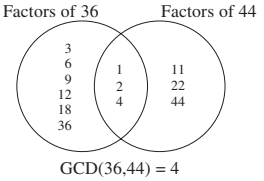
4. b. Let n be a whole number then $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10 = 5(n + 2)$. Thus, the sum is divisible by 5. **5.** Check divisibility by 3 and 8, $24 \nmid 4152$. **6.** Answers vary. For example, 16. To obtain five divisors we raise a prime to the (5 − 1) power. **7.** 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144
8. $n = a \cdot 10^2 + b \cdot 10 + c$
$= a \cdot (99 + 1) + b \cdot (9 + 1) + c$
$= 99 \cdot a + 9 \cdot b + a + b + c$
Since $9 \mid 99a$ and $9 \mid 9b$, then $9 \mid [(99 \cdot a + 9 \cdot b) + (a + b + c)]$ if, and only if, $9 \mid (a + b + c)$. **9.** If 1009 is prime, $17 \nmid 1009$. We know $17 \mid 17$, so $17 \nmid (1009 + 17)$.
11. a. Because $(7 \cdot 11 \cdot 13 \cdot 17) + 17 = 17[(7 \cdot 11 \cdot 13) + 1]$ and $17 \mid 17[(7 \cdot 11 \cdot 13) + 1]$, $(7 \cdot 11 \cdot 13 \cdot 17) + 17$ is composite. **b.** Because $(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10) + k$ can be factored as in part (a) for the given values of k, $10! + k$ is composite. **12.** We first show that among any three consecutive odd whole numbers, there is always one that is divisible by 3. For that purpose, suppose that the first whole number in the triplet is not divisible by 3. Then by the division algorithm, that whole number can be written in the form $3n + 1$ or $3n + 2$ for some whole number n. Then the three consecutive odd whole numbers are $3n + 1$, $3n + 3$, $3n + 5$ or $3n + 2$, $3n + 4$, $3n + 6$. In the first triplet $3n + 3$ is divisible

by 3, and in the second $3n + 6$ is divisible by 3. This implies that if the first odd whole number is greater than 3 and not divisible 3, then the second or the third must be divisible by 3, and hence cannot be prime. **18.** No, they are the same if the numbers are equal. **19.** $\text{LCM}(a, b, c) = \text{LCM}(m, c)$, where $m = \text{LCM}(a, b)$. $\text{LCM}(a, b) = \dfrac{ab}{\text{GCD}(a, b)}$ and $\text{LCM}(m, c) = \dfrac{mc}{\text{GCD}(m, c)}$. Each of these GCDs can be found using the Euclidean algorithm. **20.** We know that $\text{GCD}(a, b) \cdot \text{LCM}(a, b) = ab$. Because $\text{GCD}(a, b) = 1$, then $\text{LCM}(a, b) = ab$. **22.** 71 lattes. Because $9869 = 71 \cdot 139$ and 71 as well as 139 are prime, she sold 71 lattes at \$1.39 each.

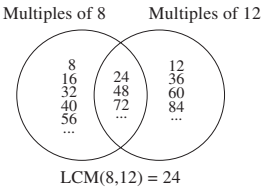
Answers to Now Try This

1. If $5 \nmid a$ and $5 \nmid b$ then $5 \nmid (a + b)$ is not always true. For example, $5 \nmid 8$ and $5 \nmid 12$, but $5 \mid (8 + 12)$. The statement is sometimes true for example $5 \nmid 7$ and $5 \nmid 12$ and $5 \nmid (7 + 12)$. **3. a.** There will never be other entries in column 1 because 1 is the only number with one factor. Other numbers have at least the number itself and 1. **b.** 49, 121, 169 **c.** 81 **d.** 64 **e.** The square numbers have an odd number of factors. Factors occur in pairs; for example, for 16 we have 1 and 16, 2 and 8, and 4 and 4. When we list the factors, we list only the distinct factors, so 4 is not listed twice, thereby making the number of factors of 16 an odd number. Similar reasoning holds for all square numbers. **f.** Answers vary. For example, only square numbers are listed in column 3; 2 is the only even number that will ever be in column 2, and column 2 contains prime numbers. **6. a.** The leftmost area contains the factors of 24 that are not factors of 40. The center (intersection) area is the factors of both 24 and 40. The rightmost area represents the factors of 40 that are not factors of 24. **b.** 8

c.



7.



Answer to the Preliminary Problem

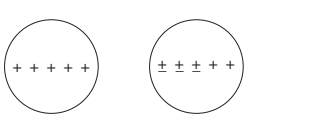
$\text{LCM}(23, 28, 33) = 21{,}252$ days after peak all three cycles will be at their peak again.

CHAPTER 5

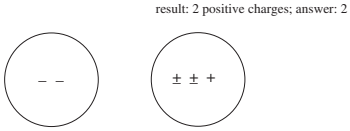
Assessment 5-1A

1. f. $^-(a + b)$ or $^-a + {}^-b$

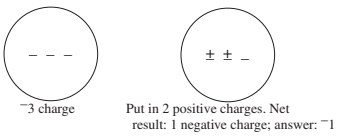
4. a.



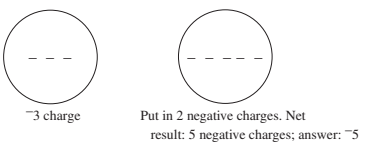
b.



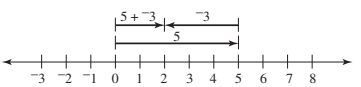
c.



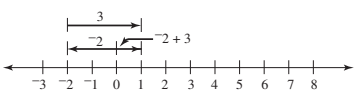
d.



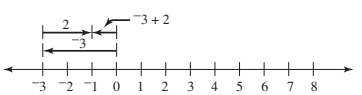
5. a.



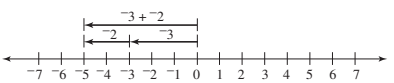
b.



c.



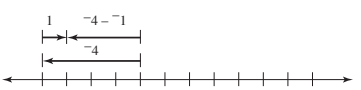
d.



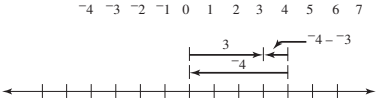
6. a. $7 + {}^-13 = {}^-(|{}^-13| - |7|)$
$= {}^-(13 - 7)$
$= {}^-(6)$
$= {}^-6$

b. $^-7 + {}^-13 = {}^-(|{}^-7| + |{}^-13|)$
$= {}^-(7 + 13) = {}^-(20)$
$= {}^-20$

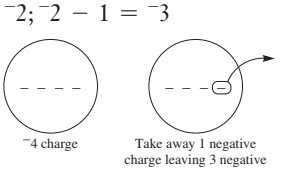
8. a. $^-2 + _ = 3$, so $_ = 5$.
b. $2 + _ = {}^-3$, so $_ = {}^-5$. **c.** $^-2 + _ = {}^-3$, so $_ = {}^-1$.
9. c. $35{,}000 + {}^-1000 = 34{,}000$ leaving an altitude of 34,000′
10. a. $^-45 + {}^-55 + {}^-165 + {}^-35 + {}^-100 + 75 + 25 + 400$
11. a.



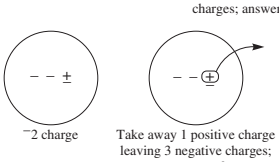
b.



12. a. $^-4 - 2 = {}^-6; {}^-4 - 1 = {}^-5; {}^-4 - 0 = {}^-4; {}^-4 - {}^-1 = {}^-3$
b. $3 - 1 = 2; 2 - 1 = 1; 1 - 1 = 0; 0 - 1 = {}^-1;$
$^-1 - 1 = {}^-2; {}^-2 - 1 = {}^-3$
13. a.



b.



15. b. $200 - 220; 200 + {}^-220; {}^-20$
16. a. $3 - (2 - 4x) = 3 + {}^-(2 + {}^-(4x)) =$
$3 + ({}^-2 + 4x) = (3 + {}^-2) + 4x = 1 + 4x$
b. $x - ({}^-x - y) = x + {}^-({}^-x + {}^-y) = x + (x + y) =$
$(x + x) + y = 2x + y$
17. Let a, b, and c be any integers. We have

$a - b + c = a + {}^{-}b + c$ Theorem 5-4
$= a + {}^{-}(b + {}^{-}c)$ Theorem 5-3 and Theorem 5-4
$= a - (b - c)$ Theorem 5-4

The equation is true for all integers.

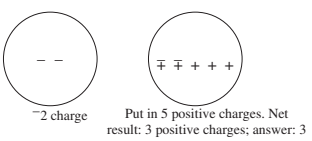
22. Answers vary. Take the opposite of the sum of the first 100 positive numbers, or 5050. The result is $^{-}5050$. **24.** Answers vary. If the east coast time is 0, then the Pacific time is $^{-}3$.

25. $55 + 10 + {}^{-}7 + {}^{-}13 = 45$

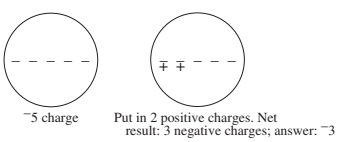
26. $1 + 0 + {}^{-}2 + 2 + {}^{-}1 + {}^{-}3 = {}^{-}3$; he is 3 under par for these holes.

Assessment 5-1B

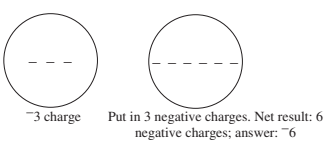
4. a.



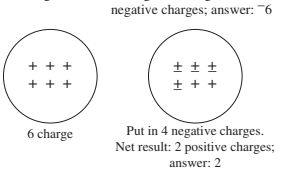
b.



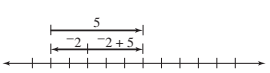
c.



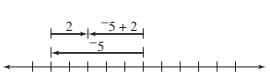
d.



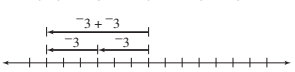
5. a.



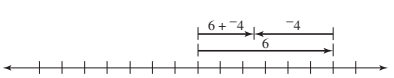
b.



c.



d.



6. a. $5 + {}^{-}31 = {}^{-}(|{}^{-}31| - |5|)$
$= {}^{-}(31 - 5)$
$= {}^{-}26$

b. ${}^{-}5 + {}^{-}31 = {}^{-}(|{}^{-}5| + |{}^{-}31|)$
$= {}^{-}(5 + 31)$
$= {}^{-}36$

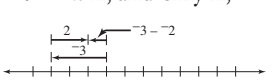
7. a. ${}^{-}3 - 5 = {}^{-}3 + {}^{-}5 = {}^{-}8$ **b.** $5 - ({}^{-}3) = 5 + 3 = 8$

8. a. ${}^{-}3 - 5 = x$ if, and only if, ${}^{-}3 = 5 + x$. Thus, $x = {}^{-}8$.

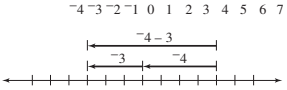
b. $5 - ({}^{-}3) = x$ if, and only if, $5 = {}^{-}3 + x$. Thus, $x = 8$.

c. ${}^{-}2 - {}^{-}3 = x$ if, and only if, ${}^{-}2 = x + {}^{-}3$. Thus, $x = 1$.

11. a.



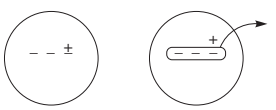
b.



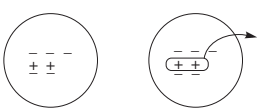
12. a.
$^{-}2 - 1 = {}^{-}3$
$^{-}2 - 0 = {}^{-}2$
$^{-}2 - {}^{-}1 = {}^{-}1$
$^{-}2 - {}^{-}2 = 0$
$^{-}2 - {}^{-}3 = 1$

b.
$3 - 2 = 1$
$2 - 2 = 0$
$1 - 2 = {}^{-}1$
$0 - 2 = {}^{-}2$
$^{-}1 - 2 = {}^{-}3$
$^{-}2 - 2 = {}^{-}4$
$^{-}3 - 2 = {}^{-}5$

13. a.



b.



16. a. $4x - 2 - 3x = 4x + {}^{-}2 + {}^{-}3x = 4x + {}^{-}3x + {}^{-}2 = x + {}^{-}2 = x - 2$ **b.** $4x - (2 - 3x) = 4x + {}^{-}(2 + {}^{-}3x) = 4x + {}^{-}2 + {}^{-}({}^{-}3x) = 4x + 3x + {}^{-}2 = 7x + {}^{-}2 = 7x - 2$

17. a. ${}^{-}x - y = {}^{-}x + {}^{-}y = {}^{-}y + {}^{-}x = {}^{-}y - x$ **b.** No. Because ${}^{-}x - y = {}^{-}x + {}^{-}y$ and ${}^{-}y - x = {}^{-}y + {}^{-}x$, the property in part (a) is just the commutative property of addition.

25. Answers vary. If x represents the amount of memory used $x + {}^{-}3.6 - {}^{-}7.2 = x + 3.6$ MB; she gained an extra 3.6 MB of memory. **26. b.** All integers except 0 **d.** $x < 1$

28. $y = \begin{cases} x - 6 & \text{if } x \geq 6 \\ {}^{-}x + 6 & \text{if } x < 6 \end{cases}$ **30. b.** False; let $x = {}^{-}1$.

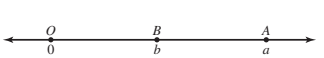
Mathematical Connections 5-1

Reasoning

1. Answers vary. He could have driven 12 mi in either direction from milepost 68. Therefore, his location could be either at the $68 - 12 = 56$ milepost or at the $68 + 12 = 80$ milepost.

2. Answers vary. The logic is flowed. However, by definition of subtraction $(a - b) + b = a$. Also using the associative property of addition and the definition of an additive inverse, we have:
$(a + {}^{-}b) + b = a + {}^{-}b + b$
$= a + 0$
$= a$
Hence, $(a + {}^{-}b) + b = a + {}^{-}b + b$. Adding ^{-}b to both sides of the last equation, we obtain $a - b = a + {}^{-}b$. **3.** Two numbers are additive inverses if, and only if, their sum is 0. We have:
$(b - a) + (a - b) = b + {}^{-}a + a + {}^{-}b$
$= b + 0 + {}^{-}b$
$= b + {}^{-}b$
$= 0$

4. No; if $x < 0$, then ^{-}x is positive. **5.** Answers vary. Students may substitute different integers for a and b, plot them on a number line, and show that in each case the distance between the points is $|a - b|$. They should give at least three examples: when both integers are positive, one is positive and the other negative, and both are negative. A more general approach is as follows: If $a > b > 0$, then $AB = OA - OB = a - b = |a - b|$.

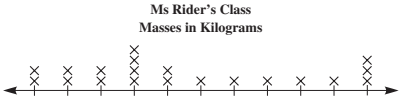


during the last 2 yr. In such a case, most of the pickups have been on the road for only 2 yr, and therefore the given information might imply, but would not substantiate, that the average life of a pickup is around 10 yr. **4.** The horizontal axis does not have uniformly sized intervals and both the horizontal axis and the graph are not labeled. **5.** The three-dimensional drawing distorts the graph. The result of doubling the radius and the height of the can is to increase the volume by a factor of 8. **6.** One would need more information; for example, do men drive more than women? **7.** No, because the mean can be affected by a few high salaries. **8.** The first graph shows a steep increase in the price of the LCD TV over the six months, which the advertising agency may want to show for the benefit of the store. **9.** Answers vary. **a.** This bar graph would have perhaps 20 accidents as the baseline and some larger number (e.g., 50) as its maximum. Then 24 in 2011 would be 4 units above the baseline while 30 in 2015 would appear as 4.5 times or 450% taller than 24 of 2011, when in fact it is just 58% higher. **b.** This bar graph would have zero accidents as its baseline and some larger number (e.g., 100) as its maximum value. The effect would be to show values from 24 to 38 as lying a line with a small positive slope **10.** Answers vary. An example is 1, 1, 1, 1, 20, 30, 40, 50. **11.** A median might be misleading, depending on the number of data points given. Also the mean would not be sufficient. A report of the mean, median, and standard deviation would be the most helpful of all "averages" studied. **12.** The student is incorrect and, in addition, is misusing percentages. She mixes percentages of "effectiveness" with percentages of times taking the drug. **13.** Answers vary. **a.** You probably would not have a representative sample. This is a *convenience sample*. **b.** You probably would not. The students eating in the cafeteria could likely be a biased sample. **c.** Assign students numbers from 1 to n, where there are n students. Choose your sample using random-number selection methods. Make sure your sample is big enough. **14.** The data are indicators of passenger complaints and not an overall airline rating. Larger airlines may have more complaints. The numbers are not percentages. The number of complaints might depend on how many flights each airline has per day. **15.** Answers vary. You might conclude that the less you spend the better your students perform on the SAT exam in verbal and math scores. **16.** Only part (b) might not be helpful.

Chapter 10 Review

1. i. If the average is reported as 2.41 children, the mean is being used. The mode would not have a decimal number, and the median would be either a whole number or one ending in .5. ii. If the average is 2.5, then either the mean or the median might have been used. **2.** 12 **3. a.** Mean = 30, median = 30, mode = 10. **b.** Mean = 5, median = 5, modes = 3, 5, 6. **c.** Mean = 150, median = 150, mode = 100, 200 (bi-modal) **d.** Mean = 13.01, median = 12.15, mode (none)
4. a. Range = 50, variance ≈ 371.4, standard deviation ≈ 19.3, mean absolute deviation ≈ 17.14, IQR = 40. **b.** Range = 8, variance = 5.2, standard deviation = 2.28, mean absolute deviation = 1.8, IQR = 3. **c.** Range = 100, variance = 2500, standard deviation = 50, mean absolute deviation = 50, IQR = 100. **d.** Range = 18.5, variance ≈ 17.34, standard deviation ≈ 4.16, mean absolute deviation ≈ 2.18, IQR ≈ 2.35.

5. a.



b. Ms Rider's Class Masses in Kilograms

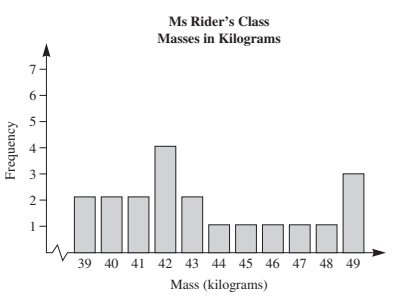
3	99
4	001122223345678999

4|0 represents 40 kg

c. Ms Rider's Class Masses in Kilograms

Mass	Tally	Frequency
39	\|\|	2
40	\|\|	2
41	\|\|	2
42	\|\|\|\|	4
43	\|\|	2
44	\|	1
45	\|	1
46	\|	1
47	\|	1
48	\|	1
49	\|\|\|	$\frac{3}{20}$

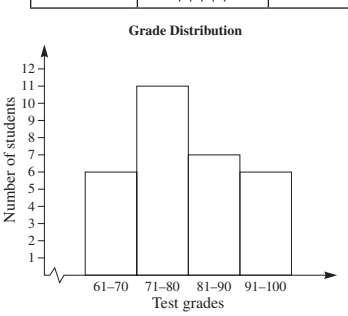
d.



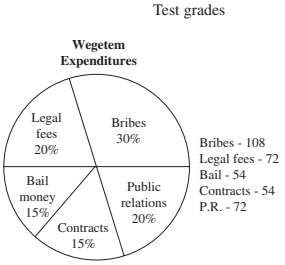
6. a. Test Grades

Classes	Tally	Frequency
61–70	𝍸\|	6
71–80	𝍸𝍸\|	11
81–90	𝍸\|\|	7
91–100	𝍸\|	6

b.

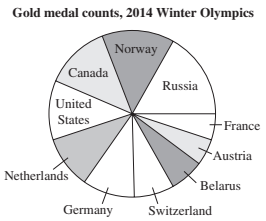


7.



8. The graph is missing axis labels: The bars are not of equal width. **9.** $2840

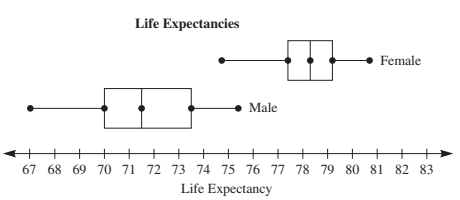
10.



11. a. Life Expectancies of Males and Females

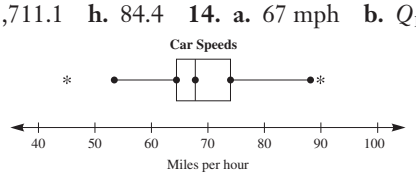
Females		Males
	67	1446
	68	28
	69	156
	70	0049
	71	02235578
	72	01145
	73	1689
7	74	123599
9310	75	14
86	76	
88532	77	
9999854332211	78	
99655443210	79	
42	80	
7 \| 74 \| represents 74.7 years old		\| 67 \| 1 represents 67.1 years old

b.



c. 75% of the women's data is significantly above that of men; women are expected to live longer. **12.** Larry was correct because his average was $3.2\overline{6}$, while Marc's was $2.7\overline{3}$. **13. a.** 360 **b.** None **c.** 350 **d.** About 108.2 **e.** 320 **f.** 200 **g.** About 11,711.1 **h.** 84.4 **14. a.** 67 mph **b.** $Q_1 = 64, Q_3 = 74$

c.



d. 30% **e.** About 50% of people checked drove between 65 and 75 mph. **f.** 64; 74; 67 **15. a.** Positive **b.** 170 lbs **c.** 67 in. **d.** 64 in. **e.** 50 lbs **16. a.** Collette is more consistent. There is less deviation from the mean in her scoring. **b.** Collette: 18–30; Rudy: 10–38 **c.** Collette scored more than might have been expected. **17.** 2.5% **18. a.** The score for each **b.** Same as (a) **c.** Same as (a) **d.** 0 **e.** 0 **19.** The bars would be approximately the same height in a vertical bar graph. **20. a.** 19 lbs. **b.** It is nearly impossible for this to happen. **c.** Since the company claims a person can lose up to 6 lbs, any average loss from 0 to 6 is legitimate. **21.** The bar graph is more appropriate. Line graphs are used to show change over time. The line graph indicates there are values between the colors, which is not true. **22.** No reportable single number would be meaningful for such a pool. **23.** Answers vary. A reasonable way to make the claim is to pull boxes at random after they are packed and weigh the contents. If more than 98% of the boxes' contents have weights within 1 g of 138 g, the claim is reasonable. The choices of 98% and 1 g are arbitrary but reasonable choices to test this claim. **24. a.** Answers vary. Perhaps to be with those of comparable age a typical senior might choose the first. **b.** The first **25.** The areas representing the two categories appear roughly the same size. **26.** Answers vary. **a.** One way would be to leave the television on, even if no one is watching. **b.** They show very popular shows during "ratings sweeps" periods. **27.** Answers vary. For example, graphs may show area or volume instead of relative size; another is to select a horizontal baseline that will support the point being made. **28. a.** About 3 billion **b.** About 6 billion **c.** The population doubled **d.** About 9.3 billion **e.** About 55% **29. a.** 25 **b.** 475 **c.** 0.16 **30.** 475 **31. a.** 525 **b.** 600 **c.** 675 **32.** The March 31 Goal is portrayed almost 3 times greater than the numbers as of March 27, which it is not. **33.** The first person has the calculator and picks a large random number. The first person then records the number, adds his salary to it, and passes the calculator to the second person. That person adds his salary and passes it to the next person. This continues for all five people. When the calculator is returned to the first person, he subtracts the random number that was used and divides by 5 to obtain the mean salary.

Answers to Now Try This

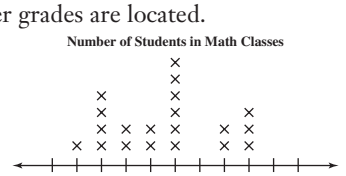
1. Quick Check: 2a-biased, 2b-fair.

2. Ages of Presidents at Death

4	69	
5	36778	
6	003344567778	4\|9 represents 49 years old
7	0112347889	
8	01358	
9	0033	

3. It appears that the fifth-period class did better. We can see that there are more scores grouped toward the bottom, which is where the higher grades are located.

4. a. 2.



a. 3.

Age	Tally	Frequency
20–29	𝍸 \|\|\|	8
30–39	\|\|\|\|	4
40–49	\|\|\|	3
50–59	\|\|	2
60–69	\|\|	2
70–79	\|	1

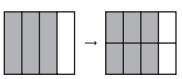
There are fewer negative differences than positive differences so that would be a case for saying the experiment worked. However, the sum of all the differences is

$$3 + {}^{-}3 + 2 + {}^{-}5 + 0 + 4 + 2 + 2 + 1 + {}^{-}7 = {}^{-}1.$$

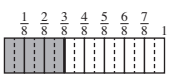
Because the sum of all the differences is negative, it would be very difficult to make a case for the experiment being a success.

CHAPTER 6

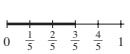
Assessment 6-1A

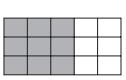
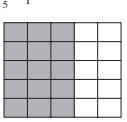
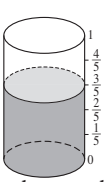
4. a. $\frac{2}{3}$ **b.** $\frac{4}{6} = \frac{2}{3}$ **c.** $\frac{6}{9} = \frac{2}{3}$ **d.** $\frac{8}{12} = \frac{2}{3}$. The diagrams illustrate the Fundamental Law of Fractions. **8. a.** $\frac{4}{43}$ **b.** $\frac{17}{43}$ **d.** $\frac{39}{43}$ **15.** Answers vary.  **17. a.** 

26. n cannot be 0; otherwise there would $\frac{0}{0}$ in the theorem.

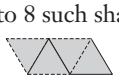
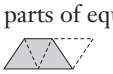
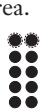
22. a. $\frac{6}{16} = \frac{3}{8}$; $\frac{6}{32{,}000} = \frac{3}{16{,}000}$ **b.** $\frac{10}{100} = \frac{1}{10}$ **c.** $\frac{15}{60} = \frac{1}{4}$ **d.** $\frac{8}{24} = \frac{1}{3}$ **27.** $n = \frac{n}{1}$ for all integers n, so every integer can be written as a fraction. **28.** Drawing is only a representation, not to scale. 

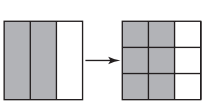
Assessment 6-1B

4. Answers vary. **a.**  **b.**

c.  **d.**  **e.**  **f.** 

5. Answers vary. **a.** No, the parts do not have to be equal. **b.** Yes, because the square can be divided into 8 such shaded triangles of equal area. **c.** Yes, because the circle can be divided into 8 such shaded parts of equal area. **6.** Answers vary.

a.  **b.**  **c.**  **d.** 

15. Answers vary.  **21. a.** $\frac{12}{16} = \frac{3}{4}$ **b.** $\frac{5}{100} = \frac{1}{20}$ **c.** $\frac{25}{60} = \frac{5}{12}$ **d.** $\frac{16}{24} = \frac{2}{3}$ **24.** $n = \frac{n}{1}$, where n is an integer. Because the set of integers is infinite, so is this representation.

29. a.
Area of $a = \frac{1}{4}$
Area of $b = \frac{1}{4}$
Area of $c = \frac{1}{16}$
Area of $d = \frac{1}{8}$
Area of $e = \frac{1}{16}$

b.
Area of $a = 1$
Area of $b = 1$
Area of $c = \frac{1}{4}$
Area of $d = \frac{1}{2}$
Area of $e = \frac{1}{4}$

Area of $f = \frac{1}{8}$
Area of $g = \frac{1}{8}$

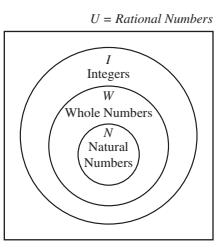
Area of $f = \frac{1}{2}$
Area of $g = \frac{1}{2}$

Mathematical Connections 6-1

Reasoning

1. Answers vary. The fractions are using different bases with one being \$1 and the other being 1 hour, or 60 minutes. The fractions are each $\frac{1}{4}$ and are equivalent, but in context with the different bases, they represent different quantities. **2.** Not necessarily. The classes may have different numbers of students. If one has 21 students and the other 24, then in the first there are 7 girls and in the second 8 girls. **3.** The new fraction is equal to $\frac{1}{2}$.

$$\frac{ar_1}{br_1} + \frac{ar_2}{br_2} + \frac{ar_3}{br_3} + \ldots + \frac{ar_n}{br_n} = \frac{ar_1 + ar_2 + ar_3 + \ldots + ar_n}{br_1 + br_2 + br_3 + \ldots + br_n} = \frac{a(r_1 + r_2 + r_3 + \ldots + r_n)}{b(r_1 + r_2 + r_3 + \ldots + r_n)} = \frac{a}{b}$$

4. 

$N \subset W \subset I \subset U$

Open-Ended

5. Answers vary. **6.** Answers vary.
7. Answers vary depending upon the windows chosen.

Cooperative Learning

8. Answers vary. **9.** Answers vary.

Connecting Mathematics to the Classroom

10. $\frac{0}{6}$ is not in simplest form. A fraction $\frac{a}{b}$ is in simplest form if and only if $\text{GCD}(a, b) = 1$; however, $\text{GCD}(0, 6) = 6$. The simplest form is $\frac{0}{6} = 0$. **11.** The first student's approach is correct. What the second student has done is to treat the problem as if it had been $\left(\frac{1}{5}\right)\left(\frac{5}{3}\right) = \frac{1}{3}$, when in reality, the problem is $\frac{15}{53}$. One cancels factors, not digits. **12.** According to Theorem 6-4, there is another rational number between any two given rational numbers. $\frac{9999}{10000}$ is one rational number between $\frac{999}{1000}$ and 1. **13.** Answers vary. The student was probably thinking that more pieces meant more pizza. A pizza (or circle) could be cut into 6 pieces, then each piece could be cut into 2 pieces; This shows the amount of pizza did not change from these last cuts, only the number of pieces changed. **14.** Answers vary; for example, she could show $\frac{3}{4}$ of the faces by picking 3 of the 4 large faces and 3 of the 4 small faces. **15.** Answers vary. **a.** Suppose the rational numbers are $\frac{2}{16}$ and $\frac{1}{4}$. In this case, $\frac{1}{4} > \frac{2}{16}$ and Iris

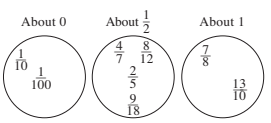
is incorrect **b.** Suppose the rational numbers are $\frac{2}{3}$ and $\frac{1}{2}$. In this case, $\left(\frac{2}{3}\right) > \left(\frac{1}{2}\right)$ and Shirley is incorrect. **16.** The whole in this case is the 3 sets of 5 dots. Since 2 sets are shaded, $\frac{2}{3}$ of the whole is shaded, and Steve is not correct. If each individual dot was considered, then there are 15 dots in the whole and 10 are shaded. Therefore, $\frac{10}{15} = \frac{2}{3}$ of the dots are shaded.

17. If you write each fraction over a common denominator, it is easier to compare them. Thus we have $\frac{3}{8} = \frac{9}{24}$ and $\frac{2}{3} = \frac{16}{24}$. Because $\frac{16}{24} > \frac{9}{24}$, then $\frac{2}{3} > \frac{3}{8}$.

18. Because the fraction of students passing in each class was $\frac{4}{5}$, the argument could be made that they did equally well on the test.

Assessment 6-2A

4. b. $\frac{1}{6}$, low **d.** $\frac{1}{2}$, high **6.**



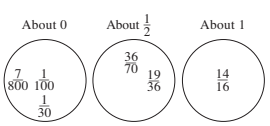
15. Answers vary. **a.** $\frac{1}{2} + \frac{3}{4} \in Q$ **b.** $\frac{1}{2} + \frac{3}{4} = \frac{3}{4} + \frac{1}{2}$

c. $\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4} = \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$ **16.** $1, \frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \frac{10}{6}, \frac{11}{6}, 2$

17. b. $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$

Assessment 6-2B

6.



12. d. Not necessarily, the total number of dollars might have been greater in 2012 than in 2014, but the fraction of the total dollars was greater in 2014. **16.** $1, \frac{7}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, 3$

19. d. No, $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$, so the numerator will always be less than the denominator.

20. Answers vary. **a.** About $\frac{1}{736}$ **b.** About $\frac{1}{2}$ **c.** About $\frac{2}{127}$

22. $\frac{303}{360} = \frac{101}{120}$ **23.** $\frac{7}{100}$

Mathematical Connections 6-2

Reasoning

1. Answers vary. For example, $\frac{1}{3} + \frac{1}{4}$ or $\frac{1}{12}$ does not represent the amount received since the fractions did not come from the same size "whole." **2. a.** No, but it is easier to write in simplest terms if we do choose a least common denominator. **b.** No. For example, $\frac{1}{3} + \frac{1}{6} = \frac{3}{6}$, which is not in simplest form. **3.** Answers vary. $5\frac{3}{4} = \frac{5}{1} + \frac{3}{4} = \frac{5 \cdot 4 + 1 \cdot 3}{1 \cdot 4} = \frac{5 \cdot 4 + 3}{4}$ **4.** No, Kara spent $\frac{1}{3}$ of $\frac{1}{2}$ of her allowance on Sunday, so she has $\frac{1}{2} - \frac{1}{3} \cdot \frac{1}{2} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6}$, or $\frac{1}{3}$, of her allowance left.

5. Answers vary. For example, $3\frac{3}{4} + 5\frac{1}{3} = \frac{15}{4} + \frac{16}{3} = \frac{45 + 64}{12} = \frac{109}{12} = 9\frac{1}{12}$ or $3\frac{3}{4} + 5\frac{1}{3} = (3 + 5) + \left(\frac{3}{4} + \frac{1}{3}\right) = 8\frac{13}{12} = 9\frac{1}{12}$. In the second method, the numbers are smaller to work with. **6. a.** Yes. If a, b, c, and d are integers, $b \neq 0, d \neq 0$, then $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ is a rational number.

b. No; for example, $\frac{1}{2} - \frac{1}{4} \neq \frac{1}{4} - \frac{1}{2}$.

c. No; for example, $\frac{1}{2} - \left(\frac{1}{4} - \frac{1}{8}\right) \neq \left(\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{8}$.

d. No. If there is an identity for subtraction, it must be 0, since only for 0 does $\frac{a}{b} - 0 = \frac{a}{b}$. However, in general, $0 - \frac{a}{b} \neq \frac{a}{b} - 0$, and hence there is no identity. **e.** No; since there is no identity, an inverse cannot be defined.

Open-Ended

7. Answers vary. For example, Mike ate a third of a pizza, leaving $\frac{2}{3}$ of the pizza for Ann. Ann gave $\frac{1}{4}$ of the entire pizza to her little brother. What fraction of the pizza was left for Ann?

8. a. Answers vary, for example, $\frac{2}{3}$ and $\frac{1}{3} \cdot \frac{b - a}{b}$.

b. Answers vary, for example, $\frac{2}{7}, \frac{4}{7}$, and $\frac{1}{7}$. **c.** Answers vary, for example, $\frac{9}{8}$ and $\frac{1}{7}$. **9.** Answers vary.

a. i. $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}; \frac{1}{3} = \frac{1}{6} + \frac{1}{6}$

ii. $\frac{1}{7} = \frac{1}{8} + \frac{1}{56}; \frac{1}{7} = \frac{1}{14} + \frac{1}{14}$

b. $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1) - n}{n(n+1)} = \frac{1}{n(n+1)}$

c. Rewriting part (b) as a sum gives $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$.

d. Answers vary. $\frac{1}{17} = \frac{1}{18} + \frac{1}{17(18)} = \frac{1}{18} + \frac{1}{306}$

Cooperative Learning

10. Answers vary. Depending on the people interviewed, students may hear an answer like the following from a teacher, "I use fractions in determining total grades for my classes. For example, if a paper is $\frac{1}{2}$ of the grade and a test is another $\frac{1}{3}$ of the grade, I need to know what fractional part of the grade is yet to be determined.

Connecting Mathematics to the Classroom

11. Answers vary. Kendra's picture shows that $\frac{1}{3}$ of the 3-square whole combined with $\frac{3}{4}$ of the 4-square whole gives $\frac{4}{7}$ of a

7-square whole. To do what Kendra is trying to do, she would need to use the same size whole rather than 3 different wholes. When $\frac{1}{3} + \frac{3}{4}$ are added, the same whole must be used.

12. Answers vary. For example, the teacher is thinking each rectangle, which is divided into 4 equal parts, represents a whole divided into fourths, so the picture is $2\frac{3}{4}$. Ken is thinking the three rectangles represent the whole and 11 of 12 parts are shaded. **13.** Answers vary. Jill is confusing this with adding fractions with common denominators. If a, b, and c are positive integers, then $\frac{a}{b+c} < \frac{a}{b}$, but $\frac{a}{b} + \frac{a}{c} > \frac{a}{b}$. A numerical example such as $\frac{1}{2} + \frac{1}{3} \neq \frac{1}{5}$ is sufficient to disprove Jill's claim. **14. a.** Like digits are being cancelled. **b.** Numerators and denominators are both being added. **c.** Numerators and denominators of fractional portions are both being subtracted.

Review Problems

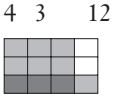
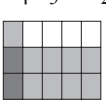
15. a. $\frac{2}{3}$ **b.** $\frac{13}{17}$ **c.** $\frac{25}{49}$ **d.** a where $a \neq {}^{-}1$ **e.** Simplified **f.** $a + b$ where $a \neq b$ **16. a.** Equal **b.** Not equal **c.** Equal **d.** Not equal **17.** It is possible because the fraction $\frac{27}{103}$ is equal to $\frac{54}{206}$, meaning that there would be 54 bones in both hands.

18. There are infinitely many nonzero integers n, such that $\frac{3}{5} = \frac{3n}{5n}$.

19. $\frac{2}{5}$ **20.** $\frac{{}^{-}1}{10} = \frac{{}^{-}10}{100}$ and $\frac{{}^{-}10}{100} < \frac{{}^{-}1}{100}$ because $\frac{{}^{-}10}{100} + \frac{9}{100} = \frac{{}^{-}1}{100}$, and $\frac{9}{100}$ is a positive number. Thus, $\frac{{}^{-}1}{100}$ is farther to the right on a number line (and greater than) $\frac{{}^{-}1}{10}$.

Assessment 6-3A

1. Answers vary. **a.** $\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$ **b.** $\frac{2}{4} \cdot \frac{3}{5} = \frac{6}{20}$

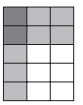
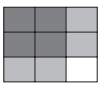
2. Answers vary. **a.**  **b.** 

19. a., b., c., e., and f. are false; counterexamples vary. **d.** True; $ab \neq 0$; use Theorem 6-19a. **22. d.** $\left(\frac{4}{5}\right)^{10}$

23. Answers vary. $2\left(\frac{a}{b}\right) = \frac{a}{b} + \frac{a}{b} < \frac{a}{b} + \frac{c}{d}$ because $\frac{a}{b} < \frac{c}{d}$. Thus, $2\left(\frac{a}{b}\right) < \frac{a}{b} + \frac{c}{d}, \frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right)$. Also, $\frac{1}{2}\left(\frac{a}{b} + \frac{a}{b}\right) < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{1}{2}\left(\frac{c}{d} + \frac{c}{d}\right) = \frac{1}{2}\left(\frac{2c}{d}\right) = \frac{c}{d}$. Thus, $\frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}$.

24. b. No; this only describes the fraction of students, not the total number of either males or females. **25.** Answers vary. **a.** About $\frac{1}{11}$ **b.** About $\frac{1}{6}$

Assessment 6-3B

1. a. $\frac{2}{3} \cdot \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$ **b.** $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ **2. a.**  **b.** 

4. b. Answers vary. $\frac{y^2 - x^2}{xy}$ **6. d.** $\frac{{}^{-}3}{2}$ or ${}^{-}1\frac{1}{2}$

19. a., c., e. are false; counterexamples vary.

b. True since $(ab)^{-m} = \frac{1}{(ab)^m} = \frac{1}{a^m b^m} = \frac{1}{a^m} \cdot \frac{1}{b^m}$. **d.** True since $2(a^{-1} + b^{-1})^{-1} = 2 \cdot \frac{1}{a^{-1} + b^{-1}} =$ $2 \cdot \frac{1}{\frac{1}{a} + \frac{1}{b}} = 2 \cdot \frac{1}{\frac{a+b}{ab}} = 2\frac{ab}{a+b} = \frac{2ab}{a+b}$.

f. True since $\left(\frac{a}{b}\right)^{-1} = \frac{1}{\frac{a}{b}} = \frac{b}{a}$. **20. a.** No integer solution

26. Answers vary. **a.** About $\frac{3}{67}$ **b.** About $\frac{2}{7}$ **29.** $\frac{1}{10}$ for stocks and $\frac{3}{20}$ for bonds

Mathematical Connections 6-3

Reasoning

1. Never less than n. $0 < \frac{a}{b} < 1$ implies $0 < 1 < \frac{b}{a}$. The last inequality implies $0 < n < n\left(\frac{b}{a}\right)$. Therefore $n \div \left(\frac{a}{b}\right) = n\left(\frac{b}{a}\right) > n$. **2.** B is the closest. Because both C and D are positive and less than 1, their product is less then either. **3.** The second number must be the reciprocal of the first and must be a positive number less than 1.

Open-Ended

4. Answers vary. **5.** Answers depend on the class. **6.** Answers depend on the class.

Cooperative Learning

7. The answers depend on the size of the bricks and the size of the joints. In all likelihood, the measurements will be made in fractions of inches for the size of the joints. The size of the bricks may be done in inches. An alternative is to measure in centimeters. All measurements are approximate and some rounding or estimation may occur. **8.** Answers vary depending on the state and year chosen. **9.** Answers depend on the group demographics.

Connecting Mathematics to the Classroom

10. Answers vary. For example, Bente is probably thinking of $12\frac{1}{4} \div 3\frac{3}{4}$ as $\frac{12 + \frac{1}{4}}{3 + \frac{3}{4}} = \frac{12}{3} + \frac{\frac{1}{4}}{\frac{3}{4}}$. In general, $\frac{a+b}{c+d} \neq \frac{a}{c} + \frac{b}{d}$, and so this will not work. The correct answer is determined by $12\frac{1}{4} \div 3\frac{3}{4} = \frac{49}{4} \div \frac{15}{4} = \frac{49}{15} = 3\frac{4}{15}$. **11.** Answers vary.

For example, $\frac{1}{2}$ of a number x is equivalent to $\frac{1}{2} \cdot x$ or $\frac{x}{2}$, whereas dividing a number x by $\frac{1}{2}$ is equivalent to $\frac{x}{\frac{1}{2}} = \frac{x}{1} \cdot \frac{2}{1} = \frac{2x}{1} = 2x$. Therefore, they are not the same. **12.** Answers vary. For example, yes she is correct, since $\frac{3}{4} \cdot \frac{2}{5} = \frac{3 \cdot 2}{4 \cdot 5} = \frac{3 \cdot 2}{5 \cdot 4} = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{5} \cdot \frac{1}{2} = \frac{3 \cdot 1}{5 \cdot 2} = \frac{3}{10}$. **13.** Answers vary. He is right. If a is divided by 2, we get $\frac{a}{2} = \left(\frac{1}{2}\right)a$. **14. a.** If instead of fractions, the question involved whole numbers, Jim would likely have an easier time. If 2 packages of sugar fill $\frac{1}{2}$ c, then 1 packet would fill $\frac{1}{2} \div 2$c, or $\frac{1}{4}$ c. If instead of 2 packages we have $\frac{3}{4}$ of a package, we would fill $\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$ of a cup. Another approach is to set up an equation: if x is the number of cups of sugar in one package of sugar, then in $\frac{3}{4}$ of a package there are $\frac{3}{4}x$ c, which is given to be $\frac{1}{2}$ c. So $\frac{3}{4}x = \frac{1}{2}, \frac{4}{3} \cdot \frac{3}{4}x = \frac{4}{3} \cdot \frac{1}{2}, x = \frac{2}{3}$c. **b.** Because in 2 c there are 4 half cups, we would need $\frac{3}{4} \cdot 4$, or 3 packages of sugar. **c.** One loaf will require $\frac{1}{3} \div 2 = \frac{1}{6}$ of a cup. Hence three loaves will require $3 \cdot \frac{1}{6} = \frac{1}{2}$ of a cup. **d.** 1 c of sugar will make $1 \div \frac{3}{4}$, or $\frac{4}{3} = 1\frac{1}{3}$ gal. Hence 2 c will make $2 \cdot 1\frac{1}{3} = 2\frac{2}{3}$ gal. **e.** 17 dolls, with $1\frac{1}{8}$ in. of ribbon left over. **15.** The student is correct when dividing a positive number by a number greater than 1. However, when the division is by a number between 0 and 1, the opposite happens. Recall that 12 divided by 4 means "How many 4's are there in 12?" $12 \div 4 = 3$ because $12 = 4 \cdot 3$. $5 \div \frac{1}{2}$ means "How many $\frac{1}{2}$s are there in 5?" $5 \div \frac{1}{2} = 10$ because $5 = \frac{1}{2} \cdot 10$. **16.** The student is wrong unless $n = 0$ or $p = m$. The Fundamental Law of Fractions holds only for multiplication. For example, $\frac{7+3}{2+3} \neq \frac{7}{2}$. **17.** Jillian is correct if a fractional answer is needed.

Review Problems

18. a. $\frac{25}{16}$ or $1\frac{9}{16}$ **b.** $\frac{25}{18}$ or $1\frac{7}{18}$ **c.** $\frac{5}{216}$ **d.** $\frac{259}{30}$ or $8\frac{19}{30}$ **e.** $\frac{37}{24}$ or $1\frac{13}{24}$ **f.** $\frac{-39}{4}$ or $-9\frac{3}{4}$ **19.** 120 **20. a.** $\frac{3x^2 + y^3}{x^2y^2}$ **b.** $\frac{az - by}{xy^2z}$ **c.** $\frac{2ab - b^2}{a^2 - b^2}$ **21. a.** This is not correct. **b.** This is not correct **c.** This is correct as long as $a \neq 0$.

Assessment 6-4A

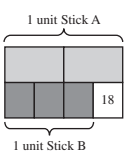
3. a. 2:5. Because the ratio is 2:3, there are $2x$ boys and $3x$ girls; hence, the ratio of boys to all students is $\frac{2x}{2x + 3x} = \frac{2}{5}$. **13.** Answers vary. $\frac{36\text{ oz}}{12¢} = \frac{48\text{ oz}}{16¢}; \frac{12¢}{16¢} = \frac{36\text{ oz}}{48\text{ oz}}; \frac{16¢}{12¢} = \frac{48\text{ oz}}{36\text{ oz}}$ **20.** The ratio between the mass of the gold in the ring and the mass of the ring is $\frac{18}{24}$. If x is the number of ounces of pure gold in the ring which weighs 4 oz, we have $\frac{18}{24} = \frac{x}{4}$. Hence, $x = \frac{(18 \cdot 4)}{24}$, or 3 oz. Consequently, the price of the gold in the ring is $3 \cdot \$1800$ or \$5400. **22. b.** Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r$. Then, $a = br, c = dr, e = fr$.

So, $a + c + e = br + dr + fr$

$a + c + e = r(b + d + f)$

$\frac{a + c + e}{b + d + f} = r$.

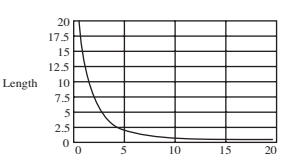
23. As seen in the drawing, $\frac{1}{2}$ of stick A is $\frac{2}{3}$ of stick B, and stick B is 18 cm shorter than stick A.



We have the following: Stick B is $3 \cdot 18$ unit sections while stick A is $4 \cdot 18$ unit sections. Thus, the lengths of the sticks are A, 72 cm, and B, 54 cm.

Assessment 6-4B

10. a.



11. Answers vary. For example,

x	4	8	12	16
y	20	40	60	80

17. a. 50:13

Mathematical Connections 6-4

Reasoning

1. a. 525 cm **b.** For the first set, $\frac{\text{footprint length}}{\text{thighbone length}} = \frac{40}{100} = \frac{20}{50}$; that is, a 50-cm thighbone would correspond to a 20-cm footprint. Thus, it is not likely that the 50-cm thighbone is from the animal that left the 30-cm footprint. $\left(\text{Notice that } \frac{20}{50} \neq \frac{30}{50}.\right)$ **2.** No, the ratio of the prices is equal to the ratio of the areas and not to the ratio of the diameters. **3. a.** $\frac{a}{b} = \frac{c}{d}$ implies

$\frac{a}{b} + 1 = \frac{c}{d} + 1$, which implies $\frac{a+b}{b} = \frac{c+d}{d}$. **b.** By inverting $\frac{b}{a} = \frac{d}{c}$ and by part (a), $\frac{b+a}{a} = \frac{d+c}{c}$. Then inverting again gives $\frac{a}{a+b} = \frac{c}{d+c}$. **c.** $\frac{a}{b} = \frac{c}{d}$ implies $\frac{a}{b} - 1 = \frac{c}{d} - 1$, which implies $\frac{a-b}{b} = \frac{c-d}{d}$. From part (a) and this last result, we have $\frac{a-b}{b} \div \frac{a+b}{b} = \frac{c-d}{d} \div \frac{c+d}{d}$, which implies $\frac{a-b}{a+b} = \frac{c-d}{c+d}$. **d.** $\frac{a}{b} = \frac{c}{d}$ if, and only if, $ad = bc$. Also, $\frac{d}{c-d} = \frac{b}{a-b}$, if and only if, $d(a-b) = b(c-d)$, $ad - bd = bc - bd$, $ad = bc$.

4. Yes, she is correct. Oneway for her to tell this is to realize that since $15 \cdot 7 \neq 15 \cdot 8$, then $15/7 \neq 15/8$. **5.** No, the dimensions don't vary proportionately because $4/6 \neq 5/7$ and $4/6 \neq 8/10$ and $\frac{5}{7} \neq \frac{8}{10}$. This can be seen by cross multiplication or just by reducing all three fractions to simplest terms—$\frac{2}{3}$, $\frac{5}{7}$, and $\frac{4}{5}$—and noticing that no two are equal. **6.** No, they can't form a proportion because for this to happen it would be necessary for $ab = ab + b^2$ and this cannot happen when $b \neq 0$.

7. Answers vary; for example, let m be the number of adult men living in the condo and w be the number of adult women. The number of married men is equal to the number of married women, so $\frac{2}{3}m = \frac{3}{4}w$. The ratio of married people to the total adult population is

$$\frac{\frac{2m}{3} + \frac{3w}{4}}{m+w} = \frac{\frac{2m}{3} + \frac{2m}{3}}{m+w} = \frac{\frac{4m}{3}}{m+w}.$$

Because $\frac{2m}{3} = \frac{3w}{4}, w = \frac{8m}{9}$. Thus,

$$\frac{\frac{4m}{3}}{m+w} = \frac{\frac{4m}{3}}{m + \frac{8m}{9}} = \frac{\frac{4m}{3}}{\frac{17m}{9}} = \frac{12}{17}, \text{ which is the desired ratio.}$$

Open-Ended

8. Answers vary, but the discussion should include the fact that two equal ratios form a proportion. **9.** Answers vary depending on the current numbers. The filibuster rules will have to be checked on a government website. **10.** Check a website on fertilizers. The ratio will be explained there. **11.** Answers vary. For example, the rectangle that is the most pleasing to the eye has its sides in the golden ratio, which is approximately 1.618:1.

Cooperative Learning

12. Answers vary.

Connecting Mathematics to the Classroom

13. Mary has not paid attention to matching the units of the corresponding parts of the proportion. **14.** Yes, she is correct. Because each ratio is equal to the same number, they must equal each other and hence form a proportion. **15.** The way Al set up the proportion assumes the tree is 15 ft tall. The correct proportion comparing the object to its shadow is

$$\frac{5 \text{ ft}}{\frac{3}{2} \text{ ft}} = \frac{x}{15 \text{ ft}}, \quad \text{so } x = 50 \text{ ft}.$$

16. Collect data on the ratio of arm length to nose length by measuring the arms and noses of other students in her class. Find the average ratio, then use it to find the expected length of the nose of the Statue of Liberty. **17.** Google "Singapore math curriculum, bar models."

Review Problems

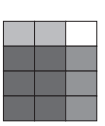
18. The numerator is 6 and the denominator is 1. **19.** $\frac{3}{4}$ is a proper fraction because the numerator and denominator are both positive with the numerator less than the denominator.

20. Any integer can be written as a rational number with the integer as the numerator and the denominator 1.

21. The statement is not true in general; for example, $\frac{25}{35} \neq \frac{2}{3}$ because $75 \neq 70$. **22.** At 6:00, the hands on the clock form a straight line, but the second hand is on 12. After that, the hands form a straight line approximately every 1 hr, 5 min, and 27 sec. So after 6:00, the minute and hour hands form a straight line at 7:05:27, 8:10:55, 9:16:22, 10:21:49, 11:27:16, 12:32:44, 1:38:11, 2:43:38, 3:49:05, and 4:54:33. **23.** No **24.** $99 \cdot 96 < 98 \cdot 97$ so $\frac{99}{98} < \frac{97}{96}$. **25. a.** $\frac{1}{6}$ of a day **b.** $\frac{ab}{a+b}$ hours. In 1 hr Amal can finish $\frac{1}{a}$ of the job, while Sharif can finish $\frac{1}{b}$ of the job in 1 hr. If they work together, they can finish $\frac{1}{a} + \frac{1}{b}$ of the job in 1 hr. If it takes them x hr to finish the job working together, then $x\left(\frac{1}{a} + \frac{1}{b}\right) = 1$ or $\left(\frac{a+b}{ab}\right)x = 1$. Hence, $x = \frac{ab}{a+b}$.

Chapter 6 Review

1. c.



2. Answers vary; for example, $\frac{10}{12}, \frac{15}{18}, \frac{20}{24}$.

6. $-2\frac{1}{3}, -1\frac{7}{8}, 0, \left(\frac{71}{140}\right)^{300}, \frac{69}{140}, \frac{1}{2}, \frac{71}{140}, \left(\frac{74}{73}\right)^{300}$

7. Yes. By the definition of multiplication and the commutative and associative laws of multiplication, we can do the following:

$$\begin{aligned}\frac{4}{5} \cdot \frac{7}{8} \cdot \frac{5}{14} &= \frac{4 \cdot 7 \cdot 5}{5 \cdot 8 \cdot 14} \\ &= \frac{4 \cdot 7 \cdot 5}{8 \cdot 14 \cdot 5} \\ &= \frac{4}{8} \cdot \frac{7}{14} \cdot \frac{5}{5}\end{aligned}$$

8. Answers vary. **a.** 24, because $\frac{1}{3}(8 \cdot 9)$ is equal to $\left(\frac{1}{3} \cdot 9\right) \cdot 8 = 3 \cdot 8 = 24$. **b.** 66, because $36 \cdot 1\frac{5}{6}$ is equal to

$36 \cdot \frac{11}{6} = 6 \cdot 11 = 66$. **12.** Answers vary. For example, $\frac{76}{100}$ and $\frac{78}{100}$. **17.** It is not reasonable to say that the university won $\frac{3}{4} + \frac{5}{8}$, or $\frac{11}{8}$, of its basketball games. The correct fraction cannot be determined without additional information but it is between $\frac{5}{8}$ and $\frac{3}{4}$. **22.** $\frac{^-12}{10}$ is greater than $\frac{^-11}{9}$ because $^-12 \cdot 9 > {}^-11 \cdot 10$. Alternatively, $\frac{^-12}{10} - \frac{^-11}{9} = \frac{^-108}{90} - \frac{^-110}{90} = \frac{2}{90}$, which is positive; therefore $-\frac{12}{10} > \frac{^-11}{9}$. **26.** Answers vary. For example, the problem is to find how many $\frac{1}{2}$-yd pieces of ribbon there are in $1\frac{3}{4}$ yd. There are 3 pieces of length $\frac{1}{2}$ yd with $\frac{1}{4}$ yd left over. This $\frac{1}{4}$ yd is $\frac{1}{2}$ of a $\frac{1}{2}$-yd piece. Therefore, there are 3 pieces of $\frac{1}{2}$-yd ribbon and 1 piece that is $\frac{1}{2}$ of the $\frac{1}{2}$-yd piece or $3\frac{1}{2}$ of the $\frac{1}{2}$-yd pieces. Thus, "3 pieces and $\frac{1}{4}$ yd left" and "$3\frac{1}{2}$ pieces" are correct answers. **32.** The ratio of hydrogen to the total is 1:9. Therefore, $\frac{1}{9} = \frac{x}{16}$ implies $x = 1\frac{7}{9}$ oz. **33.** No, the ratio depends on how many chips came from each plant. **40.** The ratio of professors of other political persuasions to liberals is 1:7 to 1:9. **42.** The cup that has 1 oz cream **44.** No, the will is impossible because the fractions of cats to be shared do not add to 1. If the woman had x cats then the number of cats her will directs to be distributed is $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{9}x = \frac{17}{18}x$, but the sum should be $1x$, or $\left(\frac{18}{18}\right)x$.

Answers to Now Try This

1. Consider two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where $\frac{a}{b} < \frac{c}{d}$. By the denseness property of rational numbers we can find a rational number x_1 between the two fractions. Since $\frac{a}{b} < x_1$, there is a rational number x_2 between $\frac{a}{b}$ and x_1. We next can find a rational number x_3 between $\frac{a}{b}$ and x_2 and so on. This process can be repeated indefinitely, and hence we obtain infinitely many rational numbers $x_1, x_2, x_3, \ldots$ between $\frac{a}{b}$ and $\frac{c}{d}$. **2.** Because $\frac{a}{b} < \frac{c}{d}$ with $b > 0$ and $d > 0$, Theorem 6-3 implies $ad < bc$. Adding ab to each side of the inequality, we now have $ab + ad < ab + bc$. Thus, by factoring $a(b + d) < b(a + c)$. **3.** $\frac{3}{4}$ is greater than $\frac{1}{2}$, $\frac{1}{2} + \frac{1}{2} = 1$, so $\frac{3}{4} + \frac{1}{2} > 1$. $\frac{4}{6}$ is less than 1, so it cannot be the correct answer for $\frac{3}{4} + \frac{1}{2}$. **4.** Answers vary. For example, consider the following. If Caleb has \$10.00, how many chocolate bars can he buy if **a.** the price of one bar is \$2.00? **b.** the price of one bar is $\$\frac{1}{2}$? For (a) the answer is $10 \div 2$ or 5. For (b) the answer is $10 \div \frac{1}{2}$, which is the same as finding the number of $\frac{1}{2}$s in 10. Since there are two halves in 1, in 10 there are 20. Hence Caleb can buy 20 bars.

5. $$\frac{a \div c}{b \div d} = \frac{\frac{a}{c}}{\frac{b}{d}} = \frac{a}{c} \cdot \frac{d}{b} = \frac{a}{c} \cdot \frac{d}{b} = \frac{\frac{a}{c} \cdot \frac{d}{b}}{1} = \frac{ad}{bc} = \frac{a}{b} \div \frac{c}{d}$$

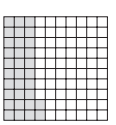
Answer to Preliminary Problem

The area of the holes is $\frac{1}{4} + \frac{1}{7} = \frac{11}{28}$ of the whole washer. Thus the area of the rubber in the finished product is $1 - \frac{11}{28} = \frac{17}{28}$ of the whole washer. If the area of the original piece of rubber is $1\frac{3}{8}$, then the area of the finished washer is $\left(\frac{17}{28}\right)\left(1\frac{3}{8}\right) = \left(\frac{17}{28}\right)\left(\frac{11}{8}\right) = \frac{187}{224}$ in.^{2} or about $\frac{5}{6}$ in.^{2}

CHAPTER 7

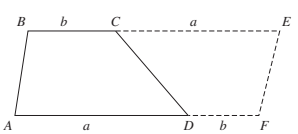
Assessment 7-1A

1. a. $0 \cdot 10^0 + 0 \cdot 10^{-1} + 2 \cdot 10^{-2} + 3 \cdot 10^{-3}$
b. $2 \cdot 10^2 + 0 \cdot 10^1 + 6 \cdot 10^0 + 0 \cdot 10^{-1} + 6 \cdot 10^{-2}$
c. $3 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0 + 0 \cdot 10^{-1} + 1 \cdot 10^{-2} + 0 \cdot 10^{-3} + 3 \cdot 10^{-4}$ **d.** $0 \cdot 10^0 + 0 \cdot 10^{-1} + 0 \cdot 10^{-2} + 0 \cdot 10^{-3} + 1 \cdot 10^{-4} + 3 \cdot 10^{-5} + 2 \cdot 10^{-6}$ **5. a.** $\frac{109}{250}$ **b.** $\frac{629}{25}$ **c.** $\frac{^-316{,}027}{1000}$ **d.** $\frac{140{,}951}{5000}$ **e.** $\frac{^-43}{10}$ **f.** $\frac{^-6201}{100}$ **6.** A rational number in simplest form can be written as a terminating decimal if, and only if, the prime factorization of the denominator contains no primes other than 2 or 5. **a.** Terminating **b.** Terminating **c.** Terminating **d.** Terminating **e.** Not terminating **f.** Terminating **7. a.** 0.8 **b.** 3.05 **c.** 0.5 **d.** 0.03125 **e.** Not possible **f.** 0.2128 **8.** One hour is 60 minutes, so 7 minutes would be $\frac{7}{60}$ of an hour. This is not a terminating decimal because $\frac{7}{60}$ is in simplest form and 3 is a factor of the denominator. **9.** Answers vary. Many values between 0 and 100 are composed of whole-number powers of 2 and 5, thus dividing 100 and being capable of being expressed as a two-digit decimal. These numbers include the coin designations of 1, 5, 10, 25, and 50 cents, but there are others which could have been used, such as 2¢ or 20¢. **10. a.** 13.492, 13.49199, 13.4919, 13.49183 **b.** $-1.4053, -1.45, -1.453, -1.493$ **12.** There are 32 of 100 squares shaded, representing $\frac{32}{100}$ of the whole grid $= 0.32$ of the grid.

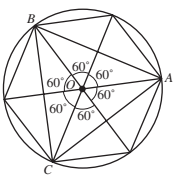


Assessment 12-3A

1. a. Construction **b.** Construction **2.** Construction **3.** Construction **4.** 25 **5. a.** 9 **b.** 62° **6. a.** Right triangle **b.** The altitude is the extension of the cable from vertex A to the ground. **7. a.** The perpendicular bisectors of the sides of an acute triangle meet inside the triangle. **b.** The perpendicular bisectors of the sides of a right triangle meet at the midpoint of the hypotenuse. **c.** The perpendicular bisectors of the sides of an obtuse triangle meet outside the triangle. **d.** Construction **8. a.** The point is equidistant from all vertices because it is on all three perpendicular bisectors. Being at the intersection of two of the perpendicular bisectors forces the point to be equidistant from all three vertices. **b.** Same as part (a). **9. a.** If the rectangle is not a square, it is impossible to construct an inscribed circle. The angle bisectors of a rectangle do not intersect in a single point. **b.** Possible. The center of the circle is the intersection of the diagonals (which are also the angle bisectors of the vertices), and the radius of the circle is the distance from the center to any of the sides. **c.** Possible. The intersection of the three longest diagonals is the center of the circle. **10.** Construction **11. a.** Construction **b.** 40¢ **12.** The figure is a rhombus. The diagonals bisect each other, so the figure is a parallelogram. Because in addition the diagonals are perpendicular to each other, it is a rhombus. **13.** Construction **14.** 2″ **15.** The center is where the angle bisectors of the angles meet (the incenter). The radius is the perpendicular distance from the incenter to a side. **16.** 3″ **17.** Answers vary. **a.** If the parallelogram is not a rectangle, cut along an altitude whose foot lies on the opposite side of the parallelogram. If the parallelogram is a rectangle, cut along any line through the point where the diagonals meet such that the line is not a diagonal and is not parallel to any side. **b.** Make a copy of a given trapezoid $ABCD$ and put it upside down next to $\overline{CD}$ as shown. More precisely, extend $\overline{BC}$ so that $\overline{CE} = a$ and extend $\overline{AD}$ so that $DF = b$. Because $\overline{BE} \parallel \overline{AF}$ and $BE = AF$ (the length of each is $a + b$), $ABEF$ is a parallelogram.



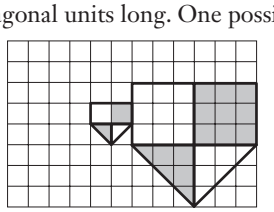
18. As close to 26 in. as the jack can close. **19. a.** Possible **b.** Infinitely many noncongruent rectangles can be constructed. The endpoints of two segments bisecting each other and congruent to the given diagonal determine a rectangle, but since the segments may intersect at any angle, there are infinitely many such rectangles. **c.** Not possible because the sum of the measurements of the angles would be greater than 180°. **d.** There is no parallelogram because the fourth angle must also be a right angle. **20. a.** Construct an equilateral triangle and bisect one of its angles. **b.** Bisect a 30° angle, then add the 30° and 15° angles or bisect a right angle. **c.** Combine 60° and 15° angles or combine 45° and 30° angles. **21.** Construction **22. a.** The point is determined by the intersection of the angle bisectors of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Because the point is on the angle bisector of $\angle BAD$, it is equidistant from its sides. Because it is on the perpendicular bisector of $\overline{BC}$, it is equidistant from B and C. **b.** The point is determined by the intersection of the angle bisector of $\angle A$ and $\angle B$. **23.** Answers vary. One way is to construct six 60° central angles with common vertex at the center O. Then construct three 120° central angles with vertex O. The sides of these angles intersect the circle at points A, B, and C, which are vertices of an equilateral triangle. $\triangle ABC$ is equilateral because $\triangle AOB$, $\triangle BOC$, and $\triangle AOC$ are congruent (by SAS). Another approach is to construct a regular inscribed hexagon and then choose any vertex of the hexagon and two other nonconsecutive vertices.



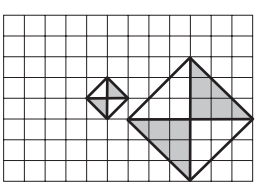
Assessment 12-4A

1. 6710 km **2.** 5.55 in. **3.** Yes. Corresponding angles are congruent and sides are proportional. **4.** Yes, they are similar. It is given that $\angle BAD \cong \angle B_1A_1D_1$. Because the pairs $\angle ABC$ and $\angle BAD$ as well as $\angle A_1B_1C_1$ and $\angle B_1A_1D_1$ are supplementary, it follows that $\angle ABC \cong \angle A_1B_1C_1$. Thus the rhombi have congruent corresponding angles. Because in a rhombus adjacent sides are congruent, $\dfrac{AD}{AB} = 1 = \dfrac{A_1D_1}{A_1B_1}$. This implies $\dfrac{AD}{A_1D_1} = \dfrac{AB}{A_1B_1}$ and therefore ratios of corresponding sides are proportional. **5. a.** Yes, AAA similarity. **b.** Yes; sides are proportional and angles are congruent. **c.** No; two rectangles that are not squares are similar if, and only if, the corresponding sides are proportional. **d.** Always similar because the ratio of the corresponding longer sides equals the ratio of the corresponding shorter sides and the angles are congruent. **6.** Make all dimensions three times as long; for example, in part (c) each side would be three diagonal units long. One possible solution set is shown here:

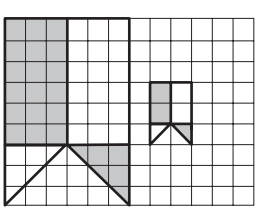
a.



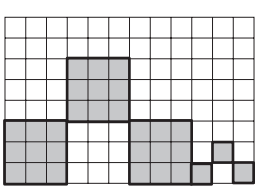
c.



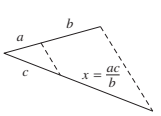
b.



d.



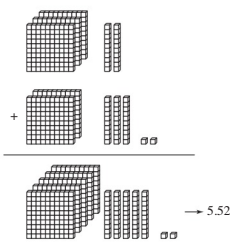
7. a. i. $\triangle ABC \sim \triangle DEF$ by AA similarity **ii.** $\triangle ABC \sim \triangle EDA$ by SAS similarity or AA similarity **iii.** $\triangle ACD \sim \triangle ABE$ by AA similarity **iv.** $\triangle ABE \sim \triangle DBC$ by SAS similarity **b. i.** $\frac{2}{3}$ **ii.** $\frac{1}{2}$ **iii.** $\frac{4}{3}$ **iv.** $\frac{2}{3}$ or the reciprocals **8. a.** 7 **b.** $\frac{24}{7}$ **9.** $x = 14, y = 18$ **10. a.** $\frac{9}{6} = \frac{3}{2}$ **b.** $\frac{6}{15} = \frac{2}{5}$ **c.** $\frac{9}{y}$ **11.** Construction **12.** Construction as follows:



Because $340 \neq 304$, the decimals are not equal. **18.** Answers vary. The second sequence is the term-by-term sum of the first sequence and the arithmetic sequence $1, 1, 1, \ldots$. Because $\frac{1.01}{1.1} \neq \frac{1.001}{1.01}$ it is not a geometric sequence. In general a sum of a geometric sequence and an arithmetic sequence is neither geometric nor arithmetic.

Assessment 7-2A

1.



2. a. $\frac{86}{10} + \frac{231}{10} + \frac{92}{100} = \frac{860}{100} + \frac{2310}{100} + \frac{92}{100} = \frac{3262}{100} = 32.62$

b. $\frac{232}{100} + \frac{21008}{1000} = \frac{2320}{1000} + \frac{21008}{1000} = \frac{23328}{1000} = 23.328$

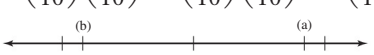
3. a. $\frac{5}{10} \cdot \frac{6}{10} = \frac{30}{100} = 0.30$ **b.** $\frac{203}{1} \cdot \frac{3}{100} = \frac{609}{100} = 6.09$

c. $\frac{3}{1000} \cdot \frac{6}{1000} = \frac{18}{1{,}000{,}000} = 0.000018$ **4.** Answers vary.

a. 2989.536; 934.23 has 2 decimal places, and 3.2 has 1 decimal place so we move $2 + 1 = 3$ decimal places in the answer. **b.** The product is about $900 \cdot 3 = 2700$, so the decimal point is placed as 2989.536. **13.** Approximately $28.58 Canadian **15.** Approximately 8.64 L **18.** A finite geometric sequence is one with a constant ratio between terms and a finite number of terms. 0.2222 can be expressed as a sum: $\frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10{,}000}$; note that the denominators are powers of ten. Thus to make it a geometric sequence with constant ratio $\frac{1}{10}$, it could be written as

$$\frac{2}{10} + \left(\frac{2}{10}\right)\left(\frac{1}{10}\right) + \left(\frac{2}{10}\right)\left(\frac{1}{10}\right)^2 + \left(\frac{2}{10}\right)\left(\frac{1}{10}\right)^3.$$

19.



22. 153.45 cm

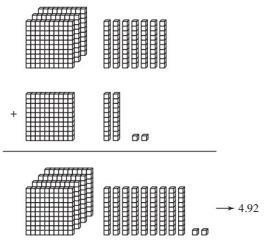
24. a. Approximately 13 or 14 **25. a.** Approximately 18.6 **c.** Approximately 121 **37.** Answers vary. For example,

$$40 \cdot \$8 + 40 \cdot \left(\frac{1}{4}\right)\$ = \$320 + \$10 = \$330.$$

40. a. $2 \cdot 1 + 0.25 = 2.25 = 1.5^2$
$3 \cdot 2 + 0.25 = 6.25 = 2.5^2$
A conjecture is that the next two are: $4 \cdot 3 + 0.25 = (3.5)^2$ and $5 \cdot 4 + 0.25 = (4.5)^2$. **b.** Answers vary. Since $1.5 = 2 - 0.5, 2.5 = 3 - 0.5,\ 3.5 = 4 - 0.5$, and $4.5 = 5 - 0.5$, we conjecture that for each natural numbers, $n(n-1) + 0.25 = (n - .5)^2$. This is true since $(n - .5)^2 = (n - .5)(n - .5) = n^2 - .5n - .5n + .25 = n^2 - n + .25 = n(n-1) + .25$.

Assessment 7-2B

1.



2. a. $\frac{53}{10} + \frac{132}{10} + \frac{86}{100} = \frac{530}{100} + \frac{1320}{100} + \frac{86}{100} = \frac{1936}{100} = 19.36$

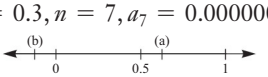
b. $\frac{542}{100} + \frac{32005}{1000} = \frac{5420}{1000} + \frac{32005}{1000} = \frac{37425}{1000} = 37.425$

3. a. $\frac{3}{10} \cdot \frac{8}{10} = \frac{24}{100} = 0.24$ **b.** $\frac{502}{1} \cdot \frac{4}{100} = \frac{2008}{100} = 20.08$

c. $\frac{4}{1000} \cdot \frac{6}{100} = \frac{24}{100{,}000} = 0.00024$ **4.** Answers vary.

a. 3780.341; 564.23 has 2 decimal places, and 6.7 has 1 decimal place, so we move the decimal point $2 + 1 = 3$ places. **b.** The product is about $600 \cdot 7 = 4{,}200$, so the decimal point is placed as 3780.341. **17.** $0.3333333 = 0.3 + 0.03 + 0.003 + \cdots + 0.0000003$ or $\frac{3}{10} + \frac{3}{10}\left(\frac{1}{10}\right) + \frac{3}{10}\left(\frac{1}{10}\right)^2 + \cdots + \frac{3}{10}\left(\frac{1}{10}\right)^6$. This is the finite sum of a geometric sequence with $a_1 = 0.3, n = 7, a_7 = 0.0000003$, and $r = \frac{0.03}{0.3} = \frac{1}{10} = 0.1$.

18.



23. a. Answers vary around 62 or 63. **28. c.** 3,000,000,000 **32.** Answers vary, for example, 10.3 because $9 \cdot 10 = 90$ and $9 \cdot 0.3$ is greater than 2 and less than 4. **36.** Answers vary, for example, $40 \cdot \$6 + 40 \cdot (1/4)\$ = \$240 + \$10 = \$250$.

Mathematical Connections 7-2

Reasoning

1. 0.03. We know that 0.76 and 0.75 differ by 0.01 so $0.76 \cdot 3$ and $0.75 \cdot 3$ differ by $3 \cdot 0.01$ or 0.03. **2.** Answers vary. For example, the multiplication of decimals is exactly like the multiplication of whole numbers if the decimal points are ignored. The difference occurs when the decimal point is placed to obtain the final answer. **3.** Answers vary. For example, many of the estimation techniques that work for whole-number division also work for decimal division. The long division algorithm is more efficient when good estimates are used. Also, estimates are important to determine whether an answer obtained by long division is reasonable. Estimation techniques can also be used to place the decimal point in the quotient when decimals are divided. **4.** Answers vary. For example, which form is easier depends on the numbers to be multiplied or divided. In most cases, it is probably easier to work with decimals because the algorithms work just as they do for whole numbers except for the placement of the final decimal point, which in turn is very easy. However, when infinite repeating decimals are involved, it is much easier to work with fractions as will be seen in the next section. **5.** Answers vary. For example, lining up the decimal points acts like using place value.

Answer to the Preliminary Problem

The distance along Ocean Drive bordering the lots is $40\,\text{m} + 34\,\text{m} + 30\,\text{m} = 104\,\text{m}$. Let x, y, and z be the ocean frontage for lots A, B, and C respectively. Using Theorem 12-15 we can set up the following proportions:

Lot A: $\dfrac{104}{130} = \dfrac{40}{x}$ and $x = 50\,\text{m}$.

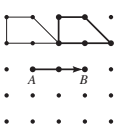
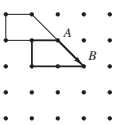
Lot B: $\dfrac{104}{130} = \dfrac{34}{y}$ and $y = 42.5\,\text{m}$.

Lot C: $\dfrac{104}{130} = \dfrac{30}{z}$ and $z = 37.5\,\text{m}$.

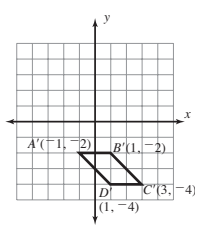
Therefore Lot A has the most ocean frontage and hence the highest selling price.

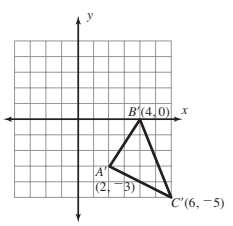
CHAPTER 13

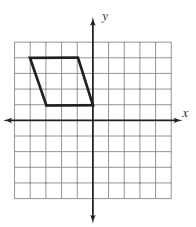
Assessment 13-1A

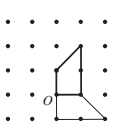
1. a.  **b.** 

2. a. Construction **b.** Construct the images B' and C' of B and C respectively. $\overline{B'C'}$ is the required image. **3. a.** $(3, {}^{-}4)$ **b.** $(0, 0)$ **c.** $({}^{-}3, {}^{-}13)$ **4. a.** $(3, {}^{-}4)$ **b.** $(0, 0)$ **c.** $({}^{-}3, {}^{-}13)$

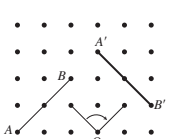
5. a. 

b. 

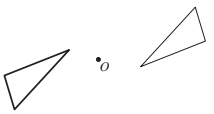
6. 

7. 

8. a. $y = 2x - 3$ **b.** $y = 2x - 7$ **9.** $y = {}^{-}2x - 1$ **10.** Choose any two points P and Q on ℓ, and construct their images P' and Q'. $\overleftrightarrow{P'Q'}$ is the desired image. **11.** *Hint:* Reverse the rotation (to the counterclockwise direction) to locate $\overline{AB}$, that is, the preimage.



12. Answers vary, but H, I, N, O, S, X, or Z could appear in such rotational words. Examples include SOS. Variations could use M and W in rotational images, for example, MOW. **13. a.** The line itself. **b.** Answers vary. **c.** ℓ and ℓ' are parallel. If P and Q are any points on ℓ and P' and Q' their respective images, then from the definition of a translation $\overline{PP'}$ and $\overline{QQ'}$ are parallel and congruent to $\overline{AB}$. Hence, $\overline{PP'}$ and $\overline{QQ'}$ are parallel and congruent. Thus, $PP'Q'Q$ is a parallelogram and therefore $\ell' \parallel \ell$. **d.** $\angle ABC$ **14.** *Hint:* An angle whose measure is 45° can be constructed by bisecting a right angle. An angle whose measure is 60° can be constructed by first constructing an equilateral triangle. These angles can be used to complete the construction. **15. a.** $({}^{-}4, 0)$ **b.** $({}^{-}2, {}^{-}4)$ **c.** $(2, 4)$ **d.** $({}^{-}a, {}^{-}b)$

16. 

17. a. $\ell = \ell$ **b.** $\ell' \perp \ell$

18. a. *Hint:* First rotate $\triangle ABC$ by angle α to obtain $\triangle A'B'C'$, and then rotate $\triangle A'B'C'$ by angle β to obtain $\triangle A''B''C''$. **b.** No **c.** Yes, by rotation about O by angle $|\alpha - \beta|$ in the direction of the larger of α and β. **19. a. i.** $({}^{-}3, 2)$ **ii.** $({}^{-}2, {}^{-}1)$ **iii.** $({}^{-}n, m)$ **b.** The image of the point with coordinates (a, b) under a half-turn with the origin as center must be on a circle with center at $(0, 0)$ and passing through (a, b). It must also be on the line connecting $(0, 0)$ with (a, b). The equation of that line is $y = \dfrac{b}{a}x$. The only point that satisfies this equation and is at the same distance from the origin as (a, b) is $({}^{-}a, {}^{-}b)$. **c.** $(b, -a)$

20. a. $y = {}^{-}x + 1$ **b.** $y = \dfrac{1}{2}x - \dfrac{1}{2}$ **c.** $y = 0$ **d.** $x = 1$

21. a. $B(h + a, k)$. **b.** Applying the Pythagorean Theorem in the triangle with hypotenuse $OA = a$ and legs of length h and k we get $h^2 + k^2 = a^2$. Then the product of the slopes of $\overline{OB}$ and $\overline{AC}$ is

$$\frac{k}{h + a} \cdot \frac{k}{h - a} = \frac{k^2}{h^2 - a^2} = \frac{k^2}{{}^{-}k^2} = {}^{-}1.$$

c. There is a turn symmetry of 180° about the point of intersection of the two diagonals. **22. a.** Construction **b.** Through B construct $\overrightarrow{BE}$ so that $\overline{BE} \parallel \overline{CD}$, $BE = CD$, and $\overrightarrow{BE}$ points in the same direction as $\overrightarrow{CD}$. Then $\overrightarrow{AE}$ is a slide arrow that corresponds to the required translation. **23. a.** Rotation by 90° or a multiple of 90° about the point of intersection of the diagonals. **b.** Rotation by 72° or a multiple of 72° about O, where O is the point of intersection of the perpendicular bisectors of the sides. **c.** Rotation by $\dfrac{360°}{n}$ and multiples of $\dfrac{360°}{n}$ about O, where O is the intersection of the perpendicular bisectors of the sides. **d.** Rotation about the center by any turn angle. **e.** No rotational symmetries **24. a.** Half-turn about the point of intersection of the diagonals. A rectangle has half-turn symmetry because the diagonals bisect each other. **b.** Same as (a) **c.** No half-turn symmetry **d.** No half-turn symmetry **25. a.** *Hint:* Find the image of the center and construct the circle with that image as the center and the same radius. **b.** The circles have a single point in common. **26.** Regular hexagon

Assessment 13-2A

1. *Construction. Hint:* Locate the images of vertices directly across (perpendicular to) ℓ on the geoboard. **2. a.** The lines containing all diameters **b.** Perpendicular bisector of the segment or the line containing the segment **c.** Any line perpendicular to the given line or the line itself **d.** Perpendicular bisectors of the sides and lines containing the diagonals **e.** None **f.** Perpendicular bisector of each side **g.** None **h.** If the kite is not a rhombus, the line containing the diagonal determined by vertices of the noncongruent angles is a line of symmetry. If a rhombus then each line containing a diagonal is a line of symmetry **i.** Perpendicular bisectors of parallel sides and three lines containing diameters determined by vertices on the circumscribed circle **3.** The original figure **4.** The image is a circle congruent to the original circle with a different center.

Assessment 7-3B

1. No, the fraction is in simplest form and the denominator has factors other than 2's and 5's. **2. a.** No, the 6's repeat after the 34 in all three cases. **6. h.** $0.\overline{210526315789473684}$

12. Answers vary. The denominator in $\frac{1}{13}$ cannot be written in the form $2^n \cdot 5^m$, where n and m are whole numbers. The fraction cannot represent a terminating decimal. It must repeat. The maximum possible length of the repetend is 12 but in fact it has period only 6. **13.** $1.3\overline{5775}$. The answer is a rational number, and there are four digits in the repetend. **15. b.** Answers vary; for example, 203.768, 203.769, and 203.7691.

19. a. i. $\frac{2}{9}$ **ii.** $\frac{2}{90}$ or $\frac{1}{45}$ **iii.** $\frac{2}{900}$ or $\frac{1}{450}$ **b.** $\frac{2}{9000}$ or $\frac{1}{4500}$

Mathematical Connections 7-3

Reasoning

1. a. Mathematically the cost is $66\frac{2}{3}¢$ but realistically, the cost is 67¢. **b.** 67¢ **c.** The cost is rounded up. **d.** Most cash registers do not allow repeating decimals, so grocery stores do not use them. **e.** See part (d). **2.** Yes. Zeros can be appended to any finite decimal to make it an infinite decimal. Also every finite decimal can be converted into an infinite decimal by using a repetend of 9. **3.** Answers vary. For example, it is easier to compute addition of fractions when there is a common denominator, as in $\frac{1}{7} + \frac{5}{7}$. When nonrepeating decimals are involved, it becomes hard to do additions because of lining up the digits using place value. When denominators are different and terminating decimals are obtained, it is easier to use decimals, as in $\frac{2}{5} + \frac{1}{4} = 0.40 + 0.25 = 0.65$.

4. $\frac{1}{11} = 0.\overline{09}, \frac{1}{111} = 0.\overline{009}, \frac{1}{1111} = 0.\overline{0009}$, so we would expect that $\frac{1}{11111} = 0.\overline{00009}$. When the divisions are performed longhand, we can see that for each successive number of 1s in the denominator, it takes one more place before a remainder of 1 occurs, and we have the initial division repeated.

Open-Ended

5. a. They each have six-digit repetends using each of the digits 1, 2, 4, 5, 7, 8 exactly once in the same cyclic order. **b.** Answers vary. For example, the patterns of the digits are cyclic; that is, they repeat in the same order. **c.** For example, $\frac{x}{17}$, where x is a natural number. The patterns of the digits repeat in cyclic order in the repetends. **6. a. i.** $0.\overline{6}$ **ii.** 1; 1 is the simplest form because it requires the fewest symbols. **iii.** $1.\overline{06}$ **b.** Because there is no right-hand end to start with, it is difficult to see how the traditional algorithm could be easily adapted into an algorithm in this context. **c.** A repeating decimal can be written as a rational number in the form $\frac{a}{b}$, where $b \neq 0$. The fractions can be multiplied and the product can be converted to a decimal. **7. a.** Answers vary. If we enter the numbers with several digits, the calculator may produce a clearly identifiable repeating decimal. For example, if we multiply 3.444444444 . . . by 2.5757575757 . . . the calculator will display the answer 8.872053872053 **b.** See part (a). **8.** Most would prefer $\frac{7}{3}$ as a solution. However when melons are priced at 3 for $7, we might prefer to know that they are $2.333 . . . each and hence actually $2.34 each. Thus the answer depends on the context.

Cooperative Learning

9. Answers vary.

Connecting Mathematics to the Classroom

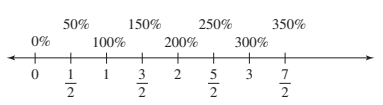
10. Answers vary. A calculator is a tool that may aid in the computation of decimals. It can help by furnishing estimates for answers. This has to do with the value of the calculator as a tool. The value of repeating decimals is a different question. One major white-collar crime dealt with rounding off fractions of pennies and depositing that money in a bank account. These fractions could have been the result of using repeating decimals and rounding. Depending upon the context, the result of not using such decimals could lead to serious errors. **11.** Yes, because every rational number can be written as a repeating decimal. For example, $0.45 = 0.44\overline{9}$. However the sum of two decimals that cannot be written as terminating decimals can still be a terminating decimal; for example $\frac{1}{11} + \frac{1}{110} = \frac{1}{10}$. **12.** Because $0.3333 \neq \frac{1}{3}$ the answer is incorrect. However, it is a good approximation. **13.** Bob has the right idea. Terminating decimal can be written with repetend 0 as in $\frac{1}{5} = 0.2 = 0.200000$. But Bob should also consider a number like $2 = 1.999999 . . .$ that has a repetend of 9. **14.** If the goal is to get an approximation, then Joe is correct. Using his method $2 \cdot 0.142857142 = 0.285714284$, which is a good approximation of $\frac{2}{7}$. However, it is not accurate as $\frac{2}{7} \neq 0.285714285714$. This difference occurs from truncating the actual value of $\frac{1}{7}$. It might be easier to see with smaller numbers. For example, if Joe truncates $\frac{1}{6}$ to 0.16 then multiplies 0.16 by 2 to obtain the value of $\frac{2}{6}$ he would obtain 0.32. However, $\frac{2}{6} = \frac{1}{3} = 0.3333 . . .$ and we can see what happens when we truncate the value. **15.** The repeating decimals $0.\overline{4}$ and $0.\overline{44}$ are equal and are just written in different ways. This can be seen by converting each to rational number form and simplifying: $0.\overline{4} = \frac{4}{9} = \frac{44}{99} = 0.\overline{44}$.

Review Problems

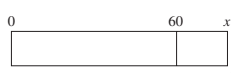
16. $22,761.95 **17.** Approximately $2.35 \cdot 10^{13}$ **18.** 0.077. The rule says that the decimal point should be placed four digits from the right of the product of 22 and 35. This product is 770, and therefore the answer is 0.0770, which equals 0.077. **19.** Answers vary. For example, **a.** 3.024 **b.** $^{-}3.024$ **c.** 134.8 **d.** 0.00713

Assessment 7-4A

2.



6.



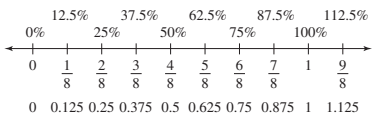
$\frac{60}{100} = \frac{x}{125}, x = 75$

18. $\frac{325}{500}$; $\frac{325}{500} = \frac{650}{1000} = \frac{65}{100} = 65\%$, while $\frac{600}{1000} = \frac{60}{100} = 60\%$. **19. b.** 10 items. You get two more items for the same price. **22.** The apprentice earns \$700, the journeyman \$1400, and the master \$2100. **25. a.** The report is essentially true. **26.** No. Since 56% is more then double 25%, \$950 should be more than double \$500, but it is not.

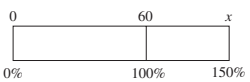
32. *Interest Rate per Period* **c.** $\frac{10}{12}\%$ or $8\frac{1}{3}\%$ **d.** $\frac{12}{365}\%$

Assessment 7-4B

2.



6.



$\frac{60}{100} = \frac{x}{150}$, $x = 90$

11. b. $1\frac{1}{2}$ in. **12.** $17\frac{11}{17}\%$ or approximately 18%. **13.** $18\frac{37}{91}\%$ or approximately 18%. **16.** $\left(\frac{9}{34}\right)\%$ or approximately 0.26% **30. c.** $\$30 \cdot 1.2 = \$36$, this works since $\$30 + 0.20 \cdot 30 = 1 \cdot 30 + 0.20 \cdot 30 = (1 + 0.20)30 = 1.2 \cdot 30$.

31. *Interest Rate per Period* **d.** $\frac{18}{365}\%$

38.

	A1's Investment	Betty's Investment
Year 1	1061.83	1070
Year 2	1127.49	1140
Year 3	1197.20	1210
Year 4	1271.22	1280
Year 5	1349.83	1350
Year 6	1433.29	1420

Mathematical Connections 7-4

Reasoning

1. Answers vary. For example, 10% of 850 is 85 and 1% of 850 is 8.5, so 11% of 850 is 93.5. **2.** No, $0.4\% = \frac{0.4}{100} = 0.004$ and $0.004 \neq 0.4$. **3.** It means that not only did you meet 100% of your savings goal, you surpassed it by 25%. If your savings goal was \$100, then you saved \$100 plus an extra \$25. **4.** Yes, 4% of $98 = \frac{4}{100} \cdot 98 = \frac{4 \cdot 98}{100 \cdot 1} = \frac{98 \cdot 4}{100 \cdot 1} = \frac{98}{100} \cdot \frac{4}{1} = 98\%$ of 4. **5. a.** Greater, because if 25% of x is 55, then x must be 4 times as great as 55, or 220. **b.** Less, because if 150% of x is 55, then x is only $\frac{2}{3}$ of 55, or $36\frac{2}{3}$. **6.** Yes; for example, 35% of 100 = 35 is greater than 55% of 50 = 27.5. **7.** The *whole* in each part is different, so 50% of the greater quantity is greater than 50% of the lesser quantity. To be equal, we would have to have the same size whole initially. **8.** Answers vary. For example, a price can increase 150%. If an item is bought for \$100, it can be sold for \$250, an increase of \$150, or 150% of the original price. The price can't decrease 150%, because 100% is the entire amount. A price cannot be less than 0. **9.** Let x be the amount invested. The first stock will be worth $(1.15x)0.85$ after 2 yr. The second stock will be worth $(0.85x)1.15$. Because each of these equals $(1.15 \cdot 0.85)x$ (commutative and associative properties of multiplication), the investments are equally good. **10.** Let a be the original value of the house. Because it depreciated 10% each year for the first 3 yr, using compound depreciation, the price after 3 yr is $a(1 - 0.10)^3$, or $a \cdot 0.9^3$. Because of compound appreciation, after another 3 yr, the value of the house is $a \cdot 0.9^3 \cdot 1.1^3$, or $a(0.9^3 \cdot 1.1^3)$, which equals approximately $a \cdot 0.9703$. Because $0.9703a < a$, the value of the house had decreased after 6 yr. The value of the house decreased by approximately 3%. **11.** If we want our money doubled, then $2 = 1(1 + 0.02)^n$, or $2 = 1.02^n$. By trial and error, we can find that 1.02^{35} is approximately 2, so it would take about 35 yr to double an investment at 2% compounded annually. **12.** No, the percentages cannot be added because each time the percent is of a different quantity. After 5 yr, the car would have depreciated approximately 67%. **13.** This is impossible because greater than 50% of the boys and 50% of the girls wanted a prom. **14.** Yes, if the class has twice as many boys as girls. **15.** Let x be the percent that Abe paid on his bill. Then $\$75x = \$56.25$, and $x = 75\%$. So Abe received a 25% discount. Let y be the percent that Barb paid. Then $\$52y = \$31.20$, and $y = 60\%$ and so her discount was 40%. Barb saved the greater percentage off the original price. **16.** Dan is correct in that the sales tax is only going up by 1 percentage point. Stan is correct since increasing the sales tax by 1% over 5% is increasing it by $\frac{1}{5} = 0.20$ or 20%.

Open-Ended

17. Answers vary. **18.** Answers vary. **19.** Answers vary. For example, **a.** 115 is 37% of what number? **b.** a is p% of what number? **20. a.** Answers vary. **b.** Let a be an initial cost of an item that depreciates at a rate of p% each period of time. If n is the number of periods and $C(n)$ is the value (as a function of n) after n periods, then $C(n) = a\left(1 - \frac{p}{100}\right)^n$ **21.** Answers vary.

Cooperative Learning

22. Answers vary. **23.** Answers vary. **24.** Answers vary.

Connecting Mathematics to the Classroom

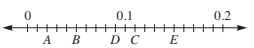
25. Jamal needs to understand that in order to compare percents the whole must be the same. Ask him to calculate the number of girls if Washington had 300 students (48% of 300 = 144), and Jefferson had 200 students (52% of 200 = 104). In this case, Washington School had more girls. **26.** Prices can increase 120%; for example, a shirt that cost \$10 can increase in cost to \$22 for a 120% increase. However, in an election 100% of the votes is all that is possible. **27.** $3\frac{1}{4}\% = 3\% + \frac{1}{4}\% = \frac{3}{100} + \frac{\left(\frac{1}{4}\right)}{100} = 0.03 + 0.0025 = 0.0325$. Knowing that $\frac{1}{4} = 0.25$, the student incorrectly wrote $\frac{1}{4}\% = 0.25$, not 0.25%. **28.** Let s denote

the amount of salary. After a p% increase, the new salary is $s\left(1 + \frac{p}{100}\right)$. When this amount is decreased by q%, the result is $s\left(1 + \frac{p}{100}\right)\left(1 - \frac{q}{100}\right)$. Similarly, if the initial salary is first decreased by q% and then the new amount is raised by p%, the final salary is $s\left(1 - \frac{q}{100}\right)\left(1 + \frac{p}{100}\right)$. Because the two expressions are equal by the commutative property of multiplication, the student is right. **29.** No, she is not correct. Men make 25¢ more for every 75¢ that women make, so men make $\frac{25}{75} = 33\frac{1}{3}$% more than women. **30.** Because the formula for compound interest is $A = P(1 + r)^n$ and $P = d$ dollars and $r = 1$, then we have $A = d(1 + 1)^n = d \cdot 2^n$. Thus, the student is correct.

Review Problems

31. a. 21.6 lb **b.** 48 lb **32. a.** $\frac{418}{25}$ **b.** $\frac{3}{1000}$ **c.** $\frac{^-507}{100}$ **d.** $\frac{123}{1000}$ **33. a.** $4.\overline{9}$ **b.** $5.0\overline{9}$ **c.** $0.4\overline{9}$ **34.** $\frac{3}{12{,}500}$ **35.** $\frac{8}{33}$ **36. a.** 208,000 **b.** 0.00038

Chapter 7 Review

1. b. $D(0.09), E(0.15)$ 

3. A fraction in simplest from, $\frac{a}{b}$, can be written as a terminating decimal if, and only if, the prime factorization of the denominator contains no primes other than 2 or 5. **5. a.** $\frac{4}{7} = 0.\overline{571428}$ **b.** $\frac{1}{8} = 0.125$ **c.** $\frac{2}{3} = 0.\overline{6}$ **d.** $\frac{5}{8} = 0.625$ **6. a.** $0.28 = \frac{7}{25}$ **b.** $^-6.07 = \frac{^-607}{100}$ **c.** $0.\overline{3} = \frac{1}{3}$ **d.** $2.0\overline{8} = \frac{94}{45}$

9. $1.451\overline{9}, 1.45\overline{19}, 1.45\overline{19}, 1.4519, 0.13\overline{401}, ^-0.134, ^-0.13\overline{401}$.

11. a. Answers vary. 0.105, 0.104, 0.103, 0.102, 0.101

b. Answers vary; 0.0005, 0.001, 0.002, 0.004 **19.** A discount of d% means the customer pays $1 - \frac{d}{100}$ for the purchase. Discounts of 5%, 10%, and 20% mean the customer pays 0.95, 0.90, and 0.80, respectively, of cost. Their product, 0.684, is the same in any order; therefore, there is no difference. **22.** Answers vary. If the dress was originally priced at \$100, 60% off would result in a sale price of \$40. Then the additional 40% off coupon would result in a final price of \$24. The reasoning could be applied to the actual list price of the dress. A percentage has to be a percentage of something, in this case her 40% coupon gives her 40% off the already discounted price, not 40% off the list price before the 60% discount. **23.** The store is losing on the item (9%).

Answers to Now Try This

2. a. Answers vary.

b. i. $3.6 \cdot 1000 = 3.6 \cdot 10^3 = \left(3 + \frac{6}{10}\right)10^3 =$

$3 \cdot 10^3 + \frac{6}{10} \cdot 10^3 = 3 \cdot 10^3 + 6 \cdot 10^2 = 3 \cdot 10^3 + 6 \cdot 10^2 + 0 \cdot 10^1 + 0 \cdot 1 = 3600$. Thus, we see that multiplication by 1000 results in moving the decimal point three places to the right. **ii.** In general, multiplication by 10^n, where n is a positive integer, results in moving the decimal point n places to the right. **3.** Answers vary. For example, using the front digits the first estimate is \$2 + \$0 + \$6 + \$4 + \$5 = \$17. Next, we adjust the estimate. Because \$0.89 + \$0.13 is about \$1.00 and \$0.75 + \$0.05 is \$0.80 and \$0.80 + \$0.39 is about \$1.20, the adjustment is \$2.20, and the estimate is \$19.20.

Answer to the Preliminary Problem

The new copy will be 6.75 cm by 13.5 cm. The one-stage setting should be $(1.5)(1.5) = 2.25$ or 225%. If a specific number such as 100 is used, then 150% of 100 = 150. We ask, "what percent of 150 is 100?" Solving $\frac{100}{150} = \frac{x}{100}$ gives $x = 66\frac{2}{3}$; $66\frac{2}{3}$%.

CHAPTER 8

Assessment 8-1A

1. Answers vary 0.232232223. . . **8. b.** Answers vary. Three such numbers are: 0.546010010001. . . , 0.547010010001. . . , and 0.548010010001. . . . Then $0.\overline{54} < 0.546010010001\ldots < 0.547010010001\ldots < 00.548010010001\ldots < 0.\overline{55}$ **c.** Answers vary. $\frac{\sqrt{3}}{4}, \frac{\pi}{7}$, and $\frac{\sqrt{2}}{3}$ are three such irrational numbers.

9. Answers vary. For example, between any two rational numbers we could find three irrational numbers. Because there are infinitely many disjoint intervals bounded by rational numbers, there must be infinitely many irrational numbers.

13. e. All real numbers greater than zero

16. a. $5\sqrt[3]{2}, 5\sqrt[3]{4}$ **b.** $2^{3/4}, 2^{1/2}, 2^{1/4}$ or $^-2^{3/4}, 2^{1/2}, 2^{^-1/4}$

c. There are two possible geometric sequences.

$\frac{5\sqrt{15}}{3}, 5, \sqrt{15}, 3$ $\quad -\frac{5\sqrt{15}}{3}, 5, ^-\sqrt{15}, 3$

d. There are two possible geometric sequences.

$\frac{2\sqrt{6}}{3}, ^-2, \sqrt{6}, ^-3$ $\quad -\frac{2\sqrt{6}}{3}, ^-2, ^-\sqrt{6}, ^-3$

21. $4^3 = 64$
$4.6^3 \approx 97.3$
$4.7^3 \approx 103.8$
$5^3 = 125$
So, an integer approximation of $\sqrt[3]{103}$ is 5.

Assessment 8-1B

1. Answers vary. **8. a.** Answers vary. For example, $3 + \frac{\sqrt{2}}{3}, 3 + \frac{\sqrt{2}}{4}, 3 + \frac{\sqrt{2}}{5}$ **b.** Answers vary. Three such numbers are: 0.562010010001. . . , 0.563010010001. . . , and 0.564010010001. . . . **c.** Answers vary. The approach described in part (b) will produce three such irrational numbers.

9. $1 + \frac{\sqrt{2}}{n}$ for $n > 2$. **10. e.** $\overline{S} = Q$, where Q is the set of rational numbers. **16. a.** $4\sqrt[3]{2}, 4\sqrt[3]{4}$ **b.** $\sqrt[3]{2}, \sqrt[3]{4}$ **c.** $4, 2\sqrt{2}$ **d.** There are two possible geometric sequences.

$\frac{\sqrt{2}}{2}, 1, \sqrt{2}, 2$ $\quad -\frac{\sqrt{2}}{2}, 1, ^-\sqrt{2}, 2$

21. $12^3 = 1728$
$12.6^3 \approx 2000.4$
$12.7^3 \approx 2048.4$
$13^3 = 2197$
So, an integer approximation of $\sqrt[3]{2001}$ is 13.

Mathematical Connections 8-1

Reasoning

1. Assume on the contrary that $\sqrt[3]{2}$ is rational. Then $\sqrt[3]{2} = \frac{a}{b}$, where a and b are positive integers. Then $2 = \frac{a^3}{b^3}$ and $2b^3 = a^3$, which by the fact given in the problem implies that 2 appears as a factor on the right side of the equation a number of times that is a multiple of 3; on the left-hand side, there is one more 2 than a multiple of 3. This contradicts the uniqueness in the Fundamental Theorem of Arithmetic. **2.** The value of $\sqrt{3}$ on a calculator is approximate because $\sqrt{3}$ is an irrational number and a nonterminating and nonrepeating decimal. **3.** False, for example, $\sqrt{64 + 36} \neq \sqrt{64} + \sqrt{36}$. The values of a and b for which it is true are $a = 0$ and b any real number or $b = 0$ and a any real number.

$$\sqrt{a + b} = \sqrt{a} + \sqrt{b} \text{ if, and only if,}$$
$$(\sqrt{a + b})^2 = (\sqrt{a} + \sqrt{b})^2$$
$$a + b = a + b + 2\sqrt{a}\sqrt{b}$$
$$2\sqrt{ab} = 0$$
$$a = 0 \quad \text{or} \quad b = 0$$

4. Since π is irrational, its decimal expansion does not repeat. Therefore $\pi \neq \frac{22}{7}$. **5. a.** Assume on the contrary that $\sqrt{p}$ is rational. Then $\sqrt{p} = \frac{a}{b}$. Then $p = \frac{a^2}{b^2}$ and $p \cdot b^2 = a^2$, which implies that p appears as a factor on the right side of the equation an even number of times; on the left-hand side, it appears an odd number of times. This contradicts the Fundamental Theorem of Arithmetic.

b. $\sqrt{\frac{7}{2}} = \frac{a}{b}$
$\frac{7}{2} = \frac{a^2}{b^2}$
$7b^2 = 2a^2$, which by the assumption in part (a) implies that 2 appears on the right side of the equation an odd number of times; on the left hand side, it appears an even number of times. This contradicts the Fundamental Theorem of Arithmetic.

6. $\left(\frac{4}{25}\right)^{-1/3} = \left(\frac{25}{4}\right)^{1/3}$, and $\left(\frac{4}{25}\right)^{-1/4} = \left(\frac{25}{4}\right)^{1/4}$.
Using the fact that for numbers $x > 1$, the bigger the exponent, the bigger is x raised to that exponent, we get:
$\left(\frac{25}{4}\right)^{1/4} < \left(\frac{25}{4}\right)^{1/3}$, Therefore
$\left(\frac{4}{25}\right)^{-1/4} < \left(\frac{4}{25}\right)^{-1/3} = \left(\frac{25}{4}\right)^{1/3}$.

7. a. Since the algebra is correctly applied, the conclusion that follows is $\frac{3}{\sqrt{5}} - \frac{2}{\sqrt{3}} > 0 \Rightarrow \frac{2}{\sqrt{3}} < \frac{3}{\sqrt{5}}$.

b. $\frac{6}{\sqrt{7}} - \frac{5}{\sqrt{6}} = \frac{6\sqrt{6} - 5\sqrt{7}}{\sqrt{42}} = \frac{\sqrt{216} - \sqrt{175}}{\sqrt{42}} > 0.$
This proves that $\frac{5}{\sqrt{6}} < \frac{6}{\sqrt{7}}$.

Open-Ended

8. Answers vary. For example: **a.** 1, 2, 3, 4, 5, . . .
b. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots$ or $\pi, \pi + 1, \pi + 2, \pi + 3, \ldots$

9. a. Answers vary. For example, $\frac{6}{15}\sqrt{2}, \frac{7}{16}\sqrt{2}$, and $\frac{8}{17}\sqrt{2}$ are three such irrational numbers. **b.** Answers vary. For example $\left(\frac{n}{n+1}\right)\frac{3}{7}\sqrt{2}$ for $n \geq 5$

Cooperative Learning

10. Answers vary. **a.** When a number between 0 and 1 is raised to larger and larger exponents, the results approach 0.
b. Answers vary. **c.** Answers vary. **11.** $3.7^{2.4} = 3.7^{\frac{24}{10}} = 3.7^{\frac{12}{5}} = \sqrt[5]{3.7^{12}}$

Connecting Mathematics to the Classroom

12. To be a rational number a number must be able to be written in the form $\frac{a}{b}$ with both a and b integers and $b \neq 0$. However $\sqrt{2}$ is not an integer. There is no Distributive Property of Exponentiation over Addition, as the student is trying to use. **13.** Using the Distributive Property of Multiplication over Addition, we can show that $(x + y)^2 = x^2 + y^2 + 2xy$. Hence, $(\sqrt{a} + \sqrt{b})^2 = (\sqrt{a})^2 + (\sqrt{b})^2 + 2\sqrt{a}\sqrt{b} = a + b + 2\sqrt{ab}$. Thus, $2\sqrt{ab}$ is missing from the student's answer.

14. Answers vary. $\sqrt{\sqrt{\sqrt{2}}} = \left(\left(2^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 2^{\frac{1}{8}} = \sqrt[8]{2}$

15. The equation $x^2 = 2$ has two solutions: $\sqrt{2}$ and $-\sqrt{2}$, which is compactly written as $\pm\sqrt{2}$, but can be misleading. If $\sqrt{2}$ had two values, there wouldn't be a need to use the "$\pm$" notation infront of it. **16.** If $a > 0$ the inequality is equivalent to each of the following: $(\sqrt[3]{a})^6 < (\sqrt{a})^6, a^2 < a^3, 1 < \frac{a^3}{a^2} = a$. Hence $\sqrt[3]{a} < \sqrt{a}$ if and only if $a > 1$. **17.** $^-x > 0$ for $x = {}^-9$, which is the solution **18.** $\sqrt{a}\sqrt{b} = \sqrt{ab}$ only if $\sqrt{a}$ and $\sqrt{b}$ are real numbers. However, $\sqrt{^-1}$ is not a real number.

Assessment 8-2A

4. d. $q \cdot 2^n$ **f.** $4s + 15{,}000$ dollars
11. c. $\sqrt{3} + 198.5; (\sqrt{3} + 0.5) + 2(n - 1)$
13. $d = \frac{\sqrt{3} - \sqrt{2}}{100}; a_1 = \sqrt{2} - \left(\frac{99}{100}\right)(\sqrt{3} - \sqrt{2})$
17. a. $\sqrt{2}, \sqrt{3}, \sqrt{2} + \sqrt{3}, \sqrt{2} + 2\sqrt{3}, 2\sqrt{2} + 3\sqrt{3}, 3\sqrt{2} + 5\sqrt{3}, 5\sqrt{2} + 8\sqrt{3}, 8\sqrt{2} + 13\sqrt{3}, 13\sqrt{2} + 21\sqrt{3}, 21\sqrt{2} + 34\sqrt{3}$
b. i. $L_7 = 5\sqrt{2} + 8\sqrt{3} = F_5\sqrt{2} + F_6\sqrt{3}$
ii. $L_8 = F_6\sqrt{2} + F_7\sqrt{3}$
$L_9 = F_7\sqrt{2} + F_8\sqrt{3}$
$L_{10} = F_8\sqrt{2} + F_9\sqrt{3}$

Assessment 8-2B

2. a. $\frac{4(n+25)-60}{4}+5$ **8. a.** $P = dt$ dollars
d. $C = 300 + 4n$ dollars **10.** David: $d + 10$, Matt: $2d - 10$
11. b. $1 - 197\sqrt{2}$; $(1+\sqrt{2}) - 2\sqrt{2}(n-1)$
c. $199\sqrt{3} + 100x$; $(x+\sqrt{3}) + (x+2\sqrt{3})(n-1)$
13. $r = \sqrt[10]{4} = \sqrt[5]{2}$; $a_1 = \frac{25}{2\sqrt[5]{2^4}}$ **17. g.** The sum of the first n terms is 2 less than the two terms later in the sequence. **h.** If the nth term is F_n then, $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 2$.

Mathematical Connections 8-2

Reasoning

1. Answers vary. Both students are correct. The first expression is based on the first term and the second expression is based on the middle term. **b.** Yes **c.** Yes, because the sum can be written as $(x-2d) + (x-d) + x + (x+d) + (x+2d) = 5x$.
2. a. If the seven consecutive terms of an arithmetic sequence are 0, 12, 24, 36, 48, 60, 72, then their arithmetic average is $\frac{0+12+24+36+48+60+72}{7} = 36$, which is the middle term. **b.** Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ be seven consecutive terms of an arithmetic sequence and d the common difference. Then:

$$\begin{aligned} a_2 &= a_1 + d \\ a_3 &= a_2 + d = a_1 + 2d \\ &\vdots \\ a_7 &= a_6 + d = a_1 + 6d \end{aligned}$$

Their arithmetic average is
$\frac{a_1 + (a_1+d) + (a_1+2d) + \ldots + (a_1+6d)}{7} =$
$\frac{7a_1 + 21d}{7} = a_1 + 3d$, which is the fourth term, i.e., the middle term. **3.** If the number is x and the answer a, then $\frac{4x+16}{2} - 7 = a$, $2x + 1 = a$, and therefore $x = \frac{a-1}{2}$.
Thus, the teacher subtracted 1 from each answer and divided the difference by 2.

Open-Ended

4. Answers vary.
5. Answers vary. Let $a_n = 0.999^n$. Then
$a_1 = 0.999$, $a_2 \doteq 0.998$, $a_3 \doteq 0.997$ but $a_{5000} \doteq 0.0067$.

Cooperative Learning

6. Answers vary

Connecting Mathematics to the Classroom

7. The student thinks about the distributive property of multiplication over addition and thinks that a similar property of multiplication over multiplication is true. A counterexample shows that this is wrong. **8.** Yes; for example in $\{A | A \subset W\}$, A is a variable set, any set that is a proper subset of W, the set of whole numbers. **9.** The student is right. The first statement is the commutative property of set union, and the second of set intersection. **10.** Answers vary. $5n + 4 = 5(n-1) + 9$, which has the form $a_1 + (n-1)d$. Hence, it is an arithmetic sequence with $d = 5$ and $a_1 = 9$. Notice that $5 \cdot 3^{\frac{n}{2}} = 5 \cdot (\sqrt{3})^n = 5 \cdot (\sqrt{3})^{1+(n-1)} = 5 \cdot \sqrt{3} \cdot (\sqrt{3})^{n-1}$ and hence has the form $a_1 \cdot r^{n-1}$. Consequently, this is the nth term of a geometric sequence with ratio $\sqrt{3}$ and first term $5 \cdot \sqrt{3}$.

Review Problems

11. Answers vary. For example, rational numbers may be 1.411, 1.412; irrational numbers may be $1.41 + \frac{0.001}{\sqrt{2}}$, and $1.41 + \frac{0.001}{\sqrt{3}}$. **12. a.** Irrational **b.** Irrational
c. Rational **d.** Rational **e.** Rational **f.** Irrational **13. a.** $\frac{9}{4}$
b. Not real **c.** $-\frac{27}{8}$ **14. a.** $^-25$ **b.** $\{x | x \leq 0\}$ **c.** $^-25$
15. $n = 3$

Assessment 8-3A

5. Let x be the amount the youngest receives. Then $x + 3x + x + 14{,}000 = 486{,}000$, or $5x = 472{,}000$. Youngest received \$94,400.00; oldest \$283,200.00; middle \$108,400.00. **7.** Let x be the number of nickels. Then $67 - x$ is the number of dimes. So $10(67-x) + 5x = 420$, $x = 50$. Thus, 50 nickels and 17 dimes.

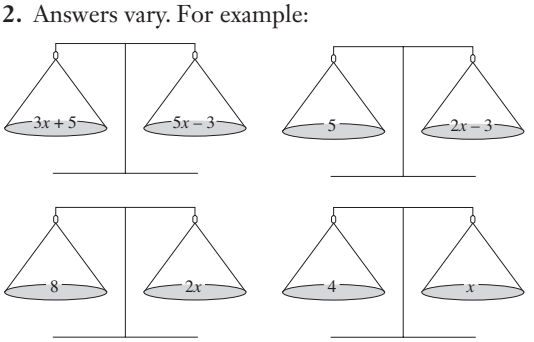
Assessment 8-3B

1. a. $\square$, since $\triangle = 2\bigcirc$ and $\square = 4\bigcirc$ **b.** $\bigcirc$ because if $2\triangle = \square$, then $\triangle < \square$. If $2\bigcirc = \triangle$, then $\bigcirc < \triangle$. Therefore, $\bigcirc < \triangle < \square$. **5.** The eldest will receive \$984,000, the middle siblings will receive \$344,000 apiece, and the youngest sibling will receive \$328,000. **7.** 40

Mathematical Connections 8-3

Reasoning

1. Both are correct. For the first student, x is the first of the three consecutive whole numbers. The second chose x to be the second of the three consecutive whole numbers.
2. Answers vary. For example:



(subtract $3x$ from both sides)
(add 3 to each side) (divide each side by 2)
3. $\sqrt{5281} \doteq 72.67$ ft **4. a.** 7 hours and 12 minutes **b.** $\frac{ab}{a+b}$ hours

Open-Ended

5. Answers vary. **a.** $-\left(\frac{x^2}{2} - 4\right) = 4 - \frac{x^2}{2}$ **b.** $x^2 + 12 = 8$
c. $x^2 = 0$ **6. a.** Answers vary; for example, $x^2 - 3 = 0$
b. Answers vary; for example, $x^3 - 2 = 0$

Cooperative Learning

7. Answers vary. **8.** Answers vary.

Connecting Mathematics to the Classroom

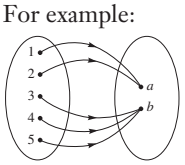
9. $x = 0$ is a solution. The student is wrong because we cannot divide by x if $x = 0$. **10.** Equations can be set up with one unknown as follows. If Jillian delivers x papers, then Abby delivers $2x$ and Brandy $2x + 50$. Thus, $x + 2x + 2x + 50 = 550, 5x = 500$, and hence $x = 100$. Jillian delivers 100 papers, Abby 200, and Brandy 250. **11.** Because there might be an error setting up the equation and hence the equation might not be a true modeling of the problem **12.** He misused the equality sign. What he wrote implies that $4x + 5 = 40$, which is not true.

Review Problems

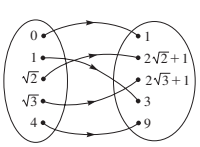
13. $x = 3y$ **14.** $10n$ **15.** Answers vary. If Jack has x CDs then Julie has $2x$ and Tyto $6x$. **16. a.** $P = 90 + 3d$ dollars **b.** $15d$ dollars **17. a.** If the middle term is x and the difference is d, then the terms are $x - d, x, x + d$. Their sum is $3x$. **b.** If the middle term is x and the ratio is r then the terms are $\frac{x}{r}, x, rx$. Their product is x^3. Thus, the product of any three consecutive terms in a geometric sequence is the middle term to the third power. **18.** $\frac{\sqrt{5}}{32}$ **19.** $\frac{\sqrt[6]{5}}{\sqrt[3]{12}}$

Assessment 8-4A

3. a. Answers will vary. For example:



4. a.

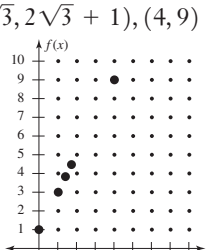


b. $(0, 1), (1, 3), (\sqrt{2}, 2\sqrt{2} + 1), (\sqrt{3}, 2\sqrt{3} + 1), (4, 9)$

c.

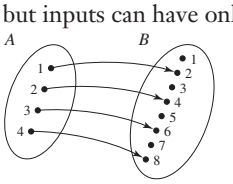
x	$f(x)$
0	1
1	3
$\sqrt{2}$	$2\sqrt{2} + 1$
$\sqrt{3}$	$2\sqrt{3} + 1$
4	9

d.

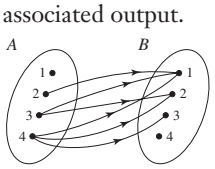


5. c. Not a function because $(4, -2)$ and $(4, 2)$ both satisfy the relation, but inputs can have only one associated output.

6. a. i.

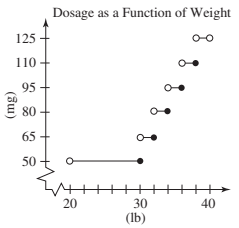


ii.



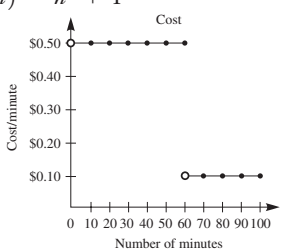
b. Part (i) is a function from A to B. For each element in A, there is a unique element in B. The range of the function is $\{2, 4, 6, 8\}$.

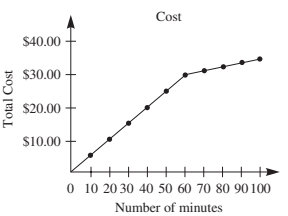
7.



8. Answers vary. **a.** $L(n) = 2n + (n - 1)$, or $3n - 1$
b. $L(n) = n^2 + 1$

9. a.

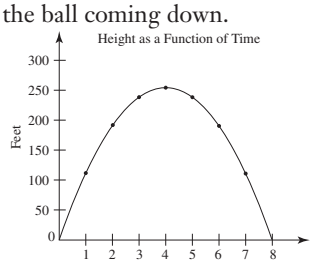




Notice that because we can't really depict 100 dots on the graphs a dot is drawn only for a multiple of 10 minutes. **b.** That the company charges for each part of a minute at the rate of \$0.50 per minute **c.** The two segments represent different charges per minute. The one representing the higher cost is steeper. **d.** $C(t) = 0.50t$ if $0 \le t \le 60, C(t) = 30 + 0.10(t - 60)$ if $t > 60$ **13. a. (a)** 7 **(b)** 35 **(c)** $7(\sqrt{10} - 5)$ **(d)** -35 **(e)** $\frac{-245}{8}$ **(f)** $7(n - 5)$ **16. a.** $\sqrt{2}$ or $-\sqrt{2}$

17. b. $\{(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)\}$ **c.** The domain is $R^+ \times R^+$, and the range is R^+.
18. e. Segments are used because the data are continuous rather than discrete. For example, there are a number of cars at 5:20 A.M.
19. a. $H(2) = 192$; $H(6) = 192$; $H(3) = 240$; $H(5) = 240$. Some of the heights correspond to the ball going up, some to the ball coming down.

b.

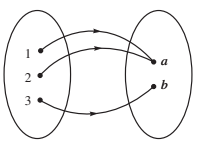


At $t = 4$ sec, the ball's height is $H(4) = 256$ ft above the ground.
20. $S(n) = (n + 1)^2 + (n + 2)n$ or $2n^2 + 4n + 1$
22. (a), (c), (d), (e), (h) are functions. (b) does not represent a function; if x, the input, is any given real number, then y is not unique, as it could be any real number y such that $y > x - 2$. **f.** Not a function; if $x = 1$ then $y = 1$ or -1. **g.** Not a function; if $x = 0, y = 1$ or -1. **24. a.** Boys: B, H; girls: A, C, D, G, I, J, E, F
b. $\{(A, B), (A, C), (A, D), (C, A), (C, B), (C, D), (D, A), (D, B), (D, C), (F, G), (G, F), (I, J), (J, I), (E, H)\}$ **26. b.** Reflexive,

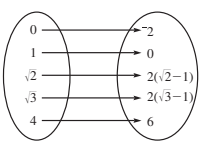
symmetric, and transitive (and so an equivalence relation) **c.** Reflexive, symmetric, and transitive (and so an equivalence relation) **d.** Reflexive, symmetric, and transitive (and so an equivalence relation)

Assessment 8-4B

1. a. Multiply the input by $^{-}3$. **2. a.** This is not a function since nothing is paired with 2. **b.** This is not a function, since the input 1 is paired with several outputs. **c.** Function **d.** Not a function because $(2, b)$ and $(2, c)$ both satisfy the relation, but inputs can have only one associated output. **3. a.** Answers vary. For example:



4. a.

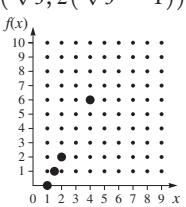


b. $\{(0, {}^{-}2), (1, 0), (\sqrt{2}, 2(\sqrt{2} - 1)), (\sqrt{3}, 2(\sqrt{3} - 1)), (4, 6)\}$

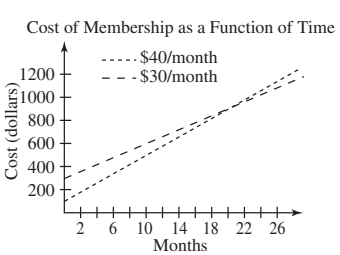
c.

x	$f(x)$
0	$^{-}2$
1	0
$\sqrt{2}$	$2(\sqrt{2} - 1)$
$\sqrt{3}$	$2(\sqrt{3} - 1)$
4	6

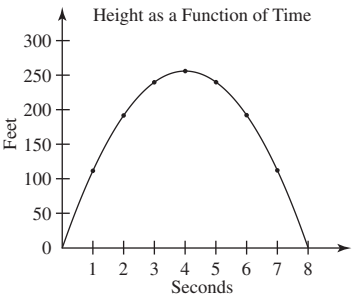
d.



5. b. This is not a function, since, for example, $f(3) = 0$ because 3 is in R and $f(3) = 1$ because 3 is in $\{3, 4, 5, 6, \ldots\}$. **d.** Not a function because $(2, 2)$ and $(2, {}^{-}2)$ both satisfy the relation, but inputs can have only one associated output in a function. **8.** Answers vary. **a.** $L(n) = n(n + 1)$ **b.** $L(n) = 2^n$ **9. a.** Cost per minute for the sixth minute of the call in the first and the total cost for a 6 min call in the second **b.** That the cost per minute is always the same regardless of the length of the conversation and that the company charges for each fraction of a minute **c.** $c = 0.45$ for the first graph and $c = 0.45t$ for the second graph **13. a. (a)** 34 **(b)** $7\sqrt{3} - 1$ **(c)** 69 **(d)** $7a - 1$ **(e)** $\frac{43}{6}$ **14. c.** All real numbers. **17. a.** 10; 6; 6; $4\sqrt{3}$; $2x + 2y$ **b.** $\{(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)\}$ **c.** No, because outputs are single numbers **18. b.** and **c.**



19. a.



At $t = 4$ sec, the ball's height is $H(4) = 256$ ft above the ground. **b.** $t = 1$ and $t = 7$ **21. a.** $(1 + 2 + 3 + \ldots + n) + n$, or $\frac{n(n + 1)}{2} + n$ **b.** $n + 2 + 2n$, or $3n + 2$ **22.** Only (b), (g) and (f) do not represent a function. (b) If x, the input, is any given real number, then y is not unique, as it could be any real number y such that $y < 20 - x$. (g) If $x = \sqrt{2}$ then $y = 1$ or $^{-}1$ (f) If $x = 1$ then $y = 1$ or $^{-}1$ **23.** All are functions except part (e), because for each input there is a single output. Domains are sets of real numbers. Part (e) has, for example, two outputs for the single input 1. **24. a.** Only (i), (iii), and (iv) have the symmetric property; if $x = a$ and $y = b$ satisfies the relation, so does $x = b$ and $y = a$. In part (ii) and part (v), $a - b = 100$, and $b = a^2$ do not imply $b - a = 100$ or $a = b^2$ respectively. These parts do not have the symmetric property. **b.** They are all functions. **25. a.** A, C, D, I, J, G are girls; B, F are boys. **b.** $\{(A, B), (A, D), (C, B), (C, A), (D, A), (A, C), (C, D), (D, C), (D, B), (G, F), (I, J), (J, I)\}$ **c.** No, because (A, B) and (A, D) are different ordered pairs with the same first component. **26. b.** Relation but not a function **c.** Relation but not a function **27. a.** Reflexive, symmetric, and transitive; an equivalence relation **d.** Reflexive, symmetric, and transitive; an equivalence relation.

Mathematical Connections 8-4

Reasoning

1. Yes; each element of A is paired with exactly one element of B. **2.** Yes, answers vary. **3. a.** This is not a function, since a faculty member may teach more than one class. **b.** This is a function (assuming only one teacher per class). **c.** This is not a function, since not every senator is paired with a committee. (Not every senator is the chairperson of a committee.) **4. a.** f is a function with inputs that are subsets of S. For each subset A of S, there is a unique output—the complement of A. If $A = S$, then $\overline{A} = \varnothing$. Hence, for the input S, the output is the empty set and vice versa. The domain is the set of all the subsets of S. The range is the same. **b.** Because the set of 20 elements has 2^{20} subsets, this is the number of elements in the domain as well as in the range because each subset has a complement. **c.** Yes; for each output there is a unique input (if B is the output, $\overline{B}$ is the input). **5. a.** $2^{3.14} = 2^{\frac{314}{100}} = \sqrt[100]{2^{314}}$ **b.** $\sqrt{3}$ can be approximated as closely as we wish by a rational number, $\frac{m}{n}$, where m and n are natural numbers. As m/n gets closer and closer to $\sqrt{3}$, the values of $2^{\frac{m}{n}}$ defined as $\sqrt[n]{2^m}$, where $\frac{m}{n}$ is in simplest form, get closer and closer to a particular real number, approximately 3.322;

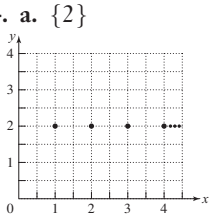
that number is $2^{\sqrt{3}}$. **c.** All real numbers as explained in part (b). **d.** All real numbers greater than 0 because 2^x is always positive. **6.** If P represents the initial investment, r the interest rate per time period (such as annually), and n the number of time periods. Then the balance B is a function of P, r, and n where: $B(P, r, n) = P(1 + r)^n$. **7.** No, justifications vary.

Open-Ended

8. Answers vary. **9.** Answers vary. **10.** Answers vary. **11.** Answers vary. **12.** Answers vary. **13.** Answers vary. Three such functions can be
$f(x) = 2x$
$g(x) = \sqrt{x}$
$h(x) = {}^{-}x$
a. $h(g(f(1))) = {}^{-}\sqrt{2}$ **b.** $h(g(f(\sqrt{3}))) = {}^{-}\sqrt{2\sqrt{3}}$
c. $h(g(f(n))) = {}^{-}\sqrt{2n}$

Cooperative Learning

14. a. $\{2\}$
b.



c. $f(x) = 2$ **d.** $\dfrac{4x + 8}{4} - x = 2$ **e.** Answers vary. **f.** Answers vary. **15.** Answers vary. **16.** Answers vary. One example is $a = 1$ and $b = 3$ and $f(x) = x^2$. Then $f(a + b) = f(1 + 3) = f(4) = 4^2 = 16$. However $f(a) + f(b) = f(1) + f(3) = 1^2 + 3^2 = 10$ and therefore $f(a + b) \neq f(a) + f(b)$. It should be pointed out that $f(x)$ is "f of x", not $f \cdot x$.

Connecting Mathematics to the Classroom

17. The student is wrong; the ordered pair (x, y) is considered a single input. **18.** It is a function. Each input has exactly one output. **19.** One way is to show that for a given output y there is a unique input. If $y = 3x + 5$ and y is an output, then the unique input is $\dfrac{y - 5}{3}$, a real number if y is a real number. **20.** If the domain does not include the numbers between two inputs, the points corresponding to the two inputs should not be connected. **21.** The first set represents a function because for each x there is a unique y. The second is not a function because for example if $x = 4$ there are two values of y; 2 or ${}^{-}2$.

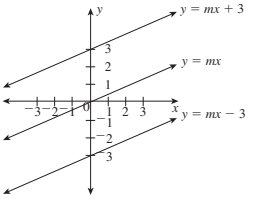
Review Problems

22. a. $\dfrac{2\sqrt{2}}{1 + \sqrt{2}}$ **b.** $\dfrac{12}{3 - 2\sqrt{2}}$ **c.** $1 \pm \sqrt{2}$ **d.** $\dfrac{3}{2}$ **e.** $\{x \mid x \geq 1\}$ **f.** $\{x \mid x \leq 1\}$ **23.** 4 hr **24. a.** Answers vary. For example, 1.8, 1.85 **b.** Answers vary. For example, $\dfrac{11 + 3^{-1/2}}{13}, \dfrac{11 + 2^{-1/2}}{13}$

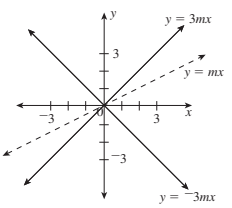
25. Answers vary. For example, $\sqrt{3} + {}^{-}\sqrt{3} = 0$, a rational number.

Assessment 8-5A

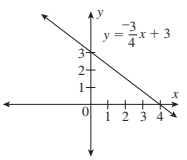
1. a. and **b.** The graph of $y = mx + 3$ contains the point $(0, 3)$ and is parallel to the line $y = mx$. Similarly, the graph of $y = mx - 3$ contains the point $(0, {}^{-}3)$ and is parallel to $y = mx$.



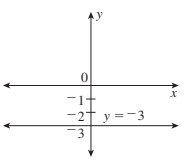
c. and **d.**



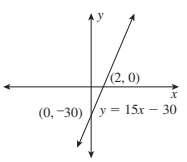
2. a.



b.



c.



3.

x-intercept	*y*-intercept
a. 4	3
b. None	${}^{-}3$
c. 2	${}^{-}30$

4. a. Using $(0, 32)$ and $(100, 212)$, the slope is $(212 - 32)/(100 - 0) = \dfrac{9}{5}$. So $F = \left(\dfrac{9}{5}\right)C + b$. Substitute in the point $(0, 32)$. Thus, $b = 32$, and the equation is $F = \left(\dfrac{9}{5}\right)C + 32$. **b.** $C = \dfrac{5}{9}(F - 32)$

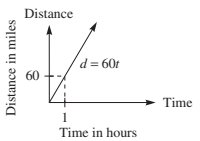
5. a. $y = \left(\dfrac{1}{3}\right)x$; slope $\dfrac{1}{3}$, y-intercept 0

b. $y = {}^{-}x + 3$; slope ${}^{-}1$, y-intercept 3

c. $y = \left(\dfrac{1}{3}\right)x$; slope $\dfrac{1}{3}$, y-intercept 0

6. b. $y = \left(\dfrac{1}{2}\right)x$ **8. b.** $x > 0$, $y < 0$; x and y are real numbers. **11. d.** Slope does not exist. **12. a.** $y = \dfrac{1}{3}x + \dfrac{5}{3}$ **b.** $y = 2$ **c.** $y = x$ **d.** $x = \sqrt{5}$ **13.** Answers vary, depending on estimates from the fitted line; for example: **a.** From the fitted line, estimate point coordinates of $(50, 8)$ and $(60, 18)$; the slope is $\dfrac{10}{10} = 1$. Use the point $(50, 8)$ and substitute $T = 50$ and $C = 8$ into $C = 1T + b$ (i.e., an equation of the form $y = mx + b$), so $8 = 1(50) + b$; $b = {}^{-}42$; the equation is then $C = T - 42$. **15. a.** The lines have the same x-intercept, ${}^{-}3$. **b.** The lines have the same x-intercept, 1. **c.** The lines have the same slope, ${}^{-}2$. **d.** The lines have the same slope, ${}^{-}1$.

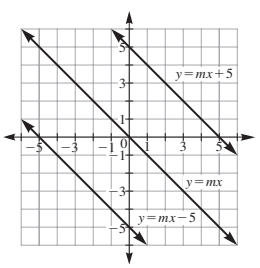
16. a.



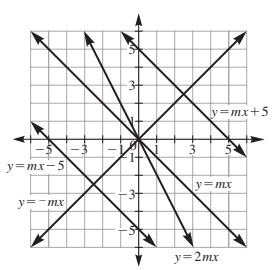
17. c. Infinitely many solutions of the form $\left(x, \frac{2}{3}x + \frac{1}{3}\right)$

Assessment 8-5B

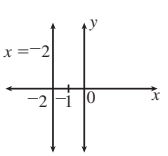
1. a. and **b.** The graph of $y = mx$ contains the point $(0, 0)$ and is parallel to the line $y = mx + 5$. The graph of $y = mx - 5$ contains the point $(0, {}^{-}5)$ and is parallel to the line $y = mx$.



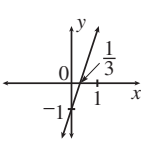
c. and **d.**



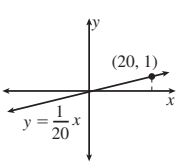
2. a.



b.



c.

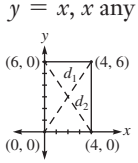


3. a. x-intercept 2, no y-intercept **b.** x-intercept $\frac{1}{3}$; y-intercept ${}^{-}1$ **c.** x-intercept 0, y-intercept 0

5. b. $y = \frac{3}{4}x + \frac{7}{4}$; slope $\frac{3}{4}$; y-intercept $\frac{7}{4}$

8. b. $y = x$, x any real number

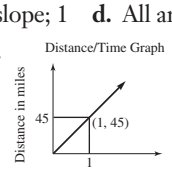
9. a.



c. Answers vary. The equation of d_1 is $y = \frac{3}{2}x$. The equation of d_2 is $y = \frac{{}^{-}3x}{2} + 6$. **10. a.** $y = {}^{-}2$ **b.** $x = {}^{-}4$ **12. a.** $y = \frac{1}{9}x + \frac{13}{9}$ **b.** $x = {}^{-}3$ **c.** $y = 20{,}000x - 19{,}990$ **d.** $y = \sqrt{5}$

14. The coordinates of A, B, and C are $(1, 0)$, $(2, {}^{-}2)$, and $(9, 5)$, respectively. **15. a.** The lines have the same x-intercept, 3. **b.** The lines have the same x-intercept, 3. **c.** The lines have the same slope; 1 **d.** All are non-vertical lines through the point $(1, 1)$.

16. a.



17. a. No solution **b.** Infinitely many solutions of the form $\left(x, \frac{2}{3}x - \frac{1}{6}\right)$ where x is any real number. **c.** Unique solution of $(2, 0)$ **d.** Infinitely many solutions of the form $\left(x, \frac{3}{2}x + \frac{1}{2}\right)$ where x is any real number. **18.** $\left(\frac{16}{3}, \frac{8}{3}\right)$ is the common intersection point of lines containing the segments: $y = 8x - 40$; $y = \frac{{}^{-}4}{7}x + \frac{40}{7}$, $y = \frac{1}{2}x$.

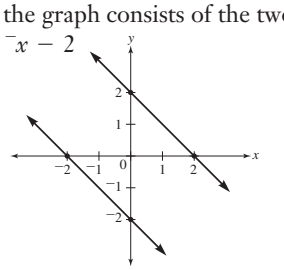
Mathematical Connections 8-5

Reasoning

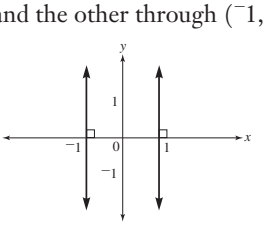
1. No (unless $r = 1$), because the slopes between two successive points are not the same. **2.** Answers vary. Explanation 1: If two distinct lines have the same slope m, then the equations are $y = mx + b$ and $y = mx + c$ for some real numbers b and c (with $b \neq c$). To show that the lines are parallel, it is sufficient to show that the lines do not intersect; that is, that the system of equations has no solution. Indeed, if we try to solve the equations, we get $mx + b = mx + c$. Because $b \neq c$, this equation has no solution. Explanation 2: With $y = mx + b$ and $y = mx + c$, each is a vertical shift of the same line, $y = mx$.

3. a. $\left(\frac{a - b}{m - n}, m\left(\frac{a - b}{m - n}\right) + b\right)$ **b.** As long as $m \neq n$ the lines will have different slopes and thus intersect in a single point.

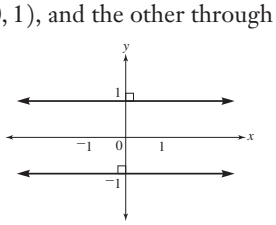
4. The lines intersect at a common point. **5. a.** $x + y = 2$ or $x + y = {}^{-}2$. Hence the graph consists of the two parallel lines $y = {}^{-}x + 2$ or $y = {}^{-}x - 2$



b. $x + y = 1$ or $x + y = {}^{-}1$. Hence the graph consists of the two parallel lines $y = {}^{-}x + 1$ or $y = {}^{-}x - 1$. The graph is similar to the one in part (a) except that one of the lines contains the points $(1, 0)$ and $(0, 1)$ and the other the points $({}^{-}1, 0)$ and $(0, {}^{-}1)$.
c. $x = 1$ or $x = {}^{-}1$. Thus the graph consists of two vertical lines one through $(1, 0)$, and the other through $({}^{-}1, 0)$



d. $y = 1$ or $y = {}^{-}1$. Thus the graph consist of two horizontal lines one through $(0, 1)$, and the other through $(0, {}^{-}1)$.



6. Because the coordinates can be irrational numbers, so can the ratio that is the slope. For example, the slope of the line through the points $(2, \sqrt{2})$ and $(1, 0)$ is $\frac{\sqrt{2} - 0}{2 - 1}$ or $\sqrt{2}$.

Open-Ended

7. Answers vary. **8. a.** Answers vary. **b.** Answers vary.

Cooperative Learning

9. Answers vary. **10. a.** All have the solution $x = {}^{-}1, y = 2$. **b.** Same as in part (a). **c.** Answers may vary. If a, b, c are consecutive terms in an arithmetic sequence, then $x = {}^{-}1, y = 2$ satisfy equation $ax + by = c$ because $a({}^{-}1) + b \cdot 2 = c$ is equivalent to $b - a = c - b$.

Connecting Mathematics to the Classroom

11. The slope of a vertical line is undefined because division by 0 is impossible. When a line is closer and closer to being a vertical line, the absolute value of the slope is getting bigger and bigger. **12.** Jonah needs to be reminded about order of operations; $5 - 5x \neq (5 - 5)x$. **13.** A line with undefined slope is perpendicular to the x-axis. Two lines perpendicular to another line are parallel. **14.** The system will have infinitely-many solutions because one equation will be a constant multiple of the other and therefore they will represent the same line.

In the case of the given equations, $\frac{2}{{}^{-}4} = \frac{{}^{-}3}{6} = \frac{{}^{-}5}{10} = \frac{{}^{-}1}{2}$ and hence $2 = ({}^{-}4)\left(\frac{{}^{-}1}{2}\right), {}^{-}3 = 6\left(\frac{{}^{-}1}{2}\right), {}^{-}5 = 10\left(\frac{{}^{-}1}{2}\right)$.

Substituting into the first equation we get

$$0 = 2x - 3y - 5 = ({}^{-}4)\left(-\frac{1}{2}\right)x + 6\left(-\frac{1}{2}\right)y + 10\left(-\frac{1}{2}\right).$$

Dividing by $-\frac{1}{2}$, we get ${}^{-}4x + 6y + 10 = 0$, which is the second equation. Thus, the equations represent the same line.

Review Problems

15. a. $\sqrt{0.\overline{9}} = 1$ **b.** $x^2 + y^2 = 36$ **c.** 1 **16.** 2.45

17. a. $x = \frac{2}{4\sqrt{2}} = \frac{\sqrt{2}}{4}$ **b.** 9 or ${}^{-}9$ **c.** $x < \frac{{}^{-}\sqrt{7}}{3}$

18. a. $2\sqrt{7}$ **b.** $7 - \sqrt{7}$ **c.** ${}^{-}5\sqrt{7}$ **19.** $x = \frac{12 + \sqrt{2}}{3}$

20. a. $\{x | x \in R \text{ and } x \geq {}^{-}1\}$ **b.** $\{x | x \in R \text{ and } x \leq 0\}$

Chapter 8 Review

6. b. $\frac{1}{\sqrt[4]{4}}, \frac{1}{\sqrt[4]{16}}, \frac{1}{\sqrt[4]{64}}$ or $\frac{{}^{-}1}{\sqrt[4]{4}}, \frac{1}{\sqrt[4]{16}}, \frac{{}^{-}1}{\sqrt[4]{64}}$

14. a. If n is the original number, then each of the following lines shows the result of performing the instruction:

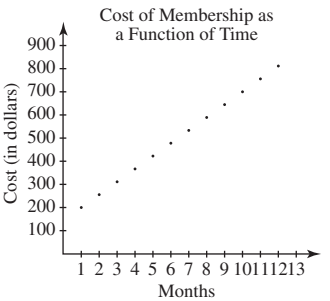
$$\begin{gathered} n \\ n + 17 \\ 2(n + 17) = 2n + 34 \\ 2n + 30 \\ 4n + 60 \\ 4n + 80 \\ n + 20 \\ n \end{gathered}$$

b. Answers will vary. For example, the next two lines could be subtract 65 and then divide by 4.

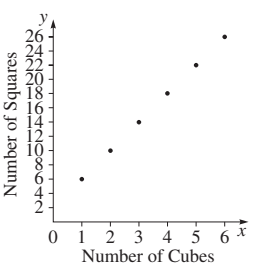
20. c. $\{x | x \geq 0, x \text{ any real number}\}$ **21. a.** This is not a function, since one student can have two majors. **b.** This is a function. The range is the subset of the natural numbers that includes the number of pages in each book in the library. **c.** This is a function. The range is $\{6, 8, 10, 12, \ldots\}$. **d.** This is a function. The range is $\{0, 1\}$. **e.** This is a function. The range is N.

22. a. $C(x) = 200 + 55(x - 1)$

b.

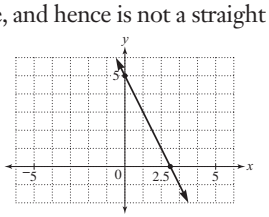


c. In the ninth month, the cost exceeds \$600. **d.** In the 107th month **25. b.** The graph consists of points on the line $y = 4x + 2$ for $x = 1, 2, 3, 4, 5, 6$

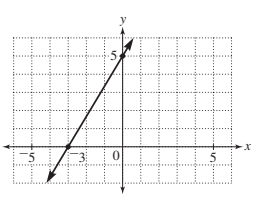


c. $y = 4x + 2$ **d.** The graph does not contain all the points on the line, and hence is not a straight line. The points lie along a line.

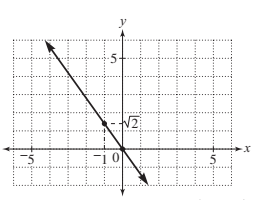
26. a.



b.



c.



27. a. Line $\overleftrightarrow{CD}$ has equation $4x + 3y - 12 = 0$. **b.** $\left(0, \frac{8}{7}\right)$

Answers to Now Try This

1. a. Let g_k be the kth approximation. Then if $g_1 = 4$ then $g_2 = \frac{1}{2}\left(\frac{13}{4} + 4\right) \doteq 3.625$, $g_3 = \frac{1}{2}\left(\frac{13}{3.625} + 3.625\right) \doteq 3.605$. Because $g_4 \doteq 3.605$, 3.605 is accurate to 3 decimal places.

b. The algorithm for finding $\sqrt{a}$ is given by the recursive formula $g_{n+1} = \frac{1}{2}\left(\frac{a}{g_n} + g_n\right)$, where g_1 is the initial guess.

2. a. The approach works as shown.

$$\begin{aligned} \sqrt{\sqrt{\sqrt{a}}} = \left(\left(a^{1/2}\right)^{1/2}\right)^{1/2} &= \left(a^{1/4}\right)^{1/2} \\ &= a^{1/8} \\ &= \sqrt[8]{a} \end{aligned}$$

b. For $n = 2^k$, where k is a positive integer. As shown in part (a),

by repeatedly applying the square-root function we get $\sqrt[2^k]{a}$.
3. For a number like $8^{0.\overline{3}}$, one meaning is $8^{1/3}$ or $\sqrt[3]{8}$. Any repeating or terminating decimal can be converted to a number in $\frac{a}{b}$ form where a, b are integers, and $b \neq 0$. For a number like $8^{0.101001\cdots}$, one interpretation is as $8^{\left(\frac{1}{10}+\frac{0}{100}+\frac{1}{1000}+\frac{0}{10000}+\frac{0}{100000}+\frac{1}{1000000}+\cdots\right)}$ or $8^{10} \cdot 8^{0} \cdot 8^{\frac{1}{1000}} \cdot 8^{0} \cdot 8^{0} \cdot 8^{\frac{1}{1000000}\cdots}$. Each part could be evaluated and an approximation of the total value could be obtained.
5. a. After 10 hours, there are $2 \cdot 3^{10} = 118{,}098$ bacteria, and after n hours, there are $2 \cdot 3^n$ bacteria. **b.** After 10 hours, there are $2 + 10 \cdot 3 = 32$ bacteria, and after n hours, there are $2 + 3n$ bacteria. We see that after 10 hours each term of the geometric sequence is greater than the corresponding term of the arithmetic sequence. In general when $r > 1$ and the first term is positive, the terms of a geometric sequence eventually increase faster than the corresponding terms of any arithmetic sequence. **6. a.** $\square = 3, \triangle = 9$ **b.** $\square = 4, \triangle = 2$
7. If Bob delivered b papers then Abby delivered $3b$ papers and Connie $3b + 13$ papers.

$$\begin{aligned} b + 3b + 3b + 13 &= 496 \\ b &= 69 \\ a = 3b &= 207 \\ c = 3b + 13 &= 220 \end{aligned}$$

8. a. It is a function from the set of natural numbers to $\{0, 1\}$, because for each natural number there is a unique output in $\{0, 1\}$. **b.** It is a function from the set of natural numbers to $\{0, 1\}$, because for each natural number there is a unique output in $\{0, 1\}$. **9. a.** Girls: A, C, D, F, G, I; boys: B, J **b.** E and H
10. a. If $m = 0$, the line is the x-axis. **b.** As m increases from 0, the slope is positive and the graph of the line becomes steeper as m becomes larger. **c.** As m decreases from 0, the slope is negative and the graph of the line becomes steeper as the absolute value of m becomes larger. **11.** If there were no restriction on the domain, a_n is replaced by y, and n is replaced by x, the equation becomes $y = a_1 + (x - 1)d$ or $y = dx + (a_1 - d)$. The slope of this line is d and the y-intercept is $a_1 - d$. **12. a.** All the points on a horizontal line have the same y-coordinate. Thus, two points on a horizontal line have the form (x_1, y_1) and (x_2, y_2), where $y_2 = y_1$. The slope is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$.
b. For any vertical line, $x_2 = x_1$ and therefore $x_2 - x_1 = 0$. If we attempted to find the slope, we would have to divide by 0, which is impossible. Hence, the slope of a vertical line is not defined.
13. a. The graphs are lines that intersect at $x = 4$ and $y = 3$.
b. The lines are parallel and not equal and hence the system has no solution. Algebraic approach: Assuming that there is a solution x and y, we multiply the first equation by 2 and add it side-by-side to the second. Therefore, x and y must satisfy $0 \cdot x + 0 \cdot y = 5$. However, no x and y satisfies this equation (a solution would imply $0 = 5$). This contradicts our assumption that the original system has a solution. **c.** The equations represent the same line. If we divide both sides of the second equation by $^-2$ we get the first equation. Hence there are infinitely many solution: x is any real number and $y = 2x - 1$.

Answer to the Preliminary Problem

a. If the length of the sheet of paper is l and the width is w, then the ratio of length to width is $\frac{l}{w}$. The ratio of length to width of the half sheet is $\frac{w}{l/2} = \frac{2w}{l}$. Because the ratios need to be equal, we have:

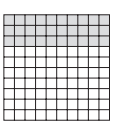
$$\begin{aligned} \frac{l}{w} &= \frac{2w}{l} \\ \frac{l^2}{w^2} &= 2 \\ \left(\frac{l}{w}\right)^2 &= 2, \frac{l}{w} = \sqrt{2}. \end{aligned}$$

b. When two sheets are joined along their lengths, the ratio of the length of the longer side to the shorter is $\frac{2w}{l}$. We have seen in part (a) that $\frac{l}{w} = \frac{2w}{l}$. Hence $\frac{2w}{l} = \sqrt{2}$.

CHAPTER 9

Assessment 9-1A

2. The probability is not likely because there is only one whole number 0 in the sample space that is not a natural number. **8. a.** Answers vary, for example, choose one cell of the grid at random.

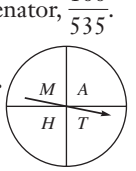


b. 70%, $P(\text{No rain}) = 1 - P(\text{Rain}) = 1 - 0.30 = 0.70$
16. No, she is not correct. The probability of drawing a white ball from box #1 is $\frac{3}{4} = \frac{6}{8}$. The probability of drawing a white ball from box #2 is $\frac{5}{8}$. Thus, the probability of drawing a white ball from box #1 is greater. **17.** The probability is $\frac{1}{2}$ because it is a fair coin. The coin has no memory, so the probability of the event of a head on the 16th toss is the same as the probability of the event of a head on any toss regardless of this history.

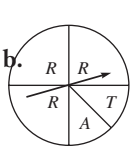
Assessment 9-1B

1. c. No. This is an experiment with chance elements in it, and it likely will not happen that we get exactly the predicted number of successes on any given set of trials of the experiment. **d.** Yes, if there are a large number of trials. **3.** There are 435 members of the House of Representative and 100 senators. The probability that the congressperson drawn is a member of the House of Representatives is $\frac{435}{535}$ which is greater than the probability this congressperson is a senator, $\frac{100}{535}$.
8. Answers vary.

a.



b.



14. a. Not possible. Suppose x white, x black, and x red balls are added to the box. Then the probability of the event of drawing a black ball is $\frac{3 + x}{(5 + x) + (3 + x) + (2 + x)} = \frac{3 + x}{10 + 3x}$. There is no x for which this ratio is $\frac{1}{3}$, because $\frac{3 + x}{10 + 3x} = \frac{1}{3}$ is

equivalent to $9 + 3x = 10 + 3x$, which has no solution. **b.** Yes, add five balls of each color. Reason: the solution of the equation $\frac{3 + x}{10 + 3x} = 0.32$ is $x = 5$. **15.** The spinner is not a good tool. $P(\text{boy}) = P(\text{odd number}) = \frac{4}{7} \neq \frac{1}{2}$.

18. $P(\text{vowel}) = \frac{12}{50} = \frac{6}{25}$; $P(\text{consonant}) = \frac{38}{50} = \frac{19}{25}$, so I am more likely to choose a state whose name starts with a consonant.

21. $\frac{1}{6} - \frac{16}{100} = \frac{50}{300} - \frac{48}{300} = \frac{2}{300} = \frac{1}{150}$

Mathematical Connections 9-1

Reasoning

1. No; 0 and 1 are not the lead digits of any normal phone number. **2.** All probabilities are less than or equal to 1 and greater than or equal to 0. If events A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B) = 0.8 + 0.9 = 1.7$, which is impossible. Therefore, events A and B are not mutually exclusive. **3.** Bobbie's reasoning is not correct. The outcomes of making a basket or missing it may not be equally likely; for most players, they would not be equally likely. To calculate her experimental probability of the event of making a free throw, divide the number of shots made by the number attempted. **4.** Answers vary. E is the letter most used in the English language. **5.** The probability that a 6 would appear 1000 times in a row with a fair die is so close to 0 that we can conclude this is not a fair die.

Open-Ended

6. a. $\frac{1}{4}$ **b.** Answers vary. **c.** Answers vary. For example, when people pick a number, their choices are not random. They may have a favorite number or they may choose to avoid the extreme values. **d.** Answers vary. **7.** Answers vary depending on the book selected. **8.** Answers vary. For example, batting averages can be interpreted as experimental probabilities that a hitter will get a hit, or a free-throw shooting percentage can be interpreted as experimental probability that a basketball player will make a free throw. Insurance companies use experimental probabilities to determine rates. **9.** Answers vary. For example, event A is an impossible event such as rolling a 10 on a single roll of a standard die. Event B has low probability, such as the chance of rain being 20%. Event C is marked at approximately 0.5, so this might be something like obtaining a head when tossing a fair coin one time. Event D has a high probability of happening, but it is not certain (for example, tossing a number less than 6 on a toss of a standard die). Event E has probability 1, so it has to happen (for example, tossing either a head or a tail on the toss of a fair coin). Event F has probability greater than 1, but this cannot happen, so no event is possible.

Cooperative Learning

10. a. Answers vary. **b.** Answers vary. **c.** The theoretical probabilities for the differences are $P(0) = \frac{6}{36} = \frac{1}{6}$; $P(1) = \frac{5}{18}$; $P(2) = \frac{2}{9}$; $P(3) = \frac{1}{6}$; $P(4) = \frac{1}{9}$; $P(5) = \frac{1}{18}$; $P(6) = 0$.

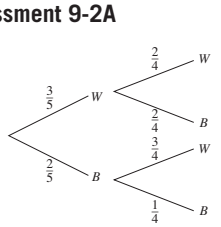
d. Using the probabilities from part (c), the 18 markers should be distributed from left to right as 3, 5, 4, 3, 2, 1. **11. a.** Answers vary. **b.** The person who receives 4 times the value of the number on the die wins when 1, 2, and 3 are tossed. The person who receives the square of the number showing wins when a 5 or 6 is tossed. When a 4 is tossed, the game is a draw. The person who receives 4 times the value of the die has a greater chance of winning, and therefore the game is not fair. **12. a.** Answers vary. **b.** Answers vary. **c.** Answers vary.

Connecting Mathematics to the Classroom

13. A fair coin has "no memory." Hence the probability of a tail on each toss is $\frac{1}{2}$ regardless of how many tails have appeared in previous tosses. **14.** If the four regions corresponding to the colors were determined by equal-sized angles, the events of the spinner's landing on each of the regions would be equally likely, and the student would be correct. However, since the four angles are not the same sizes, the events are not equally likely, and the student is wrong. **15.** Tossing three heads on the first three tosses of a coin does not imply the coin is unfair. Only when a fair coin is tossed a much greater number of times can we expect to get approximately equal numbers of tails and heads. The probability of getting the same outcome is three tosses is $\frac{1}{4}$, not all that unlikely. It would be premature to conclude that the coin is unfair on this evidence. **16.** For an experiment with sample space S with equally likely outcomes, the probability of an event A is defined by $P(A) = \frac{n(A)}{n(S)}$. Because $n(S)$ is never negative and $n(A)$ is never negative, $P(A)$ can never be negative. **17.** The probability of an event is the ratio and does not necessarily reflect the number of elements in the event or in the sample space. For example, if $n(S) = 20$ and $n(A) = 12$, then $P(A) = \frac{12}{20}$, which could also be reported as $P(A) = \frac{3}{5}$. **18.** Joe's conjecture is incorrect. Each of the numbers 1 through 4 has approximate probability $\frac{1}{4}$ of occurring because each angle where the arrow is located seems to measure 90 degrees and the spinner has the same chance of landing in any of the regions. **19.** No, because the areas of the different sectors of the dartboard are unequal, we would not expect the probabilities to be equal. **20.** Mari needs to be asked to think about the number of pages with words beginning with each letter. Because those numbers of pages are not equal, the probabilities cannot be equal.

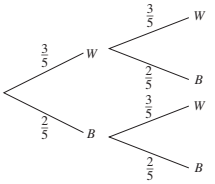
Assessment 9-2A

1. a.



The probability of the event that two balls are of different colors is $\frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{5}$.

b. $\frac{12}{25}$

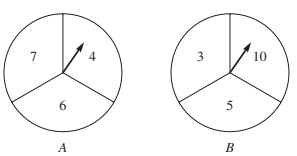


8. a. The first spinner. If you choose the first spinner, you win if, and only if, the spinning combinations are as follows:

Outcome on spinner *A*	Outcome on spinner *B*
4	3
6	3 or 5
8	3 or 5

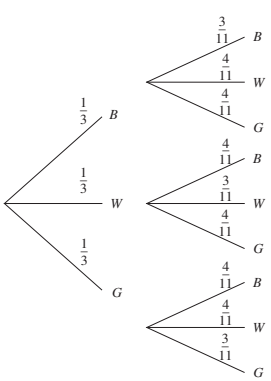
The probability of this happening is $\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{2}{3}=\frac{5}{9}$. If you choose spinner *B*, the probability of winning is only $\frac{4}{9}$.

b. Answers vary. One choice is as follows:



If you choose *A*, the probability of winning is $\frac{5}{9}$. If you choose *B*, the probability of winning is only $\frac{4}{9}$.

14. a.



b. $\frac{3}{11}$ **c.** $\frac{1}{11}$ **18.** Billie-Bobby-Billie. **19.** Answers vary. For example, 18 blue marbles may be added. **22. b.** $\frac{2}{5}$

Assessment 9-2B

1. a. $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$

4. a. Box 1, with probability $\frac{1}{3}$; Box 2 has probability $\frac{1}{5}$.

11. b. $P(\text{Region } A)=\frac{1}{25}$; $P(\text{Region } B)=\frac{3}{25}$; $P(\text{Region } C)=\frac{1}{5}$; $P(\text{Region } D)=\frac{7}{25}$; $P(\text{Region } E)=\frac{9}{25}$

17. Each room has probability $\frac{1}{2}$ of being chosen. Hence, it does not matter where the car is placed. **18.** She should serve the first one hard, and then it does not matter what she does on the second serve.

Mathematical Connections 9-2

Reasoning

1. If the die is fair, the probability of the event that Jim will roll a 3 on the next roll is $\frac{1}{6}$, which is the same probability of rolling any one of the other numbers. The results of the rolls are independent. **2.** Answers vary. If treated as independent events, finding a suspect with blond hair and blue eyes has greater probability. The probability of the event red car introduces another number that is less than 1 (and still positive) into the product. When you multiply *a*, the probability that you have for blond hair and blue eyes by a number less than 1 and greater than 0, the probability becomes smaller. **3.** Put the red ball into one of the boxes and the three white balls into the other box. In this case the probability of the event of getting the red ball is $\left(\frac{1}{2}\right)1+\left(\frac{1}{2}\right)0=\frac{1}{2}$. In the three other arrangements, the probability of getting the red ball is smaller. **4.** Scenario (2) has greater probability, $\frac{1}{2}$. **5.** The probabilities could be equal if event *B* in $P(B|A)$ is the empty set; in that case both probabilities are 0.

Open-Ended

6. Answers vary. The game could be as simple as player *A* wins if the sum of the digits is even and player *B* wins if the sum of the digits is odd. **7.** One possibility is that one die has the numbers 0, 0, 0, 3, 3, 3 and the other die has the numbers 1, 2, 3, 7, 8, 9. The probability of each outcome is $\frac{1}{12}$. **8.** Answers vary, but $\frac{3}{5}$ of the area of the board must be labeled in a certain way. **9.** Answers vary.

Cooperative Learning

10. a. Answers vary. **b.** Answers vary.

c.

x	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

d. Yes, each player has probability $\frac{1}{2}$ of winning. **11.** Answers vary. For example, one player wins with products 1, 2, 3, 4, 5, 6, 8, 10 and the other player wins with products 9, 12, 15, 16, 18, 20, 24, 25, 30, 36. Then each player has probability $\frac{1}{2}$ of winning.

12. If the first player chooses *TT* and the second *HT*, then the first player wins the game if, and only if, tails appears on the first and second flips. The probability of this happening is $\frac{1}{4}$. The probabilities of winning the game for other choices are summarized below:

		First Player's Choice			
		HH	***HT***	***TH***	***TT***
Second Player's Choice	***HH***	—	0.50	0.75	0.50
	HT	0.50	—	0.50	0.25
	TH	0.25	0.50	—	0.50
	TT	0.50	0.75	0.50	—

Of the 12 possibilities, only 8 result in choices with equally likely probabilities of winning; that is, 8 fair and 4 not. Therefore, the

game is not fair. **13. a.** The game is not fair. You should choose spinner *A*. **b.** The game is not fair. You should choose spinner *C*. It has a winning probability of $\frac{35}{99}$. **14.** No matter which die the other player chooses, you can always choose one that will beat it $\frac{2}{3}$ of the time. Therefore, the strategy for playing this game is to go second and make your choice of die based on the information in the following table.

First Person's Choice	Second Person's Choice	Probability of Second Choice Winning
A	*D*	$\frac{2}{3}$
B	*A*	$\frac{2}{3}$
C	*B*	$\frac{2}{3}$
D	*C*	$\frac{2}{3}$

Connecting Mathematics to the Classroom

15. The sample space for this event is not $\{HH, HT, TT\}$ but rather $\{HH, HT, TH, TT\}$. Consequently the probability of *HH* is $\frac{1}{4}$. The student can think of the experiment as having 3 different outcomes but the outcomes are not equally likely. **16.** Independence refers to whether one event affects another. A second outcome does not depend upon the first. For example, if you get a head on the first flip of a coin, the probability of the event of getting a head on the second flip is not changed. The term *mutually exclusive* is used to describe events that cannot occur simultaneously. For example, when you flip three coins, then the events "two or more heads" and "two or more tails" are mutually exclusive. **17.** The student is not correct. The confusion probably lies in the fact that the student thinks that probabilities are additive. The student does not understand the Multiplication Rule for Probabilities. A tree diagram for the experiment could possibly help. A partial tree diagram is

$$\xrightarrow{\frac{1}{6}} 5 \quad \xrightarrow{\frac{1}{6}} 5$$

Thus, $P(5, 5) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$. **18.** One multiplies probabilities when events are independent and we are finding the probability of their intersection. Adding probabilities occurs when events are mutually exclusive and we are finding the probability of their union. **19.** The sample space for the outcomes of the experiment is $\{BB, BW, WB, WW\}$. The probability of the event of two blacks is $\frac{1}{4}$, not $\frac{1}{2}$. **20.** Alva's reasoning is not correct. If she chooses bag A, her chances of getting a red ball are $\frac{1}{5}$. If she chooses bag B, her chances of getting a red ball are $\frac{3}{16}$. Because $\frac{1}{5} > \frac{3}{16}$, she should choose bag A.

Review Problems

21. a. v **b.** iii **c.** ii **d.** i **e.** iv **22. a.** $\frac{1}{30}$ **b.** 0 **c.** $\frac{19}{30}$ **23.** $\frac{1}{3}$ **24. a.** $\frac{9}{19}$ **b.** $\frac{1}{19}$ **c.** $\frac{13}{19}$ **d.** $\frac{10}{19}$ **25.** 10 times

Assessment 9-3A

1. Answers vary; likely this would not work because the outcomes are not equally likely. **2.** Answers vary. For example, let the digits 1 and 2 represent diamonds, digits 3 and 4 represent hearts, digits 5 and 6 represent spades, and 7 and 8 represent clubs. If 0 or 9 appear, then disregard these digits. Read the random digits in pairs to simulate two draws; repeat many times to estimate the probability. **3. a.** Let 1, 2, 3, 4, 5, and 6 represent the numbers on the die and ignore the numbers 0, 7, 8, 9. **b.** Number the persons 01, 02, 03, . . . , 18, 19, 20. Go to the random-digit table and mark off groups of two. The three persons chosen are the first three whose numbers appear. Skip over pairs of digits that exceed 20 or represent a person already chosen. **c.** Answers vary. Represent red by the numbers 0, 1, 2, 3, 4; green by the numbers 5, 6, 7; yellow by the number 8; and white by the number 9. **4.** Mark off 30 blocks of three digits in a random-digit table. Disregard 000 or any numbers that are greater than 500. Also disregard duplicates. These are the numbers of the 30 students who will be chosen for the trip. **5.** To simulate Monday, let the digits 1 through 8 represent rain and 0 and 9 represent no rain. If rain occurred on Monday, repeat the same process for Tuesday. If it did not rain on Monday, let the digits 1 through 7 represent rain and 0, 8, and 9 represent dry. Repeat a similar process for the rest of the week. **6.** Answers vary. For example, mark off blocks of two digits and let the digits 00, 01, 02, . . . , 13, 14 represent contracting the disease and 15 to 99 represent no disease. Mark off blocks of six digits to represent the three children. If at least one of the numbers of successive pairs within your 6-digit block representation of the family is in the range 00 to 14, then this represents a child in the three-child family having strep. Repeat many times to estimate the probability. **9. b.** Answers vary. For example, let the digits 0, 1, 2, 3, 4 represent a victory by team A and the digits 5, 6, 7, 8, 9 represent a victory by team B. Then go to the random-digit table, pick a starting place, and count the number of digits (games) it takes for one of the teams to win. Repeat this experiment a number of times for four games and for seven games and use the definition of probability to compute the experimental probability. **10.** One way to simulate the problem is to use a random-digit table repeatedly. Because Carmen's probability of making any basket is 80%, we could use the occurrence of a 0, 1, 2, 3, 4, 5, 6, or 7 to simulate making the basket and the occurrence of an 8 or a 9 to simulate missing the basket. Another way to simulate the problem is to construct a spinner with 80% of the spinner devoted to making a basket and 20% of the spinner devoted to missing the basket. This could be done by constructing the spinner with 80% of the 360 degrees (that is, 288 degrees) devoted to making the basket and 72 degrees devoted to missing the basket. **30.** Answers vary. Let the two-digit numbers 00, 01, 02, 03, . . . , 57 represent the 58 consonants. Choose a random place to start the random digit table. Read the numbers in pairs. If one of the pairs listed above appears in reading the table, a consonant is found. Here *y* is counted as a consonant.

Assessment 9-3B

1. Answers will vary. Let red represent a boy and blue represent a girl. **2.** Answers will vary. One possibility is to use digits 0, 1, 2, 3, 4, 5 to represent rain. **3.** Answers will vary. Consider the people to be numbered as 00, 01, 02, 03, . . . , 09, 10, 11, . . . , 24. Then go to a random-digit table. Choose a starting point. Look at sets of numbers from the starting point in pairs. For example, if you see 97 85 80 56 42 88 04 76 89 60 . . . , the first pair that represents a person is 04. Then continue in this manner until the desired four people are found. Skip over all repeats.
4. Answers will vary. Mimic the solution to exercise 3. Number people with triples 000, 001, 002, . . . , 199 and use the process described, only now using triples instead of pairs. **5.** Answers will vary. Use a random-digit table and look at pairs of digits where every odd digit represents tails and every even digit represents heads. For example, 83 represents a head and a tail while 71 represents two tails. Look at 25 such pairs to determine the experimental probability. **8.** With 100 questions, each student could expect to be asked about 14 times. **9.** Answers vary. For the simulation you choose 5 numbers from 12 possibilities. Two-digit numbers between 00 and 11 could be selected from a random number table with each number 00, 01, 02, . . . , 11 representing one of the signs. The probability should be around 0.65. **10.** Answers vary. Use a random number table and read the numbers in groups of two. A number less than 45 represents a type-O donor. **a.** Around 0.73 **b.** Around 8.9 **11. b.** The ratio of the probability of drawing a black card to the probability of drawing a non-black card **12. c.** 1023 to 1 **23. c.** To lose about a nickel per play **24.** $E = \frac{1}{6}(10) + \frac{5}{6}(-2) = 0$; therefore, you should come out about even if you play a long time.
30. Answers vary. Let the two-digit numbers 00, 01, 02, 03, . . . , 41 represent the 42 vowels. Choose a random place to start the random-digit table. Read the numbers in pairs. If one of the pairs listed above appears in reading the table, a vowel is found.

Mathematical Connections 9-3

Reasoning

1. Answers will vary. One possibility is to see Kanold, C. "Teaching Probability Theory Modeling Real Problems." *Mathematics Teacher* 85 (April 1994): 232–235. **a.** Should be about 2 **b.** 1:1
2. Answers vary. One way is to turn left when an even number is rolled, turn right when the number 1 is rolled, and move straight when 3 or 5 is rolled. **3.** 25 cents because $P(HH) = \frac{1}{4}$ so $E = \left(\frac{1}{4}\right)(\$1) = \$.25$.

Open-Ended

4. Answers vary; one could use a random-digit table with blocks of two digits, or a spinner could be designed with 12 sections representing the different months. The spinner could be spun 5 times to represent the birthdays of five people. We could then keep track of how many times at least two people have the same birthday. Repeat to estimate the probability. **5.** Answers vary. For example, use a random-digit table. Let the digits 1–8 represent a win and the digits 0 and 9 represent losses. Mark off blocks of three. If only the digits 1–8 appear, then this represents three wins in a row. **6.** Let the 10 ducks be represented by the digits 0, 1, 2, 3, . . . , 8, 9. Then pick a starting point in the table and mark off 10 digits to simulate which ducks the hunters shoot at. Count how many of the digits 0 through 9 are not in the 10 digits; this represents the ducks that escaped. Do this experiment many times and take the average to determine an answer. See how close your simulation comes to the theoretical average of 3.5 ducks.
7. Answers vary. For example, a person pays \$1.00 to play the game; he/she wins \$2.00 if coins match and nothing if they don't match. **8.** Answers vary. The premium would have to cost more than $\left(\frac{47}{100{,}000}\right)(50{,}000) = \$23.50$. **9.** Answers vary. One possibility is a game involving a spinner with five sectors of equal area numbered 1 through 5. You pay \$2.00 for playing the game. If the spinner lands on regions 1 or 2, you win \$5.00. The expected winnings in this game are $5 \cdot \frac{2}{5} - 2$, or 0.

Cooperative Learning

10. a.–c. Answers vary. **d.** The probability of the event of two boys and two girls is $\frac{3}{8}$. **11. a.** Answers vary. **b.** Answers vary. **c.** 15 **d.** $\frac{1}{1024}$ **12. a.–c.** Answers vary. **13.** Answers vary.
14. a. i. Use RANDINT(1, 6) **ii.** Use RANDINT(1, 6) + RANDINT(1, 6) **b. i.–ii.** Answers vary. **c.** Answers vary. **15.** Games and rules will vary, as will the odds and expected values of the game.

Connecting Mathematics to the Classroom

16. Answers vary. Maximilian could be correct, but students should be asked to explain how their simulations were designed and carried out. **17.** Answers vary. Suppose the odds in favor of winning a game are 1:2. Even when the outcomes are equally likely, it does not mean that out of every three games the player will win one game. It only means that if a large number of games are played, the ratio between the number of wins and the number of losses is close to $\frac{1}{2}$. However, the answer is partially correct because the ratio of wins to losses is $\frac{P(W)}{1 - P(W)} = \frac{a}{b}$, $P(W) = \frac{a}{a + b}$. **18.** The probability of the event occurring is $\frac{3}{7}$ and not $\frac{3}{4}$. The student likely is confusing odds and probability.

19. Answers vary. The probability is $\frac{1}{3}$ that an individual does not vote. Maria expects $48 - \frac{1}{3} \cdot 48$, or 32 to vote. Since she is guaranteed 24 votes and $24 > (51\%)(32)$, she could be confident of winning.

Another way to consider the situation is to consider that 32 are expected to vote, BUT perhaps $\frac{1}{3}$ of 24, or 8, that she expected to vote for her might not. In that situation, she is guaranteed $24 - 8$, or 16, votes but needs $51\%(32)$, or 17. She is missing 1 vote. Her ground is certainly less safe in this scenario.

Review Problems

20. No, one person will win about $\frac{3}{8}$ of the games and the other about $\frac{5}{8}$ of the games. **21. a.** $\frac{1}{4}$ **b.** $\frac{1}{52}$ **c.** $\frac{12}{13}$ **d.** $\frac{3}{4}$ **e.** $\frac{1}{2}$

f. $\frac{1}{52}$ **g.** $\frac{4}{13}$ **h.** 1 **22. a.** $\frac{15}{19}$ **b.** $\frac{56}{361}$ **c.** $\frac{28}{171}$ **23.** $\frac{8}{27}$ **24.** Answers vary. **a.** {1, 2, 3, 4} **b.** {Red, Blue} **c.** {(1, Red), (1, Blue), (2, Red), (2, Blue), (3, Red), (3, Blue), (4, Red), (4, Blue)} **d.** {(Blue, 1), (Blue, 2), (Blue, 3), (Blue, 4), (Blue, 5), (Blue, 6), (Red, 1), (Red, 2), (Red, 3), (Red, 4), (Red, 5), (Red, 6)} **e.** {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)} **f.** {(Red, Red), (Red, Blue), (Blue, Red), (Blue, Blue)} **25.** The blue section must have angle measure 300 degrees; the red has angle measure 60 degrees. **26.** $\frac{25}{676}$

Assessment 9-4A

14. a. $\frac{9}{133}$ **15. c.** About 0.995 **d.** About 0.001

Assessment 9-4B

17. 2,598,956 : 4 or 649,739 : 1

Mathematical Connections 9-4

Reasoning

1. Answers vary. For example, the Fundamental Counting Principle (FCP) says that to find the number of ways of making several decisions in a row, multiply the number of choices that can be made for each decision. The FCP can be used to find the number of permutations. A permutation is an arrangement of things in a definite order. A combination is a selection of things in which the order is not important. We could find the number of combinations by using the FCP and then dividing by the number of ways in which the things chosen for the permutation can be arranged. **2.** There are 10 choices for the number on the first reel, 10 choices for the number on the second reel, and 10 choices for the number on the third reel. By the Fundamental Counting Principle, there are $10 \cdot 10 \cdot 10$, or 1000, choices for the combination of the lock. **3. a.** $7! \cdot 3!$. If the family is considered a unit and each of the remaining people also a unit, we have 7 units. There are 7! ways to arrange the 7 units. For each of the 7! ways, the family unit can be arranged in 3! ways and hence the number of seating arrangements is $7! \cdot 3!$ or 30,240. **b.** $10! - 7! \cdot 3!$ or 3,598,560. We need to subtract the answer in part (a) from the number of all possible seating arrangements. **4.** 3840. Consider each couple as a unit. The five units can be arranged in 5! ways. For each of the 5! arrangements, each of the five couples can be arranged in 2 ways. Consequently there are $5! \cdot 2^5$ arrangements. **5.** Answers vary. Typically the number of permutations is greater than the number of combinations, but that is not always the case. Consider n objects chosen 1 at a time.

Open-Ended

6. a. 1,000,000 **b.** Answers vary. For example, you would first find the population of California and then experiment with using letters in the license plates. This would help because the choice is for 26 letters in a slot rather than 10 numbers. **7.** Answers vary.

Cooperative Learning

8. a. Answers vary. The first and last numbers in every row are 1. Every other number is obtained by adding the numbers immediately above. The next two rows are 1, 7, 21, 35, 35, 21, 7, 1 and 1, 8, 28, 56, 70, 56, 28, 8, 1. **b.** Answers vary. For example, the diagonal 1, 3, 6, 10, 15 are triangular numbers. **c.** The sums of the numbers in the rows are 1, 2, 4, 8, 16, 32, 64. The sum in row 10 is $2^{10} = 1024$. **d.** Yes, a similar relationship holds in all the rows for the entries in Pascal's triangle. **e.** The entries are given by ${}_nC_r$, where n is the row number and r is the position in the row, with the count starting at $r = 0$ for the leftmost entry. For example, the first three entries in row 6 are ${}_6C_0$, ${}_6C_1$, ${}_6C_2$. **9.** Answers vary.

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10. The student is confused about choosing four objects none at a time. There is one way to choose no objects. Therefore, we say ${}_4P_0 = 1$. **11.** We define 0! as 1 because it fits the formula for combinations. It is also consistent with the rest of mathematics. To define $\frac{1}{0}$ as 1 would cause many inconsistencies. If $\frac{1}{0} = 1$ and $\frac{1}{1} = 1$, then $\frac{1}{0} = \frac{1}{1}$, which should imply that $0 = 1$. This is not true. **12.** If the principle is used properly, one can compute the number of permutations. That does not imply that they are not needed. It is a concept that comes up a lot, so having a name for it and formulas to help with computations are useful.

Review

13. a. $\frac{33}{221}$ **b.** $\frac{26}{51}$ **14.** $\frac{1}{12}$ **15.** $E = \$0$ so the game is fair. **16. a.** No **b.** The expected payoff is $\frac{1}{38} \cdot \$36.00$, or approximately 95¢. Therefore, you can expect to lose on the average about 5¢ a game if you play it a large number of times. **17.** We first find the probability that all 40 children have different birthdays and then subtract the result from 1. We get

$$1 - \left(\frac{365 - 1}{365}\right)\left(\frac{365 - 2}{365}\right) \cdots \left(\frac{365 - 39}{365}\right) \approx 0.89$$

Thus, the probability that the friend wins the bet is approximately 0.89, or 89%. For 50 people, the probability rises to approximately 97%.

Chapter 9 Review

1. a. {*HHH*, *HHT*, *HTH*, *HTT*, *TTT*, *TTH*, *THH*, *THT*} **2. a.** {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday} **3.** Answers vary. There are 800 blue ones, 125 red ones, and 75 that are neither blue nor red. **6. a.** $\frac{1}{4}$ **d.** $\frac{12}{13}$ **7. a.** $\frac{64}{729}$ **11.** 4 to 48 or 1 to 12 **15.** $33\frac{1}{3}$ cents, so if 34 cents is charged the game is not fair. **16.** The expected value is \$1.50. The expected net winnings are $^{-}\$0.50$. **27.** Answers vary. For example, **a.** Randomly select digits 0–5 from a random-digit table; discard digits 6–9. **b.** Mark off two-digit numbers in a random digit table. Discard any other than 01–12. Select the months represented in the first three non-repeated numbers. **c.** Let random digits 0–2 represent red; digits 3–5 represent white; digits 6–8 represent blue; discard any 9s. **28.** These events are not equally likely. If each probability is computed, we have $P(3H) = \frac{1}{8}, P(2H) = \frac{3}{8}, P(1H) = \frac{3}{8}$, and $P(0H) = \frac{1}{8}$. **29. a.** $\frac{1}{8}$ **36. a.** 0.9999 **b.** 0.99999603 **37.** $\frac{3}{8}$

Answers to Now Try This

1. a. 1 **b.** Yes, they all sum to 1. 1 is the sum of the probabilities of all the different elements in any sample space. **c.** $\frac{1}{4}$; red represents 90 degrees of the 360 degrees where the spinner is located. **4. a.** Yes **b.** Answers vary. With replacement: any game with the same number of white and black marbles. Without replacement: 3 black and 6 white or 3 white and 6 black, 6 black and 10 white or 6 white and 10 black marbles. **c.** Without replacement: the white and black marbles need to be two consecutive triangular numbers; that is, $1 + 2 + \ldots + n$ and $1 + 2 + 3 + \ldots + n + n + 1$ (this is easier to discover using combinations introduced in Section 9-4). **5. a.** Answers vary. **b.** $\frac{3}{8}$ **c.** No, simulations will not always result in the same probability as the theoretical probability. However, if the experiment is repeated a great number of times, the simulated probability should approach the theoretical probability. **6. b.** There is one way to toss a head and one way of not tossing a head, so the odds in favor are 1 : 1. **c.** There are 4 ways to draw an ace and 48 ways of not drawing an ace, so the odds in favor are 4 : 48 or 1 : 12. **d.** There are 13 ways of drawing a heart and 39 ways of not drawing a heart, so the odds in favor are 13 : 39 or 1 : 3. **8.** Answers vary. **a.** We get an error message because 100! and 98! are too large for the calculator to handle.

b. $\frac{100!}{98!} = \frac{100 \cdot 99 \cdot 98!}{98!} = 9900$

Answer to the Preliminary Problems

Whenever any person entering the bus chooses a seat at random with both the seats assigned to the first person or to the 50th person are open, the probability is $\frac{1}{2}$ that one of those seats is chosen. At some point, one of these two seats has to be taken. If the seat chosen is the one assigned to the first person, then all the remaining people including the 50th person gets the correct seat. If the seat chosen is the one assigned to the 50th person, then the 50th person cannot get the correct seat but must choose the seat assigned to the first person. In any case, the probability that the 50th person gets the correct seat is $\frac{1}{2}$.

CHAPTER 10

Assessment 10-1A

1. Answers vary. Among the questions that the class will have to determine are the following:

- Do you count the houses all around the block or only on the side where your house is?
- Do you count the houses across the street from you? Are they considered on your block?
- What happens if you are in a new part of town where the blocks aren't developed yet?

Data to be collected will be determined by the question asked but likely will be a frequency count in the second grade. The frequency count could be shown in a bar graph and interpretations could be made about the graph. **2.** Answers vary. First and foremost will be the question of what it means to be active. Other questions to consider might include the following:

- Does this mean that they do the same type of activities but one group does it more than the other?
- Does this mean that one group sleeps more than the other? In middle school, there will be choices of both adults and sixth-grade students. Depending on understanding, a randomization might be made but likely not on a big scale. Convenient samples might be a class and its adult caregivers where answers to the question chosen could be obtained. If randomization is done, then it will likely be confined to the school itself. One unanswered question is, "What is an adult?" Is this term understood in the same way by all who do the study?

3. Among the questions the class must determine are: i. What is the definition of "classroom"? Are auditoriums, laboratories, libraries, and hallways to be counted in the definition? ii. How will the temperature be measured, that is, what instrument, level of sensitivity, and scale? iii. How often will the temperature be taken and at what time of day? iv. Is there a temperature baseline? **4.** The sample in **(i)** is more likely to be random and is more likely to provide better input to answer the question. The sample in **(ii)** would most likely sample students at only one grade level and they would be likely to have similar tastes. **5. b.** The question is biased because it uses words such as "refreshing," "ice-cold," and "hot," suggesting the person should like soda. **c.** The question is fair because it does not make any assumptions about the subjects or contain any persuasive words. **6.** Answers vary. **a.** Elementary students might blindfold the adults and simply hand them two different cans to see if the adults could tell the difference. **b.** Middle grade students might pour the liquids in unmarked cups, use a series of taste tests to eliminate random guessing, and so on. **7.** A major question is, "What does it mean to visit a country?" Does this mean that your airline landed in the country; that you spent a night there; that you spent an extended vacation there? If a country has now separated into different countries as the USSR did, do you count each part that was visited when the country was the USSR, or do you not count a visit to the new countries at all because your visit was before they became separate countries? **8.** A strong criticism of the prediction is that the sample of owners who had telephones and automobiles was not likely to be representative of voters in 1936. **9.** This type of representation is likely not to be indicative of the class at all. Representative samples of comments would be more "fair" and would give a better picture of the class. **10.** Whether or not the outlined sampling methods draw a representative sample depends in part on the study. **a.** In general, not representative because the students could have all ridden the same bus, hence come from the same general location; or, these students constitute the "early risers." **b.** In general, not representative because the students might inadvertently sort themselves by height, class, or by groups of acquaintances. **c.** In general, not representative because the sample fails to account for the students who are not interested in sports, or can't attend for one reason or another. Also, there is no mention of how to choose the 50 students. **d.** The responses in the drop box will be from those students for whom going to the drop box wasn't inconvenient or from those students who felt passionately about the topic of the questionnaire. **11.** Second-grade students are likely to make many observations which cannot be determined from the graph but from their knowledge of each

other. An example is "Alfie was the only one to wear dress shoes today." Reasonable interpretations are that on Tuesday the most popular shoes worn are tennis shoes and Crocs. A different interpretation is that on Tuesdays the most popular shoes are those with softer soles. **12. a.** Population: fans of the team to whom the stadium belongs. Sample: the 50 fans. In general, the survey method is representative. **b.** Population: students at a particular school. Sample: the first 75 students to pass through the main gate. Not representative because it doesn't account for the tardy students who may live farther away than most students (which is the reason they are not among the first to arrive). Also, the sample may neglect students entering school grounds from other entryways different from the main gate. **c.** Population: Households in town. Sample: The students at school. Not representative since it does not account for the childless households. **d.** Assume that the number of students at each grade level is the same. Population: Students at school. Sample: Groups of 5 students from each grade. The survey method is representative.

Assessment 10-1B

1. Among the questions the class must determine are: i. What defines a pet? What do you count as a pet? ii. Do you count the animals that live under the same roof as a student even though an animal belongs to a sibling or parent? iii. Do you count wild animals that may visit a residence regularly? iv. Do you count the pets that may have died in the past month? v. Which grades are of interest in the study? vi. How many students will be surveyed in each grade?

Data to be collected will be determined by the questions asked, but in the second grade will likely be a frequency count.

The frequency count could be displayed using a bar graph. Any interpretations or analysis will be based on the graph. **2.** Answers vary. A logical question is, "Could this sensitive question be asked?" If it is decided that the question could be asked about laziness, then the next question is likely to be what it means to be lazy and could the determination of laziness be done in such a way that the word *laziness* does not have to be used. Additional questions involve what population and sample to use and what is an adult. Other questions might be as follows:

- Does this mean that the people asked do the same type of activities but one group does them less than the other?
- Does this mean that one group sleeps more than the other?

3. Answers vary. You might ask the students what they think it means to be affluent? Is there a better word so that people with whom you talk understand the word better? Do you want a governmental definition of what it means to be affluent? Where could one be found? Where could you find a baseline of "normal" so that you would know what affluent means and what it means to be less than affluent? In a typical school, would one expect the students on one floor to be more affluent than those on a different floor? This is the type of question that a teacher might choose to discourage a class from asking in the school in order not to embarrass some students. But it is a typical question that a class might want to consider. **4. (i)** is more likely to be random **5. c.** The question is biased because it unfairly juxtaposes the sciences and sports; moreover, it makes it seem that the sciences are underfunded which may not be the case. **d.** The question is fair because it does not make any assumptions about the subjects or use any persuasive words. **6.** Answers vary. A major decision is what types of chocolate could be used. Does this mean a light chocolate versus a dark chocolate, milk chocolate versus other kinds, and so on? All students might find it necessary to blindfold any adults in the test so that they cannot see the chocolates and tell a difference that way. An additional question is, "What is an adult?" Middle-school students might want to try to choose adults at random for the test and would have to consider how to randomize them. **a.** Elementary students might blindfold the adults and simply hand them two different types of chocolate to see if the adults could tell the difference. **b.** Middle-grade students might rewrap pieces of chocolate in brown paper, use a series of taste tests to eliminate random guessing, and so on. **7.** Among the questions the class must determine are: i. What is a "sporting event"? School-related? Do practice games or scrimmages count? Do games in which a student participated count? Should those events that are cancelled, forfeited, or postponed part way through count? What about those in which a student leaves out of his or her own discretion? ii. What kind of sports should be included or excluded? Classify as winter or summer? With professional leagues or can they be without? Physical or mental? Should non-traditional sports such as X-games, chess tournaments, daredevil shows, vehicle racing (car, boat, bike), and so on count? **8.** There are many factors that could have affected the game. The strength of the schedule, playing the game in a different stadium, different fans, and level of preparedness could all be examined. **9.** This type of representation is likely not to be indicative of the class at all. Each choice is "good" or better. There are no negative choices. **10. a.** The sample probably would not be representative. In most elementary schools, the younger students eat first, so the sample would not be unbiased. **b.** Representative because a random choice was made. **c.** Assuming that number of students in each are the same the sample is representative. **d.** Unless the teachers are asked to provide a random list of their students, the sample won't be representative of the students at the school because teachers are inclined to name their best or most outgoing students first. **11.** It is reasonable for students to talk about the most popular types of shoes worn and the least popular. They likely will begin to identify different people's shoes, but that is added information that they might already know that has nothing to do with the graph. It is important that they recognize what the graph says and what they are adding to the conversation from outside knowledge. **12. a.** Population: students at a particular school. Sample: the 50 students interviewed. The sample will not be representative because it doesn't account for those students who are not in the habit of recycling at school or those who happen not to have anything to recycle on that given day. **b.** Population: all faculty and staff at a particular school. The sample is a census—everyone in the population—so it is representative by default. **c.** Population: presumably males attending a particular school. Sample: the 50 students interviewed. The sample is not representative, it takes into account only those males who work or study at the school library in the afternoon. **d.** Population: the cars in your city. Sample: the cars observed and noted down during the several days of the study. The sample is not representative because it deals with cars that are from one residential district.

Mathematical Connections 10-1

Reasoning

1. The U.S. Library of Congress since it has textbooks from all publishers. **2.** Answers vary. One discussion might be why it is important for students to begin discussing data at an early age. The *Statistics Framework* might also help.

Open-Ended

3. Answers vary according to grade level and sophistication. **4.** Answers vary depending on the level. In early grades, students may simply take a common page from a book and count. As students become older and more sophisticated with data, they could choose a text at random and choose a page at random and then determine how to find an average length of word to describe what they found.

Cooperative Learning

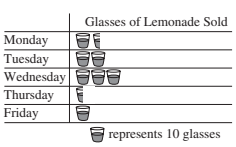
5. Answers vary. **6.** A proprietary system(s) of gathering information on U.S. audience size and makeup for television shows. In short, there are two primary ways this is accomplished, using statistical models and the newest technology. **7.** Answers vary.

Connecting Mathematics to the Classroom

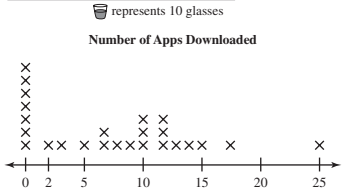
8. The answer is "Yes." A manufacturer should be able to ensure with some degree of accuracy that the product is the "same" all the time. In order to do that, tools should be calibrated with small allowances for error. **9.** An exit poll attempts to determine who or what voters in an election cast their ballot for as soon as they leave their polling station. Since the responses are provided by people, they are prone to error. **10.** Answers vary.
11. Answers vary.

Assessment 10-2A

2.

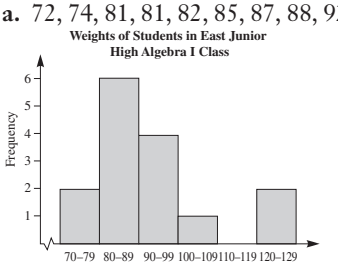


3.



4. a. 72, 74, 81, 81, 82, 85, 87, 88, 92, 94, 97, 98, 103, 123, 125

5.



6. Answers vary. **a.** A dot plot will have 6 columns, one for each outcome 1 through 6. **b.** A bar graph will have 6 bars, one for each outcome 1 through 6. The vertical axis will be scaled to show frequencies of each.

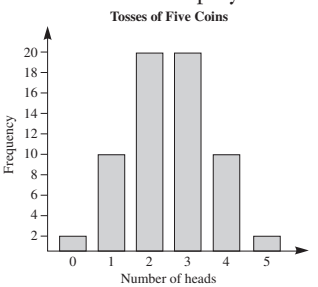
8. a. Ages of HKM Employees

1	889
2	01113333334566679
3	224447
4	1115568
5	2248
6	233

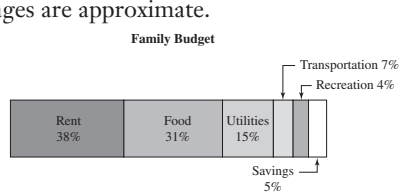
3 | 4 represents 34 years old

b. There are more employees in their 40s.

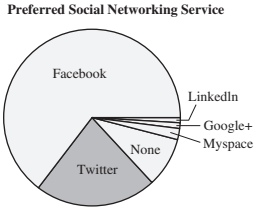
9.



10. Percentages are approximate.



11.



15. a. Answers vary. A graph of this type might be suspect. To have a survey where there are more people in their 70s–90s might mean the people are in a home for the aging. Also to have more people in their 90s than 80s reading might make one suspicious. Without more information, one cannot tell. **b.** From this flawed "survey", one would suspect that people in their 70s read the most, but the graph only depicts the number of readers of mysteries.

16. a. 1 | 3 represents $130

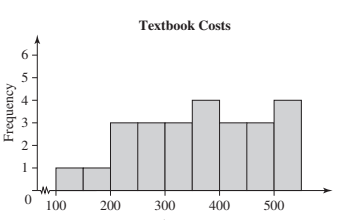
Amounts Paid for Textbooks

1	36
2	244556
3	2445556
4	244556
5	3444

b.

Amounts	Frequency
100–149	1
150–199	1
200–249	3
250–299	3
300–349	3
350–399	4
400–449	3
450–449	3
500–549	4

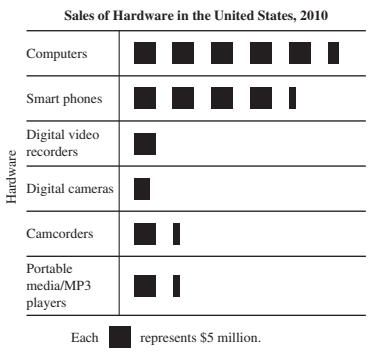
c.



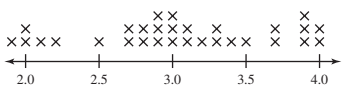
18. Measures are approximate: savings 0.8 cm; rent 2.4 cm; food 0.9 cm; auto payment 2.2 cm; tuition 1.7 cm.

Assessment 10-2B

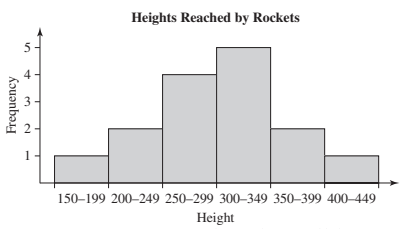
2.



3.



5.

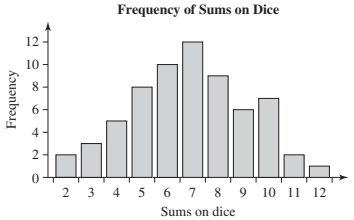


6. Answers vary. **a.** A dot plot will have 5 columns, one for each outcome 0–4. **b.** A bar graph will have 5 bars, one for each outcome 0–4. The vertical axis will be scaled to show frequencies of each. **8. a. Car Thefts**

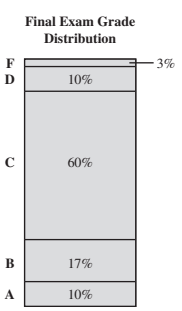
1	
2	
3	5
4	89
5	023356
6	25669
7	22885
8	2

8 | 2 represents 82 car thefts

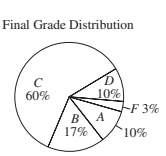
9.



10.

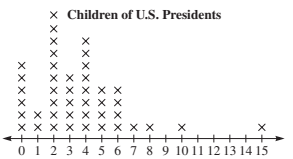


11.



12. c. Africa is $\frac{2}{3}$ the size of Asia. **f.** Approximately 59 million mi^2

15. a.

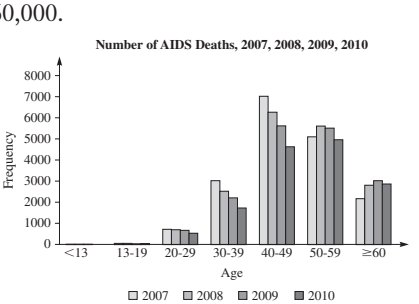


b.

Number of Children	Tally	Frequency
0	𝍸\|	6
1	\|\|	2
2	𝍸 𝍸	10
3	𝍸	5
4	𝍸 \|\|\|	8
5	\|\|\|\|	4
6	\|\|\|\|	4
7	\|	1
8	\|	1
9		0
10	\|	1
11		0
12		0
13		0
14		0
15	\|	1

16. a. The greatest profit was in the 4th quarter and the greatest loss was in the 3rd quarter. **b.** Profit; they made a profit in 3 of the 4 quarters and each of these profits was greater than the 3rd quarter loss. **c.** Yes, the income is over $60,000 in the 2nd, 3rd, and 4th quarters and for each of these quarters, the expenses are greater than $60,000.

17. a.



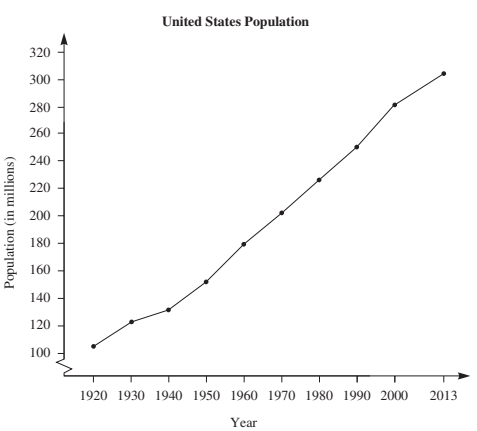
b. Answers vary. For example, the total number of deaths from AIDS decreased from 2007 to 2008. The deaths ages for ≥60 increased for two years and then decreased.

Mathematical Connections 10-2

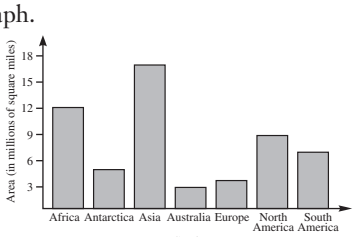
Reasoning

1. a. Answers vary. For example, in a pictograph, we can observe change over time and make comparisons between similar

situations. In a circle graph, it is hard to make these observations. **b.** Answers vary. For example, circle graphs are used when comparing parts to a whole. Circle graphs allow for visual comparisons of fractional parts. Bar graphs can't handle these comparisons as well. **c.** Answers vary. For example, in a stem and leaf plot, we can see the same shape and information as in a histogram but no data points are lost in the display as they may be in a histogram, especially when intervals are used. **2.** Answers vary. For example, if only percentages are shown on the circle graph, then the circle graph would not change. If data are shown, the numbers in the graph change but not the shape of the graph. **3.** Answers vary. For example, the percents are of a whole and the circle represents the complete whole, or 100%. Because of rounding, it may happen that the sum of the percents is close to 100% but not exactly 100% **4.** Answers vary. For example, the budget for 1 year is usually represented with a circle graph because a circle graph allows for a visual comparison of the relative sizes of fractional parts, which is what the graph of the federal budget is attempting to show. **5. a.** Answers vary. For example, a different graph may be more appropriate since we have continuous data changing over time. A line graph is given though students see this in the next section.



b. The data fall into distinct categories and are not continuous, so we use a bar graph.

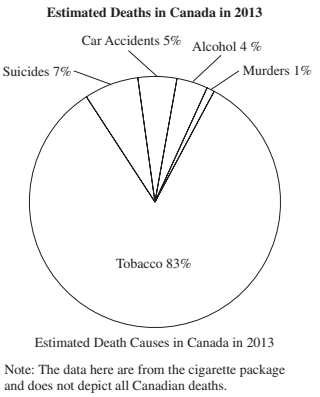


6. Answers vary. For example, the percentage share of large cars sold decreased 5.5% from 1990 to 2015. **7.** Answers vary. For example, one conclusion that could be drawn is that by year 3, students were carrying a fraction of the weight they were carrying two years prior. Side-by-side circle graphs or a triple-bar chart can be used to effectively display the data. Note that year 2 sophomores were year 1 freshman. So color-coding individual classes may be something to consider.

Open-Ended

8. Answers vary. **9.** Answers vary.

10. a.



b. Answers vary.

Cooperative Learning

11. Answers vary. A circle graph might be a logical choice with the total graph representing 100% of their working time and sectors representing teaching, service, and research. **12.** Answers vary. Expectations of word length on future pages depend on the first page chosen.

Connecting Mathematics to the Classroom

13. A stem and leaf plot does not have to be constructed with small numbers at the top. An advantage to constructing one in this way is that one can rotate the plot 90° counterclockwise to obtain a histogram. **14.** Aliene's graph is acceptable though it may be difficult for some to read and needs a title. **15.** A number like 3.14 can be represented as 3|14 where the unit of the leaf is in hundredths.

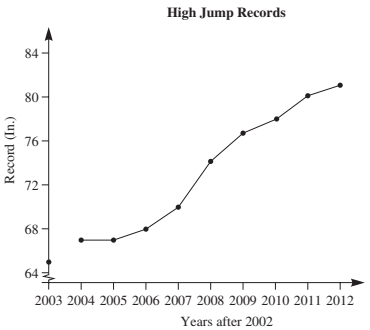
Review Problems

16. a. Fair **b.** Biased **c.** Fair **d.** Biased **e.** Biased **f.** Fair **17. (i)** With this kind of sampling, the owners should expect only those patrons who are strongly opinionated (especially those involved in a rare incident) to mail in the card. While this will certainly provide the owners with some feedback, overall it may not be enough to get at the underlying issue or issues. **(ii)** The owners should expect the same limited results as in the sampling method in part (i). There is little motivation for patrons to go through with the survey. **(iii)** This is kind of sampling is unbiased. Patrons conveniently complete the questionnaire while still in the restaurant guaranteeing honest, precise responses while it's still fresh on their minds. Patrons are also well-motivated. **(iv)** In general, this kind of sampling will be biased. Patrons may not be willing to be dissenting while face-to-face with the interviewer.

Assessment 10-3A

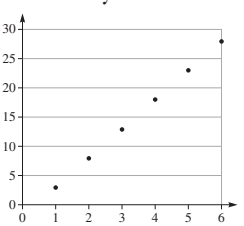
1. b. Approximately $14,000 **2. a.** About 1802 **c.** 1810–1820, because the slope of the line is steeper in this region

5.

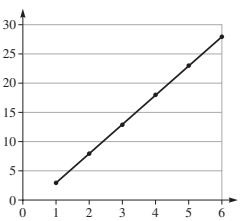


6. c. About 22 years old

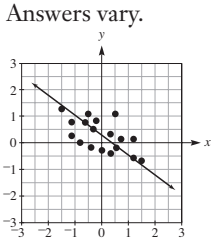
7. a.



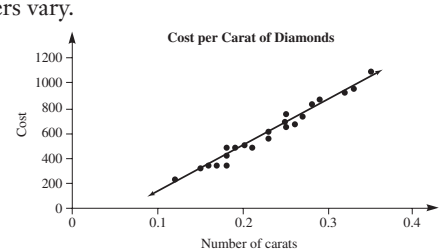
b.



8. a. i. Negative **ii.** Answers vary.



10. Answers vary.



Using ordered data and drawing a trend line, an estimate for 0.5 carat is about $1700, but it is unwise to estimate outside the given range of data.

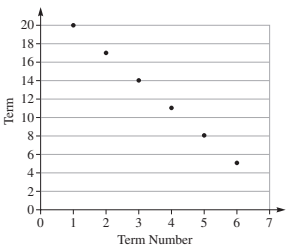
Assessment 10-3B

2. a. The number of workers per beneficiary will drop with only 1.9 workers/beneficiary in 2050. This shows that funding for social security will have problems in the future. **3. b.** Between 2000–01 and 2001–02 **c.** The trend is positive with the number of men and women participating in NCAA sports increasing. **4. a.** In both grades, the total number of spelling errors decreases with time. The 7th graders on the whole commit more errors than the 8th graders. **c.** Seventh graders may make anywhere between 10 and 15 errors. The 8th graders may commit anywhere between 0–5 errors. **5.** What happens to the temperature of ice when heated may not be intuitive at first as the temperature does not simply go up smoothly. You should have noticed two horizontal segments in your graph. This graph of temperature versus time is commonly known as the heating curve.

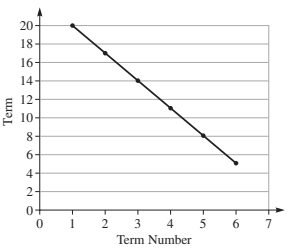
When the ice is heated, the temperature rises until it gets to 32°F. At this point, time is needed for the ice to change states (melt), so the temperature stays the same at 32°F. Then the temperature again rises until it reaches the boiling point, when time is needed for the liquid water to boil. The temperature stays the same at 212°F until the state changes to gas.

6. a. Negative association

7. a.

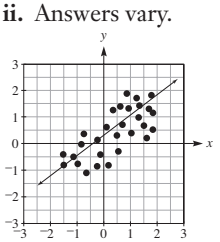


b.

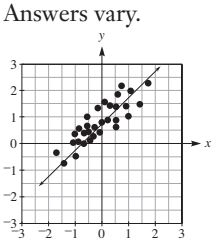


c. $y = {}^{-}3x + 23$

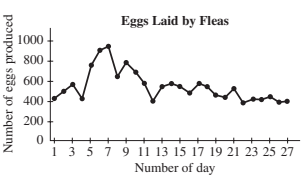
8. a. i. Positive
ii. Answers vary.



b. i. Positive
ii. Answers vary.



10. a.



b. Likely around 400 because the number of eggs for the last few days are close. **12.** There seems to be a positive linear association. **13.** Answers vary. If $0 \leq x \leq 100$ the data is almost constant.

Mathematical Connections 10-3

Reasoning

1. Answers vary. Two variables are associated if measured changes in one variable causes change in the other.

Cooperative Learning

2. Answers vary depending upon books chosen.
3. Answers vary.

Open-Ended

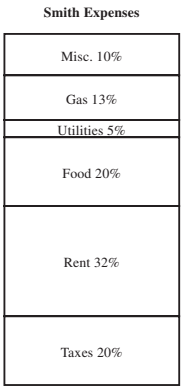
4. Answers vary. **5.** Answers vary.

Connecting Mathematics to the Classroom

6. Jacquie is incorrect. A scatterplot gives a good visual relationship between two variables. It shows if the relationship is close to a linear one, and if there is a positive, negative or no association. It also may show presence of outliers. **7.** Merle has a valid point, but we need to understand the graph construction.

Review Problems

8. The sum of the percents is greater than 100%, so the data have an error. **9.** If the miscellaneous category were only 10%, the stacked bar graph follows:



10. a. Mark McGwire home runs by season

0	399	
1		
2	249	
3	223499	
4	29	
5	28	
6	5	5 \| 2 represents 52 home runs
7	0	

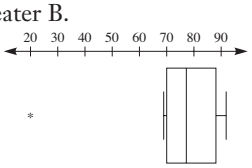
b. There are 3 possible low outliers and a gap from 9 to 22. Most scores are in the 30s. **11. a.** 21 **b.** $^{-}0.11$ **c.** 31.89 **d.** 64.85 **12.** There is a negative association. **13.** The data has a very slight positive, but strong linear association. The data are almost constant. **14. a.** Negative association **b.** Positive association **c.** Positive association

Assessment 10-4A

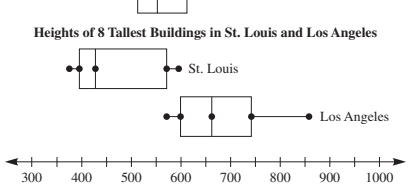
1. b. Mean = 13.94; median = 12.5; mode = 12.5
c. Mean = 0; median = 0.05; mode = (none)
2. a. Answers vary. An example is 1, 2, 3, 4, 5, 6, 6.
b. Answers vary. An example is 1, 1, 1, 5, 9, 9, 9.
3. a. iii. Bi-modal: 75 and 88. **6.** $\dfrac{100m + 50n}{m + n}$

11. a. Answers vary. The median of $32,000 reported with the interquartile range of $16,000 would be one appropriate set of choices. **b.** Based on the answer in (a), we know that at least 50% of the salaries are at least within $16,000 of $32,000. **15. a.** A—$25, B—$50 **d.** Answers vary. There is more variation at Theater B; also there are higher prices at Theater B.

16.



17. a.



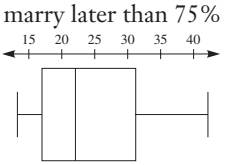
b. Answers vary. For example, in the buildings listed, Los Angeles has the tallest buildings, and about 75% of the tallest buildings in Los Angeles are taller than the tallest listed building in St. Louis. **19. g.** Approximately 9.04 **23.** $6.75 is one standard deviation below the mean; $8.25 is one standard deviation above the mean. 68% of the area under the normal curve lies within 1 standard deviation from the mean. Therefore, the probability of having a wage between $6.75 and $8.25 is 0.68. **24.** 90; two standard deviations above the mean is $79 + 2 \cdot 5.5 = 90$. Thus at least 90 will earn an A. **27.** Jill would have the higher standing since Jack would be at the 65th percentile.

Assessment 10-4B

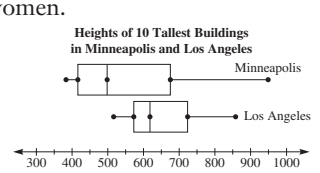
1. a. Mean = 84.4; median = 83; mode = 83
c. Mean = $^{-}0.05$; median = $^{-}2.15$; mode = (none)
d. Mean = 12.82; median = 11.99; mode = (none)
3. (b) Answers vary. For example, 70, 80, 80, 80, 80, 90.
5. d. The mean in (b) is equal to the mean in (a) plus 10 yr. The mean in (c) is equal to the mean in (a) plus 20 yr or the mean in (b) plus 10 yr. **6.** $\dfrac{h(100) + (n - h)50}{n}$

7. b. 0.60(TP) + 0.25(HW) + 0.15(FE) is the current scheme, where TP represents the term paper grade, HW represents homework average grade, and FE represents the final exam score for an individual. The percentages could be generalized with x% for the term paper, y% for the homework average, and $[100 - (x + y)]$% for the final exam grade as follows: x%(TP) + y%(HW) + $[100 - (x + y)]$%(FE).
9. a. balance beam—Olga (9.575); uneven bars—Lisa (9.85); floor—Lisa (9.925) **12. b.** The mean and median of an arithmetic sequence are the same. They are both equal to the average of the first and last terms of the sequence. **13. b.** No, the range might be affected by extreme values. **15. b.** About 21 **c.** The ranges are about the same with women slightly greater. **d.** Men in this sample first marry about 2.7 years later than women. The spread in ages of first marriage for men and women is about the same. In this sample about 75% of the men first marry later than 75% of the women.

16.



17. a.



c. Answers vary. For example, Minneapolis has more spread in its 10 tallest buildings and has taller buildings than Los Angeles. However, all of the tallest buildings in Los Angeles are higher than 50% of the tallest buildings in Minneapolis. All of the tallest buildings in Los Angeles are between 516 ft and 858 ft inclusive. The highest building in the two groups is located in Minneapolis and is 950 ft tall.

Mathematical Connections 10-4

Reasoning

1. Mode, answers vary. **2.** The answers must be the mode. One cannot compute the mean and median on non-numerical data without a numerical coding. **3.** The government probably uses the mean of data collected over a period of years.
4. a. Mean = 90, median = 90, mode = 90. **b.** Median

or mode **c.** Mean **5. a.** For example, 10, 30, 70, and 90 **b.** Choose four numbers whose mean is 50. **c.** The mean of the new numbers is 50. **6.** Answers vary. **a.** Suppose the scores were 100, 100, 50, 50. The mean is 75. **b.** Suppose the scores were 100, 100, 100, and 50. The mean is 87.5 and so her reasoning is not valid. **7.** No. To find the average speed we divide the distance traveled by the time it takes to drive it. The first part of the trip took $\frac{5}{30}$, or $\frac{1}{6}$, of an hour. The second part of the trip took $\frac{5}{50}$, or $\frac{1}{10}$, of an hour. Therefore, to find the average speed we compute $\frac{10}{\frac{1}{6}+\frac{1}{10}}$ to obtain 37.5 mph. **8.** Answers vary. The mode could be an extreme value in a data set (for example, the least value) and not represent the center at all. **9.** Answers vary. A set of data that contains wait time might be appropriate if the times were determined during peak periods of the events. In that case, the mean absolute deviation or standard deviation would be appropriate for this numerical data. **10. a.** Answers vary. For example, 3, 4, 5 and 12, 14, 16 have means 4 and 14, respectively. **b.** If $a_1, a_2, \ldots, a_n$ are the first n terms in an arithmetic sequence then using the formula for the sum, (developed in Chapter 8) we have $\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} = \left(\frac{a_1 + a_n}{2}\right)n \div n = \frac{a_1 + a_n}{2}$.

11. Justifications vary. **a.** The mean is multiplied by that constant. We illustrate the reason for three numbers x_1, x_2, x_3. If $\bar{x} = \frac{x_1 + x_2 + x_3}{3}$ and the constant is c then the new mean is $\frac{cx_1 + cx_2 + cx_3}{3} = c\left(\frac{x_1 + x_2 + x_3}{3}\right) = c\bar{x}$ **b.** The standard deviation is multiplied by the absolute value of that constant, if $s = \sqrt{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2}$, then the new standard deviation is

$$\sqrt{\frac{(cx_1 - c\bar{x})^2 + (cx_2 - c\bar{x})^2 + (cx_3 - c\bar{x})^2}{3}}$$
$$= \sqrt{c^2\left(\frac{(x_1 - \bar{x})^2 + (x_2 - x)^2 + (x_3 - \bar{x})^2}{3}\right)}$$
$$= |c|s.$$

12. The larger standard deviation implies that the data have more variability. The data are spread out more. **13. a.** Shoe sizes of students **b.** Exam scores; heights will be bimodal **c.** SAT scores

Open-Ended

14. Answers vary depending on student choice.

Cooperative Learning

15. Answers vary. **16.** Both the mean and median of each data set will increase by 5, decrease by 5, and increase five-fold, respectively.

Connecting Mathematics to the Classroom

17. The new mean is $\frac{9(10{,}000) + 20{,}000}{10} = 11{,}000$. Consequently, the new mean has increased by \$1000. The median and mode may change in special cases. **18.** The mean of two whole numbers is either a whole number or a whole number plus 0.5. If $\frac{a+b}{2} = 3.8$, then $a + b = 7.6$, which is not possible if a and b are whole numbers. **19.** Since the median is 90, at least half of the class had grades of 90 or more. Since Tom scored 80, he did not do better than half of the class. **20.** In a grouped frequency table, the precise values, of the raw data are not displayed, and hence it is impossible to conclude from the table which value occurs most often. Consequently, it is impossible to find the exact mode from the information given in a grouped frequency table. In this situation, the mode is usually given as a class interval. **21.** If the mean is less than the median, then one can be certain that there were more scores above the mean than below it. The low scores tend to be further from the mean than the high scores. **22.** No, it is not possible to have a standard deviation of $^-5$. By definition, the standard deviation is the principal square root of the variance. **23.** Mel did not really miss the cutoff by a single point. She would have had to increase her score on each of the 10 tests by a single point to reach an average of 90 or increase her total score for the 10 tests by 10 points to reach an average of 90.

Review Problems

24. Approximately, midsize—173°, large—25°, luxury—61°, small—101° **25. a.** Everest, approximately 8500 m
b. Aconcagua, Everest, McKinley
26. a. History Test Scores

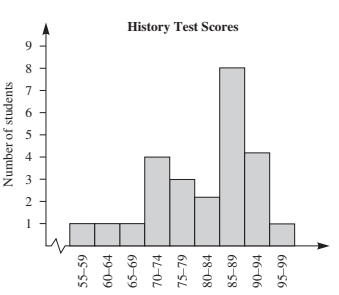
5	5
6	48
7	2334679
8	0255567889
9	00346

7 | 2 represents a score of 72

b. History Test Scores

Classes	Tally	Frequency
55–59	\|	1
60–64	\|	1
65–69	\|	1
70–74	\|\|\|\|	4
75–79	\|\|\|	3
80–84	\|\|	2
85–89	~~\|\|\|\|~~\|\|\|	8
90–94	\|\|\|\|	4
95–99	\|	1

c.



d. Approximately 115°

Assessment 10-5A

This entire assessment is subject to varying answers. Each of the following are representative possibilities. **1. a.** The ad should not be trusted without knowing more about the type of average in question, how it was obtained, and how likely it would be for someone to attain those average earnings. **b.** The results obtained from the questionnaires will suffer from a type of response bias where the respondents ignore the latter part of it. **c.** There is no mention of the size of the sample or the manner in which the sample was obtained. It is entirely possible that the sample contained only four dentists, who may have also been "cherry-picked" on top of it all. **d.** The survey will not be representative of all viewers at home. Only those viewers with mobile phones and passionate enough to send in their opinions will get counted. **e.** The question is biased as it contains persuasive words such as "much-needed." **2.** One possibility is that the temperature is always 82°F or close to it. **3.** It could very well be that most of the pickups sold in the last 10 yr were actually sold during the last 2 yr. In such a case, most of the pickups have been on the road for only 2 years, and therefore the given information would not substantiate, that the average life of a pickup is around 10 years. **4.** The horizontal axis does not have uniformly sized intervals and both the horizontal axis and the graph are not labeled. **5.** The three-dimensional drawing distorts the graph. The result of doubling the radius and the height of the can is to increase the volume by a factor of 8. **6.** One would need more information; for example, do men drive more than women? **7.** No, because the mean can be affected by a few high salaries. **8.** The first graph shows a steep increase in the price of the LCD TV over the six months, which the advertising agency may want to show for the benefit of the store. **9. a.** This bar graph would have perhaps 20 accidents as the baseline and some larger number (e.g., 50 as its maximum). Then 24 in 2011 would be 4 units above the baseline while 38 in 2015 would appear as 4.5 times or 350% taller than 24 of 2011, when in fact it is just 58% higher. **b.** This bar graph would have zero accidents as its baseline and some larger number (e.g., 100) as its maximum value. The effect would be to show values from 24 to 38 as lying a line with a small positive slope **10.** An example is 1, 1, 1, 1, 20, 30, 40, 50. **11.** A median might be misleading, depending on the number of data points given. Also the mean would not be sufficient. A report of the mean, median, and standard deviation would be the most helpful. **12.** The student is incorrect and, in addition, is misusing percentages. She mixes percentages of "effectiveness" with percentages of times taking the drug. **13.** Answers vary. **a.** You probably would not have a representative sample. This is a *convenience sample*. **b.** You probably would not. The students eating in the cafeteria could likely be a biased sample. **c.** Assign students numbers from 1 to n, where there are n students. Choose your sample using random-number selection methods. Make sure your sample is big enough. **14.** The data are indicators of passenger complaints and not an overall airline rating. Larger airlines may have more complaints. The numbers are not percentages. The number of complaints might depend on how many flights each airline has per day. **15.** Answers vary. You might conclude that the less you spend the better your students perform on the SAT exam in verbal and math scores.

Assessment 10-5B

1. a. Without knowing sales figures for the last 15 years, this claim cannot be substantiated. It may be that 95% of the manufacturer's cycles sold in the United States were sold in the last year. **b.** "Up to" is quite indefinite (6 mpg is "up to" 30 mpg). The conditions under which 30 mpg may have been realized are not stated (the car may have been rolling downhill). **c.** "Some" is indefinite (means 1 or more); moreover, here is no mention of a reference for the study. **f.** Most accidents occur in the home because a majority of time is spent in the home. This is an example of carrying an argument to a deceptive extreme. **2.** He is assuming that there are no deep holes in the river where he crosses.
3. We don't know who was surveyed, how many were surveyed, or how the question was phrased. Was the question biased? **4.** There is no title and no label on the horizontal axis. The vertical axis does not have uniform spacing.
5. The three-dimensional graph distorts the data and makes it appear that many more cars were sold in 2015 than in the other years when in fact the number of cars sold in 2013 and 2015 were approximately the same. There is no scale on the vertical axis. **6.** When the radius of a circle is doubled, the area is quadrupled, which is misleading since the number of majors has only doubled. **7.** She could have taken a different number of quizzes during the first part of the quarter than the second part.
8. There are no labels for vertical axis, so we cannot compare actual sales. Also, there is no scale on the vertical axis.
9. a. Answers vary. The survey depicts that teachers are generally happy with their mathematics textbooks. The articles (given without data) lead one to infer that national experts in mathematics are unhappy with textbooks that teachers use. **b.** Answers vary. The survey and articles do not agree. One tendency is to react that the teachers are in the classroom and are more knowledgeable about texts being used. **10.** A student would need to know the highest possible score that a person could make. Also the scores of other students would be important along with the teacher's expectations. **11.** Answers vary. **a.** At grades K–4, homework of less than 1 hr (60 min) could be justified because 75% of the teachers surveyed assign homework that takes this long on the average. Perhaps 0–30 minutes for homework would be more appropriate at this grade level because almost 50% of the surveyed teachers assign this amount. It is less easy to justify a specific amount for grades 5–8, but 71% of teachers surveyed use the range from 31 to 120 min of homework. **b.** A potential misuse at grades 5–8 might have one report that of the teachers who assign at least 91 min of homework, approximately half of those assign at least 2 hr of homework. It would be very difficult to justify at least 2 hr of homework at the K–4 level using these data. **12.** No, for example, in the data set 2, 7, 7, 7, 8, 11 the mean and the median are equal to 7. **13.** An example is 1, 2, 3, 50, 50, 50. **14.** The region may be like the Netherlands where dikes keep out the water where the land is below sea level. **15.** Comparable demographics of the sample and the population is a must. **16. a.** As a reporter, one might use the first headline because it grabs the attention—even based on the same data. **c.** If the headlines are based on the same data set as indicated, the second headline is the most informative.
17. a. $\frac{647}{649}$, or about 99.7% **b.** $\frac{622}{649}$, or about 95.8%

c. For the lung cancer group, the probability is $\frac{647}{1269}$, or about 51%. For the control group, it is $\frac{2}{29}$, or about 7%. **d.** The evidence is not conclusive. **18.** You could not automatically conclude correctly that the population of the West Coast has increased since 1790. However, based on the westward movement of the mean center of population, there would be strong suspicion that that was the case. **19.** Though the fish and dairy departments have the same monthly figures, the graphic for the fish department is longer and gives the impression that the fish department is the highest grossing department. **20.** By starting the vertical scale at 300, rather than 0, the artist of the first graphic can give the impression that figures are dramatically rising. In the second graphic, the same artist can make it seem that the same figures are increasing but slowly and steadily.

Mathematical Connections 10-5

Open Ended

1. Answers vary. In general, a graph can be made to look "steeper" or more "shallow" by changing the scale on the vertical axis. **2.** Answers vary.

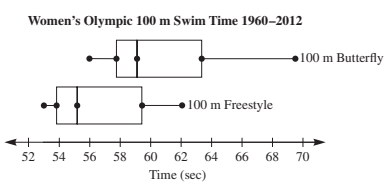
Review Problems

3. a. About 74.17 **b.** 75 **c.** 65 **d.** About 237.97 **e.** About 15.43 **f.** 13 **4.** About 27.74 **5.** About 76.7 **6.** Men's Olympic 100 m Run Times 1896–2008

9	6778899
10	0001123334568888
11	00
12	0

10 | 0 represents 10.0 sec.

7.

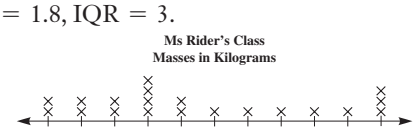


From examining the box plot, we can see that the times on the 100-m butterfly are much greater (relatively speaking) than the times on the 100-m freestyle.

Chapter 10 Review

1. i. If the average is reported as 2.41 children, the mean is being used. The mode would not have a decimal number, and the median would be either a whole number or one ending in .5. ii. If the average is 2.5, then either the mean or the median might have been used. **3. a.** Mean = 30, median = 30, mode = 10. **b.** Mean = 5, median = 5, modes = 3, 5, 6. **c.** Mean = 150, median = 150, mode = 100,200 (bi-modal) **d.** Mean = 13.01, median = 12.15, mode (none) **4. a.** Range = 50, variance ≈ 371.4, standard deviation ≈ 19.3, mean absolute deviation ≈ 17.14, IQR = 40. **b.** Range = 8, variance = 5.2, standard deviation = 2.28, mean absolute deviation = 1.8, IQR = 3.

5. a.



b. Ms Rider's Class Masses in Kilograms

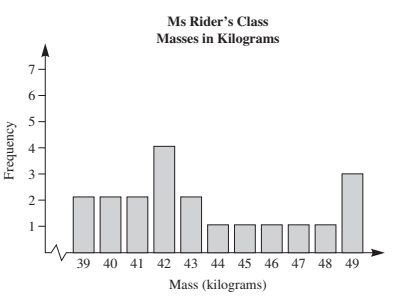
3	99
4	001122223345678999

4|0 represents 40 kg

c. Ms Rider's Class Masses in Kilograms

Mass	Tally	Frequency
39	\|\|	2
40	\|\|	2
41	\|\|	2
42	\|\|\|\|	4
43	\|\|	2
44	\|	1
45	\|	1
46	\|	1
47	\|	1
48	\|	1
49	\|\|\|	$\frac{3}{20}$

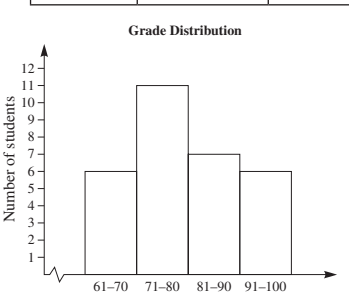
d.



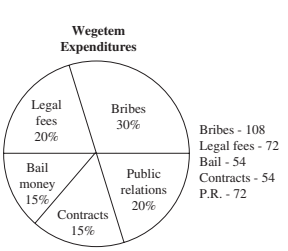
6. a. Test Grades

Classes	Tally	Frequency
61–70	𝍸\|	6
71–80	𝍸𝍸\|	11
81–90	𝍸\|\|	7
91–100	𝍸\|	6

b.

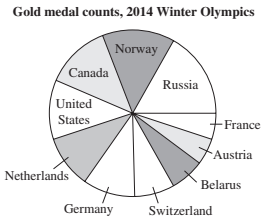


7.



8. The graph is missing a title and axis labels. The bars are not of equal width.

10.



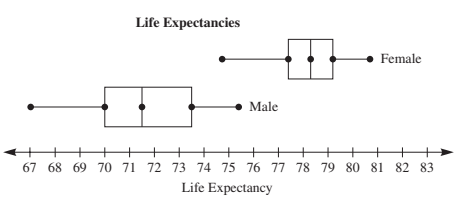
11. a. Life Expectancies of Males and Females

Females		Males
	67	1446
	68	28
	69	156
	70	0049
	71	02235578
	72	01145
	73	1689
7	74	123599
9310	75	14
86	76	
88532	77	
9999854332211	78	
99655443210	79	
42	80	

7 | 74 | represents 74.7 years old

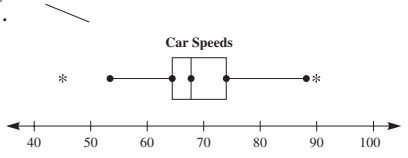
| 67 | 1 represents 67.1 years old

b.



12. Larry was correct because his average was $3.2\overline{6}$, while Marc's was $2.7\overline{3}$.

14. c.



e. About 50% of people checked drove between 65 and 75 mph. **16. a.** Collette is more consistent. There is less deviation from the mean in her scoring. **c.** Collette scored more than might have been expected. **19.** The bars would be approximately the same height in a vertical bar graph. **20. b.** It is nearly impossible for this to happen. **c.** Since the company claims a person can lose up to 6 lbs, any average loss from 0 to 6 is legitimate. **21.** The bar graph is more appropriate. Line graphs are used to show change over time. The line graph indicates there are values between the colors, which is not true. **22.** No reportable single number would be meaningful for such a pool. **23.** Answers vary. A reasonable way to make the claim is to pull boxes at random after they are packed and weigh the contents. If more than 98% of the boxes' contents have weights within 1 g of 138 g, the claim is reasonable. The choices of 98% and 1 g are arbitrary but reasonable choices to test this claim. **24. a.** Answers vary. Perhaps to be with those of comparable age a typical senior might choose the first. **25.** The areas representing the two categories appear roughly the same size. **26.** Answers vary. **a.** One way would be to leave the television on, even if no one is watching. **b.** They show very popular shows during "ratings sweeps" periods. **27.** Answers vary. For example, graphs may show area or volume instead of relative size; another is to select a horizontal baseline that will support the point being made. **32.** The March 31 Goal is portrayed almost 3 times greater than the number as of March 27, which it is not. **33.** The first person has the calculator and picks a large random number. The first person then records the number, adds his salary to it, and passes the calculator to the second person. That person adds his salary and passes it to the next person. This continues for all five people. When the calculator is returned to the first person, he subtracts the random number that was used and divides by 5 to obtain the mean salary.

Answers to Now Try This

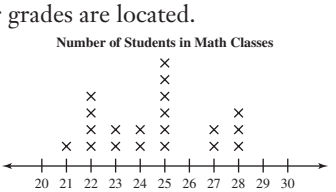
2. Ages of Presidents at Death

4	69
5	36778
6	003344567778
7	0112347889
8	01358
9	0033

4 | 9 represents 49 years old

3. It appears that the fifth-period class did better. We can see that there are more scores grouped toward the bottom, which is where the higher grades are located.

4. a. 2.



Part (a) of question 4 continued (answer to Quick Check 3).

Age	Tally	Frequency
20–29	\|\|\|\|\|\|\|	8
30–39	\|\|\|\|	4
40–49	\|\|\|	3
50–59	\|\|	2
60–69	\|\|	2
70–79	\|	1

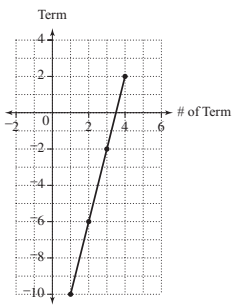


b. Yes, the interval 0–1 should be represented on the graph or a break shown as in part (a) above. **5.** Answers vary with the number of terms.

a.

# of Term	Value of Term
1	$^-10$
2	$^-6$
3	$^-2$
4	2

b.



c. $y = 4x - 14$, where x is a natural number **6. a.** $y = 212.5x$ **b.** 425 cal **c.** $y = 112.5x$ **d.** 450 cal **e.** About 1190 cal **f.** From 0 to 1 hr it will go from 0 to about 112, then from 1 hr to 4 hr it will go from 112 to about 750. **7. a.** A histogram is more appropriate to compare numbers of data that are grouped into numerical intervals. With the line graph there appears to be a frequency for *Times per month* such as 4.5, etc. **b.** Connecting the dots is visually okay, but is mathematically meaningless. **c. i.** A circle graph is appropriate for showing the division of a whole into parts. **ii.** A line graph is appropriate for showing how data values change over time. **8. a.** Answers vary. For example, all the puppies could weigh 7 lb. Then $\frac{6 \cdot 7}{6} = 7$ lb. Another possibility is 4, 6, 6, 7, 8, 9, 9. The mean for these weights is also 7. **b.** If all the scores were equal, then the mean is equal to the high score and also the low score. Otherwise, this is not possible. **c.** 5 **9.** Answers vary. For example, one set of data is 92, 94, 94, 94, 96, in which the mean, median, and mode are all 94. **10. a.** The average of 2.58 could only be a mean. To be a mode it would have to be a whole number. To be a median it would have to be a whole number or a whole number plus $\frac{1}{2}$.

b. The average of 2.5 could be a mean or a median. It could not be a mode since the mode for the number of children must be a whole number. **11.** The completed table follows:

	Overall Mean	Mean for Females	Mean for Males	Percent Females
Ramirez	218	230	205	52
Jonsey	221	224	199	88

The percentage of males in Ms. Jonsey's class is 100% − 88%, or 12%. Also the mean for Ms. Jonsey's class is determined as follows:

$$(88\%)(224) + 12\%(\text{Mean for males}) = 221$$

Solving, we find that the mean for males in Ms. Jonsey's class is 199.

For the entry for Mr. Ramirez's class, we know that the percentages of males and females must sum to 100%. If F is the percentage of females, then the percentage of males in that class must be $1.00 - F$. Further, we know that the overall mean is determined as follows:

$$(F)(230) + (1.00 - F)(205) = 218$$

Solving for F, we find that $F = 52\%$.

Finally, we observe that the means for both females and males were higher in Mr. Ramirez's class, but the overall mean is higher in Ms. Jonsey's class. **13.** Because there is no base for the percentages given, arguments are difficult to make. Assuming the base is the set of all parents, consider the following: **a.** The argument cannot be made because we do not know even if the other 82% answered this part of the survey. **b.** The argument cannot be made. The statement may be the result of considering test/learning to be two equal parts that make 18%. Hence, half of 18% is 9%. **c.** The argument cannot be made. **d.** This argument cannot be made. Possibly it could be made for 11%. **e.** The argument cannot be made. The indicated 9% does not tell us about the other 91% of parents.

Answer to the Preliminary Problem

If $g_1, g_2, \ldots, g_n$ are the ages of the n guests and c is Calvin's age, then the mean a of the $n + 1$ people's ages (n guests and Calvin) is: $a = \dfrac{g_1 + g_2 + , \ldots , + g_n + c}{n + 1}$. Hence $c = a(n + 1) - (g_1 + g_2 + , \ldots , + g_n)$. Since the guests' ages were announced, Calvin's age can be computed by multiplying the mean by the number of people at the party and subtracting the sum of the ages of all the guests.

CHAPTER 11

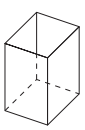
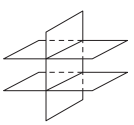
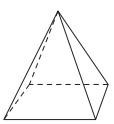
Assessment 11-1A

2.  **3. c.** False; they intersect in a line, the empty set, or a plane (if the two planes are the same plane). **e.** False, see $\overrightarrow{FA} \cup \overrightarrow{GH}$ in exercise 1. **5. a.** $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$, $\overleftrightarrow{AD}$ and $\overleftrightarrow{GH}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{DH}$, $\overleftrightarrow{AE}$ and $\overleftrightarrow{BD}$, etc. **e.** $\overline{FH}$ is the line of interaction of the two planes. $\overline{DH}$ is perpendicular to plane FGH because it is perpendicular to $\overline{HG}$ and $\overline{EH}$ in that plane. Thus, $\overline{DH}$ is perpendicular to any line through H in the plane FGH which implies $\overline{DH} \perp \overline{FH}$. If a line ℓ through H is constructed in the plane FGH and perpendicular to $\overline{FH}$, $\overline{DH}$ will be perpendicular to ℓ. Then one of the dihedral angles created by the two planes BDH and FHG measures 90°. Thus, the planes are perpendicular because a dihedral angle measures 90°. **6. d.** Answers vary. $\overleftrightarrow{AC}$ and $\overleftrightarrow{BE}$ or $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$. **8.** Answers vary. **a.** Edges of a room; vertical and horizontal parts of a window frame; some intersecting crossroads. **b.** Some branches in a tree; clock hands at 7:30; angle on a yield sign. **c.** The top of a coat hanger; clock hands at 7:15. **12. a.** $m(\angle AOB) = 22.5°; m(\angle COD) = 67.5°$ **b.** $m(\angle BOC) = 31.25°; m(\angle AOB) = 58.75°$ **c.** Assume that if the position of $\overline{OE}$ is changed, the other rays will be adjusted so that all x's are congruent and all y's are congruent. $3x$ and $3y$ will have different values, and thus, x and y will have different values. $3x + 3y$, though, will remain $180° \Rightarrow 3(x + y) = 180° \Rightarrow x + y = 60° \Rightarrow m(\angle BOC) = 60°$.

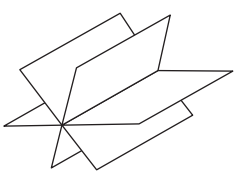
14. a.

Number of Lines \ Number of Intersection Points	0	1	2	3	4	5
2			Not possible	Not possible	Not possible	Not possible
3					Not possible	Not possible
4			Not possible			
5			Not possible	Not possible		
6			Not possible	Not possible	Not possible	

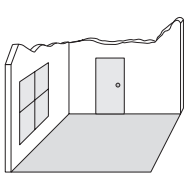
15. Perspectives vary; e.g.,

a.  **b.**  **c.** 

16. Answers vary; e.g., **a.** A real-world example would be a paddle wheel:



b. A real-world example would be the intersection of floor and two adjacent walls of a house:

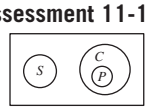


c. A real-world example would be a field irrigated with a central pivot:



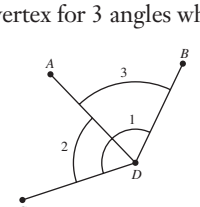
Field Irrigates with a Central Pivot

Assessment 11-1B

2. 

3. a. False; they could be skew.

b. False; the line can pass through and intersect at 1 point.
c. False; the end points differ. **e.** False; the definition requires skew lines cannot be contained in the same plane. **4.** Each point can serve as a vertex for 3 angles whose measures are less than 180°. For example:

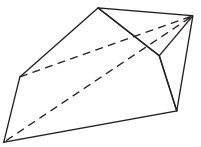
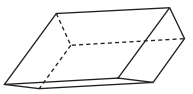


There are 4 possible vertices times 3 angles for each vertex, and thus, 12 distinct angles. **5. b.** Answers vary, for example *ABF* and *BCD*, or *ADH* and *CDH*. **c.** Answers vary, $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$ **6. e.** None of the lines in the drawing are parallel.
f. Answers vary, for example, the plane determined by *B*, *E*, and *D*.

12. a. $m(\angle DOC) = 25\frac{5^\circ}{7} \approx 25.71^\circ$

$m(\angle BOA) = 34\frac{2^\circ}{7} \approx 34.29^\circ$

13. c. ${}_nC_3 = \dfrac{n(n-1)(n-2)}{6}$ **15.** Answers vary.

a.  **b.** 

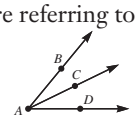
Mathematical Connections 11-1

Reasoning

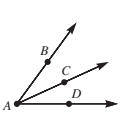
1. a. 1, a line segment has 2 endpoints and 2 points determine a line. **b.** An infinite number. There are an infinite number of points on a line and any point not on the segment can be the endpoint of a ray containing the segment. **c.** An infinite number of planes can pass through a given line. **d.** No, if collinear they can be contained in an infinite number of planes. If noncollinear, exactly one plane contains them. **2.** A line has an infinite number of points. Pick two distinct points *A* and *B* on the line and the given point *C* not on the line. Now we have 3 noncollinear points that determine a unique plane. The same is true for any two distinct points that are chosen on the line. **3. a.** Parallel lines and skew lines are similar in that they do not intersect. However, a line is parallel to itself and therefore a line does intersect itself.
b. Parallel lines and skew lines are different in that parallel lines are coplanar, and skew lines are not coplanar. **4.** No, by convention the middle letter is the vertex and points *A* and *C* are points on the rays other than *B*. **5.** Points *A*, *B*, and *C* are collinear. If two distinct planes intersect, the intersection is a line. If points *A*, *B*, and *C* are in common, then they must lie on the line of intersection. **6. a.** Answers vary. **b.** The fire is located near the intersection of the two bearing lines. **c.** Answers vary.
7. No; if all four points are collinear, then one line is determined; if only three are collinear, then four lines are determined; if no three are collinear, then six lines are determined. **8.** Yes; through *P* in plane α, draw a line *k* perpendicular to $\overleftrightarrow{AB}$. By definition, since *n* is perpendicular to α it is perpendicular to every line in α through *P*. Thus, *n* and *k* are perpendicular, which implies that planes α and β are perpendicular. **9.** Yes, 2:27:16

Open-Ended

10. Answers vary. **a.** If the points on the three rays were labeled *B*, *C*, and *D* as shown, then you can't name any angle *A* because it is not clear whether you are referring to $\angle BAC$, $\angle CAD$, or $\angle BAD$.



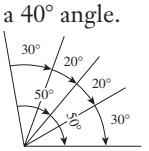
b. $\angle BAD$ and $\angle BAC$ in the given figure.



11. Answers vary. **12.** Answers vary.

Cooperative Learning

13. a. Answers vary. An angle of 20° can be drawn by tracing a 50° angle and a 30° angle, as shown in the following figure. Another 20° angle adjacent to the first 20° angle can be drawn in a similar way, thus, creating a 40° angle.



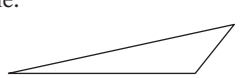
b. Yes; answers vary. **c.** All positive multiples of 10°; once a 10° angle is determined, using the 40° angle from (a) and the given 30° angle, the other multiples can be determined.

Connecting Mathematics to the Classroom

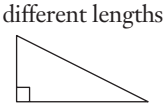
14. The segments have to lie on parallel lines. If the lines containing the segments were drawn, there would be an intersection point, and the lines would not be parallel. **15.** Two points determine a line, so only two points are used to name the line. **16.** Two segments are equal if they represent the same set of points. Both $\overline{AB}$ and $\overline{BA}$ represent the same set of points, so they are equal. **17.** From the Denseness Property, we know that for any two real numbers, there is a real number between them. Because points of a line and the set of real numbers can be put in a one-to-one correspondence, we can always find a point between any two points. Since this is true for any two points, we see that there are an infinite number of points in a segment. **18.** It is a matter of definition. Two distinct lines are parallel if they do not intersect and are in the same plane. Lines that do not intersect in exactly one points, and do not lie in a single plane are skew lines. **19.** The measure of an angle has nothing to do with the fact that rays cannot be measured. The measure of an angle is a measure of the angle's opening. In degrees, the measure is based on constructing a circle with its center at the vertex of the angle and dividing the circle into 360 congruent parts. The number of parts in the arc that the angle intercepts is the measure of the given angle in degrees. The number of parts in the intercepted arc is the same regardless of the size of the circle (or protractor). **20.** Extending the rays does not change the angle measure or the angle's opening. The angle measure is a measure of the opening, not the length of the rays. **21.** The student is incorrect. While the degree is the basic unit of angle measure, it can be further subdivided. This in itself would prove that the student is incorrect. However, many geometry books also consider a Protractor Postulate, which puts all the rays in a half-plane emanating from a point in a one-to-one correspondence with the real numbers greater than 0° and less than 180°. This would allow infinitely many rays emanating from one point. **22.** Since half-lines have no endpoints, the two half-lines have no common endpoint.

Assessment 11-2A

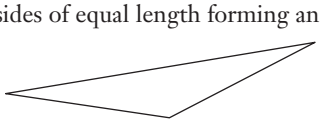
6. a. Possible; three sides of different lengths with one obtuse (i.e., greater than 90°) angle.



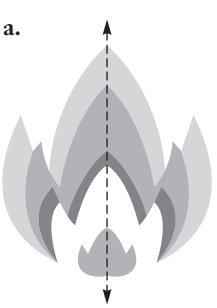
b. Possible; three sides of different lengths with a right angle.



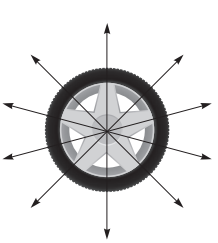
c. Impossible. An equilateral triangle has three 60° angles.
d. Impossible. An equilateral triangle has three 60° angles.
e. Possible; two sides of equal length forming an obtuse angle.



8. a.

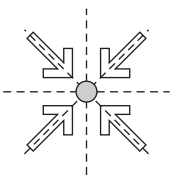


b.

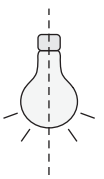


10. (i) Line symmetry: *a* and *b*; (ii) Turn symmetry: *a*; (iii) Point symmetry: *a*

11. a.



b.

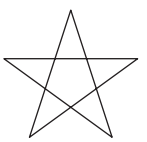


12. Answers vary.

a.



b.



c. The letter **N** has 180° rotational symmetry and no line of symmetry. **14. a.** 0, vertical and horizontal line symmetry; 3, horizontal line symmetry. Depending on how it is written the digit "1" might have line and point symmetry. **b.** 0 and 1 depending on how they are drawn. **15. a.** 180° **b.** 40°, 80°, 120°, 160°, 200°, 240°, 280°, 320°

Assessment 11-2B

4. Both (b) and (d) are convex. In both (a) and (c) points can be found so that line segments connecting these points lie partially outside the figures.

a.

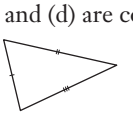


c.



Thus, (a) and (c) are concave. This is not possible for (b) and (d), so (b) and (d) are convex.

6. a.



b.

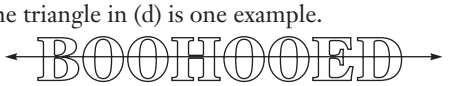


c. Not possible. A scalene triangle has no congruent sides and an equiangular triangle is an equilateral triangle, which has three congruent sides. **d.** Equilateral triangles are equiangular triangles.

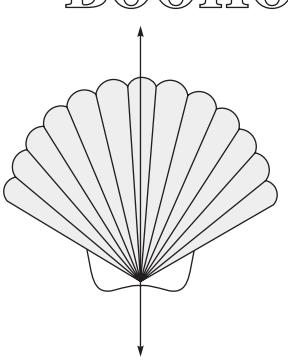


e. The triangle in (d) is one example.

8. a.



b.



11. a. 4 lines of symmetry; 90°, 180°, and 270° turn symmetry; and point symmetry **b.** Point symmetry and 180° turn symmetry (It also has 90° rotation symmetry if you ignore color.) **c.** Point symmetry; 90°, 180°, and 270° turn symmetry **d.** 4 lines of symmetry; 90°, 180°, and 270° turn symmetry; and point symmetry
14. a. 8, vertical and horizontal line symmetry **b.** 8, point symmetry

Mathematical Connections 11-2

Reasoning

1. a. Answers vary. An $8\frac{1}{2}$ by 11-in. piece of rectangular paper requires only one fold. Fold the bottom left corner up until the bottom side meets the right side. The bottom side now measures a side on the right congruent to the bottom side. Cut off the top portion of the rectangle that is not covered by the fold. **b.** Folding the square along one diagonal and then along the other creates congruent angles. Each is $\frac{360°}{4} = 90°$. The paper folding implies that the diagonals bisect each other. Unfolding and then folding the square so that opposite sides fall onto each other shows that the diagonals are congruent. **2.** No, a concave polygon "bends in"; that is, at least one angle is greater than 180° and some are less than 180° or the figure could not close. Therefore, the figure is not equiangular and so it is not regular. **3.** According to the definition in this text, isosceles triangles have at least two congruent sides. Equilateral triangles have three congruent sides; therefore, they have at least two congruent sides and are isosceles.
4. a. True **b.** False, consider the figure shown in part (d).
c. False, if the diagonal lies outside then the polygon is concave.
d. False, consider the following figure.



Open-ended

5.

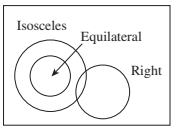
Polygon	Triangle	Quadrilateral	Pentagon
Concave	not possible		
Convex			
Regular			
Equilateral only	not possible		
Equiangular only	not possible		

6. Answers vary.

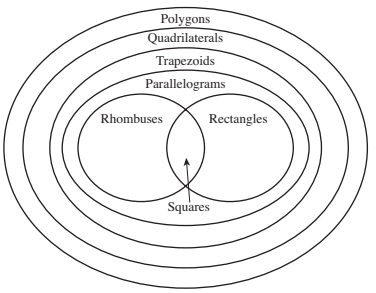
Cooperative Learning

7. Answers vary.

8.



9.



10. a. No, the figure will always have sides the same length because the length of each side is the length of the cut. **b.** No, the length of each side will be the length of the cut. **c.** Yes, cut off the corner so that it is in the shape of an isosceles triangle.
11. a. (i) All but the green triangle and the yellow hexagon (ii) The orange square and the two non-square rhombuses, blue and light orange (iii) The square and the two non-square rhombuses (iv) The green triangle, orange square, and yellow hexagon. **b.** The interior angle of the triangle is $\frac{360°}{6} = 60°$; the interior angle of the square is $\frac{360°}{4} = 90°$; the interior angle of the hexagon is $\frac{360°}{3} = 120°$; if the two trapezoids are placed together to form a hexagon, then the obtuse angles are 120° and the acute angles are 60°; if the 3 blue rhombuses are placed together, then the interior angle is $\frac{360°}{3} = 120°$; if the other rhombus is placed with the blue rhombus and square as shown, then the interior obtuse angle measures $360° - 90° - 120° = 150°$, so the acute angle is 30°. **c.** Answers vary.

Connecting Mathematics to the Classroom

12. A regular polygon is a polygon in which all the angles are congruent and all the sides are congruent. In general neither condition implies the other, and hence neither is sufficient to describe a regular polygon. For example, a rhombus that is not a square has all sides congruent, but all its angles are not congruent. A rectangle that is not a square has all its angles congruent, but not all its sides congruent. Neither a rhombus nor a rectangle is a regular polygon unless it is a square. **13.** Since an angle is a set of points determined by two rays with the same endpoint, to say that two angles are equal implies that the two sets of points determining the angles are equal. The only way this can happen is if the two angles are actually the same angle. To say that two angles are congruent is to say that the angles have the same size or measure. **14.** To be regular, all sides must be congruent but also all angles must be congruent. All angles are not congruent unless the rhombus is a square. **15.** This statement is correct. A rectangle is a parallelogram with four right angles. A square is

a parallelogram with four right angles so a square is a rectangle. A square must be a parallelogram with right angles and all sides congruent. Not all rectangles satisfy the condition of all sides congruent, so only some rectangles are squares. **16.** It is agreed that when naming polygons the order of the letters represent consecutive vertices (either clockwise or counterclockwise). A and C are not consecutive vertices. **17.** This is not correct, as not only are four congruent sides needed for a square but also four right angles. The student could be shown a picture of a non-square rhombus and then asked whether this satisfies his definition and whether it is a square. **18.** They are both correct. A rhombus with four congruent angles is a rhombus with four congruent sides and four congruent angles; it is a square. A rectangle with sides the same length has four right angles; it satisfies the definition of a square.

GeoGebra Activities

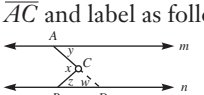
19. See online appendix.

Review Problems

20. a. 45 **b.** $\frac{n(n-1)}{2}$ **21.** $\varnothing$, 1 point, 2 points, ray, and possibly a line of the angle is a straightangle and the line is collinear with it. **22. a.** False; a ray has one endpoint. **b.** True **c.** False; by definition skew lines cannot be contained in single plane. **d.** False; $\overrightarrow{MN}$ has endpoint M and extends in the direction of point N; $\overrightarrow{NM}$ has endpoint N and extends in the direction of point M. **e.** True **f.** False; their intersection is a line.

Assessment 11-3A

8. Answers vary. **a.** Yes. A pair of corresponding angles are 50° each (note that the supplementary angles formed by lines n and ℓ are 130° and 50°). **b.** Yes. A pair of corresponding angles are 70° each (note that the supplementary angles formed by lines n and ℓ are 110° and 70°). **c.** Yes. A pair of alternate interior angles are 40° each. **d.** Yes. A pair of corresponding angles are 90° each. **e.** Yes. The marked angles are alternate exterior angles. **f.** Yes. Extend $\overline{AC}$ and label as follows:



Because x is the measure of an exterior angle in ΔBCD, $x = z + w$. But $x = y + z$ so $z + w = z + y$ implying $w = y$. Because $\angle w$ and $\angle y$ are alternate interior angles, $m \| n$.
19. b. The measure of the exterior angle equals the sum of the measures of the nonadjacent interior angles. Since $\angle 1$ and $\angle C$ are supplementary, $m(\angle 1) + m(\angle C) = 180°$. Since the sum of the measures of the interior angles of a triangle is 180°, $m(\angle A) + m(\angle B) + m(\angle C) = 180°$. Therefore, $m(\angle 1) = m(\angle A) + m(\angle B)$.
20. $m(\angle x) = \frac{180°}{11} \approx 13.8°$

$m(\angle y) = \frac{360°}{13} \approx 27.7°$

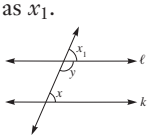
$m(\angle z) = \frac{720°}{13} \approx 55.4°$

$m(\angle w) = \frac{1080°}{13} \approx 83.1°$

21. $m(\angle 1) = 120°$, $m(\angle 2) = 130°$, $m(\angle 3) = 50°$, $m(\angle 4) = 120°$, $m(\angle 5) = 60°$, $m(\angle 6) = 50°$, $m(\angle 7) = 130°$, $m(\angle 8) = 60°$, $m(\angle 9) = 70°$, $m(\angle 10) = 50°$, $m(\angle 11) = 50°$, $m(\angle 12) = 60°$

Assessment 11-3B

1. Answers vary. Vertical angles are created by intersecting lines and are the pair of angles whose sides are two pairs of opposite rays. Vertical dihedral angles are created by two distinct intersecting planes and are the pair of dihedral angles whose sides are two pairs of opposite half-planes. **3. a.** Complementary, 70°; supplementary, 160°; vertical, 20° **b.** Complementary, $(80 - x)°$; supplementary, $(170 - x)°$; vertical, $(10 + x)°$ **8. a.** Label the angle supplementary to y as x_1.



Then $m(\angle x_1) + m(\angle y) = 180°$ because the angles are supplementary, and since $m(\angle x) + m(\angle y) = 180°$ it follows that $m(\angle x_1) = m(\angle x)$. Since $\angle x_1$ and $\angle x$ are corresponding angles, then $k \| \ell$. **b.** Extend the 30° ray from line l to line k; a triangle is formed. The interior angles are 20°, 30° (alternate interior angles), and 130° (total of 180° interior angles). $\angle x$ and the 130° angle are supplementary, thus, $m(\angle x) = 50°$.
c. Extend the 135° ray from line k to line l; a triangle is formed. The interior angles are 45° (supplement of 135°), 40° (supplement of 140°), and 95° (total of 180° interior angles). The 95° angle and $\angle x$ are supplements, thus, $m(\angle x) = 85°$.
14. a. $m(\angle x) = 36°$; $m(\angle y) = 108°$ **b.** $m(\angle x) = 120°$
c. $m(\angle x) = 93°\,20'$; $m(\angle y) = 104°$; $m(\angle z) = 114°\,40'$; $m(\angle u) = 125°\,20'$; $m(\angle v) = 136°$; $m(\angle w) = 146°40'$
19. Because the sum of the interior angles is 360°, $\angle A$ and $\angle D$ are supplementary. Also, $\angle D$ and $\angle C$ are supplementary. If we extend $\overline{AD}$ to form an alternate interior angle to $\angle C$, we see that these angles are congruent, and thus, $\overline{BC}$ is parallel to $\overline{AD}$. (Similarly, $\overline{AB}$ is parallel to $\overline{DC}$.) Thus, $ABCD$ is a parallelogram.
21. $m(\angle 1) = 70°$, $m(\angle 2) = 110°$, $m(\angle 3) = 70°$, $m(\angle 4) = 70°$, $m(\angle 5) = 70°$
22. $m(\angle 1) = m(\angle 2)$ (vertical angles)
$m(\angle 2) = m(\angle 3)$ (given)
$m(\angle 3) = m(\angle 4)$ (vertical angles)
Therefore, $m(\angle 1) = m(\angle 2) = m(\angle 3) = m(\angle 4)$ and so $m(\angle 1) = m(\angle 4)$.

Mathematical Connections 11-3

Reasoning

1. a. No. Two or more obtuse angles will produce a sum of more than 180°. **b.** Yes. For example, each angle may have measure 60°. **c.** No. It may have an obtuse or right angle as well. **d.** No. If two angles are supplementary the sum of their measures is 180°. The third angle will produce a sum greater that 180°. **2.** No. It is not possible. The sum of all the measures of the angles in ΔABC must equal 180°, but $m(\angle ABC) + m(\angle ACB) + m(\angle BAC) = 90° + 90° + m(\angle BAC) > 180°$. Hence, the situation shown in the diagram is not possible. **3.** Answers vary. No. The measures of the exterior angles corresponding to the given angles are 120°, 100°, 90°, and 45°; the sum of these is 355°. The measures

of the exterior angles of a convex polygon is 360°, so this cannot happen. **4. a.** Yes **b.** No. The sum of the measures of the interior angles of a polygon must be a multiple of 180°, and 1400° is not a multiple of 180°. **c.** Yes **5.** $m(\angle 4) = m(\angle 1) + m(\angle 2)$. We know that $m(\angle 1) + m(\angle 2) + m(\angle 3) = 180°$. We know also $m(\angle 3) + m(\angle 4) = 180°$. Therefore $m(\angle 1) + m(\angle 2) + m(\angle 3) = m(\angle 3) + m(\angle 4)$. Subtracting $m(\angle 3)$ from both sides, we have $m(\angle 4) = m(\angle 1) + m(\angle 2)$. **6. a.** Five triangles will be constructed in which the sum of the angles of each triangle is 180°. The sum of the measures of the angles of all the triangles equals $5 \cdot 180°$, from which we subtract 360° (the sum of all the measures of the angles of the triangles with vertex P). Thus, $5 \cdot 180° - 360° = 540°$. **b.** Here, n triangles are constructed, so we have $n \cdot 180°$ from which we subtract 360° (the sum of all the measures of the angles of the triangles with vertex P). Thus, we obtain $n \cdot 180° - 360°$ or $(n - 2) \cdot 180°$. **7. a.** Fold the isosceles triangle so that the congruent sides match up. The two angles will fall on top of each other. **b.** Fold the triangle so that the two congruent angles fall on top of each other. The two sides will match up. **c.** Choose a vertex and fold the triangle at that vertex so that the other two vertices fall on top of each other. Two angles of the triangle should fall on top of each other. Repeat at a second vertex. **d.** Fold the trapezoid so that its congruent sides fall on top of each other and check that the base angles fall on top of each other as well. **8. a.** Divide the quadrilateral into two triangles: ABC and ACD. The sum of the measures of the interior angles of each triangle is 180°, so the sum of the measures of the interior angles of the quadrilateral is $2 \cdot 180°$, or 360°. **b.** The formula is also true for concave polygons. **c.** Any concave pentagon can be divided into three triangles, and a concave hexagon can be divided into four triangles. In general, any concave n-gon can be divided into $(n - 2)$ triangles, so that the sum of the measures of the angles of the triangles is the sum of the measures of the interior angles of the n-gon. **9.** No. Regular hexagons fit because the measure of each vertex angle is 120°, three hexagons fit to form 360°, and the plane can be filled. For a regular pentagon, the measure of each vertex angle is 108°, so pentagons cannot be placed together to form 360° (360° is not divisible by 108°) and the plane cannot be filled. **10.** Yes. Because the sum of the measures of the marked angles is 360° and opposite angles are congruent we get $2m(\angle 2) + 2m(\angle 1) = 360°$ or $m(\angle 1) + m(\angle 2) = 180°$. From problem 8(a) of Assessment 11-3B, we $a \| b$.

11. $m(\angle DCB) + m(\angle ABC) = 180° - [(m(\angle 3) + m(\angle 4)]$
$+ 180° - [(m(2) + m(1)]$
$= 360° - (2m(\angle 3) + 2m(\angle 2))$
$= 360° - 2(m(\angle 3) + m(\angle 2))$
$= 360° - 2 \cdot 90° = 180°$

Thus, by the result in problem 8(a) of Assessment 11-3B, the rays $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are parallel.

Open-Ended

12. Answers vary, for example, 127°, 129°, 131°, 133°, 137°, 139°, 141°, 143°; their sum must equal 1080°. **13.** Answers vary; you should get a multiple of 360°. **14.** For an n-gon, the sum of the interior angles should be $(n - 2)180°$.

Cooperative Learning

15. a. If $m(\angle A) = \alpha$ and $m(\angle B) = \beta$, then $m(\angle D) = 180° - \left(\frac{\alpha}{2} + \frac{\beta}{2}\right)$.

b. If $m(\angle C)$ is always the same, no matter what the measures of $\angle A$ and $\angle B$ are, then $m(\angle D)$ is always the same.

c. Answers vary. The following is a possible solution. From (a): $m(\angle D) = 180° - \left(\frac{\alpha + \beta}{2}\right) = \frac{360° - (\alpha + \beta)}{2}$. Because $\alpha + \beta = 180° - m(\angle C)$, we get the following:

$$m(\angle D) = \frac{360° - (180° - m(\angle C))}{2}$$
$$= \frac{360° - 180° + m(\angle C)}{2} = 90° + \frac{1}{2}m(\angle C)$$

Because the answer for $m(\angle D)$ depends only on $m(\angle C)$, the conjecture is justified. **16. a.** Because congruent supplementary angles are formed. **b.** The theorem concerning the sum of the measures of the angles of a triangle. **c.** When $\angle A$ and $\angle C$ are folded, congruent supplementary angles are formed and, hence, each is a right angle. When B is folded along $\overline{BB'}$, the crease $\overline{DE}$ formed is perpendicular to $\overline{BB'}$ (again because congruent supplementary angles are formed); consequently, $\overline{DE} \| \overline{GF}$. Thus, $\angle D$ and $\angle E$ are also right angles.

d. $GF = GB' + B'F = \frac{1}{2}AB' + \frac{1}{2}B'C =$
$\frac{1}{2}(AB' + B'C) = \frac{1}{2}AC$; hence, $GF = \frac{1}{2}AC$.

Connecting Mathematics to the Classroom

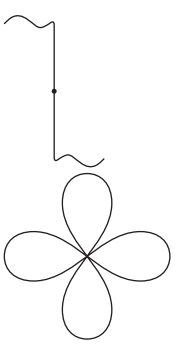
17. The only possible figures are acute triangles. To see why, let an n-gon ($n \geq 4$) have the property that each angle is acute. Then all the exterior angles would be obtuse. A quadrilateral would have more than $90° \cdot 4$, or 360°, as the sum of the measures of the exterior angles and for $n > 4$ the polygon would have more than $90° \cdot 4$ or 360° as the sum of the measures of its exterior angles, which is not possible. **18.** Such a polygon would have to have four right exterior angles which already add up to 360°. If such a decagon existed, it would have the sum of measures of exterior angles greater than 360°, which is impossible. Therefore, such a decagon does not exist. **19.** No, the sum of the measures of the three angles would be greater than 180°.

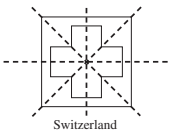
Review problems

20. a. All angles must be right angles and all diagonals are the same length. **b.** All sides are the same length and all angles are right angles. **c.** Impossible. All squares are parallelograms. **21. a.** (i) Two sets of parallel sides: A, B, C, D, E, F, G. (ii) One set of parallel sides: I, J. (iii) No parallel sides: H. **b.** (i) Four right angles: D, F, G. (ii) Two right angles: I. (iii) No right angles: A, B, C, E, H, J. **c.** (i) Four congruent sides: B, C, F, G. (ii) Two pairs of congruent sides: A, D, E, H. (iii) One pair of congruent sides: J. (iv) No congruent sides: I. **22.** The symbol reads the same forward, backwards, and upside down, reflected about a vertical line, and turned 180°.

23. Answers vary. **a.** The figure has 180° turn symmetry but no line symmetry.

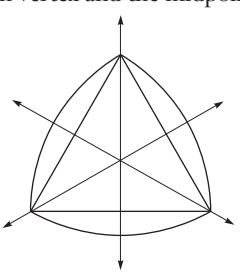
b. The figure has point symmetry, 90° turn symmetry, and vertical and horizontal line symmetry.



24. a. (i) Four line symmetries, the diagonals and horizontal and vertical lines through the center. (ii) No lines of symmetry.
b. (i) 

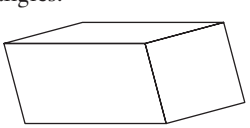
(ii) No lines of symmetry.

25. a. Turn symmetries of 90°, 180° (also called point symmetry), and 270° about the center of the square. **b.** Point symmetry about the center of the square. **26. a.** None.
b. 3, one through each vertex and the midpoint of its opposite side.

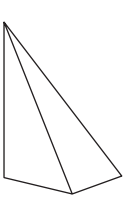


Assessment 11-4A

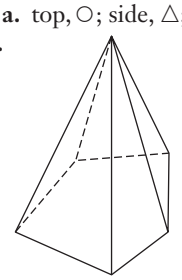
1. a. Answers vary .Quadrilateral pyramid; possibly square pyramid **b.** Quadrilateral prism; possibly trapezoidal or right trapezoidal prism **c.** Pentagonal pyramid **2. a.** (i) 6 (ii) 14 glued together **b.** (i) 24 (ii) 92 glued together **7. b.** False. No pyramid is a prism; a pyramid has one base, and a prism two bases. **d.** False. They lie in parallel planes. **e.** False. A prism must have two bases that lie in parallel planes. **8.** Answers vary, but all are possible; for example, **a.** An oblique square prism. Some faces are not bounded by rectangles.



b. An oblique square pyramid. The base is a square but a vertical line segment from the vertex does not intersect the base at its center.

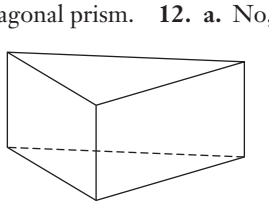


9. a. top, ○; side, △; front, △ **b.** top, ○; side, ○; front, ○
10.

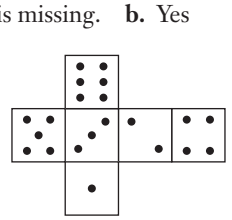


11. a. Right regular hexagonal pyramid. **b.** Right square pyramid. **c.** Cube. **d.** Right square prism **e.** Right regular hexagonal prism. **12. a.** No, one face is missing. **b.** Yes

13.



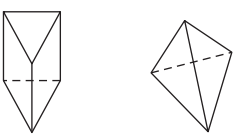
14. b.



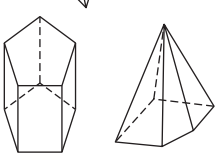
17. a. Triangle **b.** Rectangle **18. a.** There will be three pairs of parallel faces, determined by any three pairs of opposite sides of the hexagonal base.

Assessment 11-4B

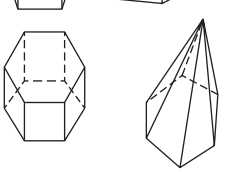
1. a. Triangular prism **b.** Quadrilateral pyramid **c.** Cylinder
2. a. (i) 7 (ii) 16 glued together **b.** (i) 28 (ii) 106 glued together
3. c. Regions formed by $\triangle BCE$, $\triangle ADF$, quadrilaterals *DCEF*, *ABEF*, *ABCD* **7. a.** False. The base can be any simple closed curve. **c.** False. They are parallelograms; if they were rectangles it would be a right prism. **8.** Answers vary; for example:
a. Prism and pyramid (respectively) with a triangle as base:



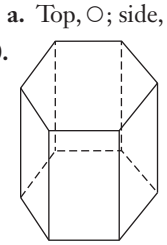
b. Prism and pyramid (respectively) with a pentagon as base:



c. Prism and pyramid (respectively) with a regular hexagon as base:

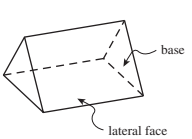


9. a. Top, ○; side, □; front, □ **b.** Top, ▭; side, ▯; front, ▯▯
10.

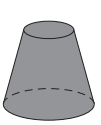


12. a. No; the faces don't match. **b.** No; there are two bottoms and no top.

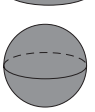
13. Answers vary.



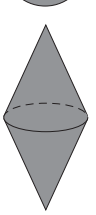
14. b. No; 17 is not a multiple of 3.
d. No; 23 is not a multiple of 2.
15. a. Truncated right circular cone:



b. Sphere:



c. Two right circular cones joined at their bases:



d. Inverted right circular cone:



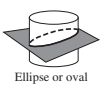
17. a. The intersection is a pentagon:



b. The intersection is a circle:



c. The intersection is an ellipse (or oval):



19. a. (i) A pyramid has $n + 1$ faces. (ii) A prism has $n + 2$ faces.
b. (i) A pyramid has $n + 1$ vertices. (ii) A prism has $2n$ vertices.
c. (i) A pyramid has $2n$ edges. (ii) A prism has $3n$ edges.
d. (i) Pyramids: $(n + 1) + (n + 1) - 2n = 2$ (ii) Prisms: $(n + 2) + 2n - 3n = 2$

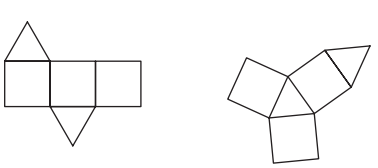
Mathematical Connections 11-4

Reasoning

1. Three. Each pair of parallel faces could be considered bases. **2.** Answers vary. For example, a cone is approximated by a many-sided pyramid and a cylinder is approximated by a many-sided prism. **3.** Both could be drawings of a quadrilateral pyramid. In (a), we are directly above the pyramid, and in (b), we are directly below the pyramid. **4.** An octagonal pyramid. To obtain 9 vertices you would need 8 on the base plus the apex. **5.** Answers vary. If the lateral faces are rectangles we have a right prism. If some lateral faces are parallelograms that are not rectangles, we have an oblique prism. **6.** 3. If there are 2 faces, then the shape is v-shaped and not closed, so we do not have a closed surface. If we have 3 faces then an intersection is possible to form a closed surface. **7.** Answers vary. **8. a.** A rectangle, $\overline{AD}$ is perpendicular to $\overline{AB}$ and to $\overline{AE}$ and therefore perpendicular to the plane *AEHB*. Since $\overline{MN} \| \overline{AD}$, $\overline{MN}$ is perpendicular to plane *AEHB*. Thus, $\overline{MN}$ is perpendicular to every line through *M* in that plane. So $\overline{MN} \perp \overline{MH}$ and $\angle NMH$ is a right angle, Similarly, the other angles of quadrilateral *MHGN* are right angles **b.** No

Open-Ended

9. Answers vary.



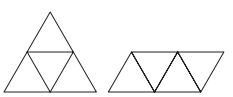
10. Answers vary. A sphere has infinitely many planes of symmetry.

Cooperative Learning

11. Answers vary. **12.** Answers vary. **13.** Answers vary. **14. a.** Parallelogram, rectangle, square, scalene triangle, isosceles triangle that is not equilateral, equilateral triangle, pentagon, hexagon, a square and its interior, trapezoid that is not a parallelogram and a parallelogram that is neither a rhombus nor a rectangle, rhombus that is not a square, and line segment if the 3 points are on an edge. **b.** Triangle, quadrilateral, line segment, triangle and its interior.

Connecting Mathematics to the Classroom

15. In a sphere the surface does not consist of polygonal regions, so it cannot be a polyhedron. **16.** Trevor is only counting the vertices, faces, and edges in a two-dimensional drawing. He should be given a real cube and asked to count the vertices, faces, and edges. He should also be shown how to draw a three-dimensional cube by drawing each base and showing the vertices, faces, and edges using dashed segments for the hidden edges. **17.** Bonnie is assuming the surface the prism is lying on is the base. This is not the case as the bases must be two congruent polygonal faces that lie in parallel planes. **18.** No, it is not a regular polyhedron as it is made up of two different regular polygons, namely pentagons and hexagons. This assumes that the soccer ball is not a sphere in which case the answer is no as the surface would not consist of polygonal regions. **19.** There are two different nets shown below (assuming that the faces are indistinguishable).



Review Problems

20. 161° 46′ 18″ **21.** Answers vary. **a.** $\overleftrightarrow{AB}$ **b.** $\overleftrightarrow{AC}$ **c.** $\overleftrightarrow{BC}$ **d.** $\overline{CD}$ **22.** No. Adjacent angles share a common side, and vertical angles do not. **23.** Answers vary. For example, the letter A. **24.** 35 **25.** 162° **26.** 110°

Chapter 11 Review

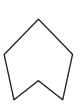
4. a. The measure of one of the dihedral angles formed by planes α and γ is $m(\angle APS) = 90°$, as given. The measure of one of the dihedral angles formed by planes β and γ is $m(\angle BPQ) = 90°$, since it is given that $\overline{PQ} \perp \overline{AB}$. **b.** $\overline{AB}$ is perpendicular to $\overline{PQ}$ and $\overline{PS}$ in plane γ.
5. Answers vary.
a. A simple closed curve:



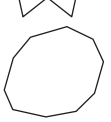
b. A closed curve that is not simple:



c. A concave hexagon:

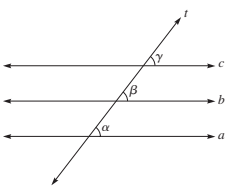


d. A convex decagon:

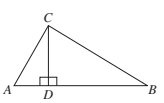


e. Draw a regular pentagon *ABCDE*. Draw line *CE* and reflect $\overline{CD}$ and $\overline{DE}$ about the line *CD*. The will create an equilateral pentagon that is not equiangular. An alternate way is to place an equilateral triangle on top of a square and erase the common edge. **f.** Draw a rectangle that is not a square. **6. a.** No. The sum of the measures of two obtuse angles is greater than 180°, which is the sum of the measures of all the angles of any triangle. **b.** No. The sum of the measures of the four angles in a parallelogram must be 360°. If all the angles are acute, the sum would be less than 360°. **16. b.** The measure of all the angles with vertices at *P* is 360°. If there are *n* triangles, then there is a total of $n \cdot 180°$ minus the 360° surrounding *P*, or $n \cdot 180° - 360° = (n - 2) \cdot 180°$. **c.** Answers vary.

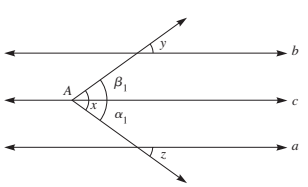
One way would be to connect B with E and connect A with F. There will then be two quadrilateral and one triangle, given an angle sum of $2 \cdot 360° + 180° = 900°$. **17. a.** Alternate interior angles are congruent by construction. $\overleftrightarrow{AB} \parallel \overleftrightarrow{BC}$ by Theorem 11-2. **b.** Corresponding angles are congruent. **c.** $m(\angle B) + m(\angle C) + m(\angle BAC) = m(\angle BAD) + m(\angle DAE) + m(\angle BAC) = 180°$ **18.** If $a = b$, or $b = c$, or $a = c$, the proof is automatic. If none of these are true, we proceed as follows. Given three parallel lines with line l intersecting them:



$a \| b$ implies $\alpha = \beta$. Next $b \| c$ implies $\beta = \gamma$. Thus, $\alpha = \gamma$ which implies $a \| c$ by congruent corresponding angles. **19.** Correct. If it is assumed that a rectangle has four right angles for a total angle measurement of 360° and that a diagonal divides it into two congruent triangles, then the sum of the measures in each right triangle is 180°.

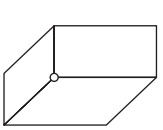


Thus, the sum of the measures of the angles in ΔACD and ΔBCD is $2 \cdot 180° = 360°$. In this sum all the angles of the original triangle are included as well as the two right angles at D, so the sum of the measures of the angles in the original triangle is $360° - 2 \cdot 90° = 180°$. **20.** One approach would be to draw a line c through A parallel to a and then prove it to be parallel to b. If $c \| a$, then $\alpha_1 = z$ (corresponding angles). Since $x = \alpha_1 + \beta_1$ and $x = \alpha_1 + \beta_1 = y + z$ then $\beta_1 = \gamma$ which implies $c \| b$. By transitivity, $a \| b$.



21. a. $\angle 3$ and $\angle 4$ are supplements of congruent angles. **b.** $m(\angle 3) = 180° - m(\angle 1)$ and $m(\angle 4) = 180° - m(\angle 2)$. If $m(\angle 1) < m(\angle 2)$ then $^-m(\angle 1) > {}^-m(\angle 2)$. $180° - m(\angle 1) > 180° - m(\angle 2) \Rightarrow m(\angle 3) > m(\angle 4)$.

23. Answers vary. Three planes that intersect in a point:



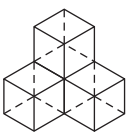
24. a. A triangle requires three points. The order of the points is not important; thus, the number of possible triangles is the number of combination of 10 points taken 3 at a time, or $_{10}C_3 = \dfrac{10!}{(10-3)!3!} = 120$

b. $_nC_3 = \dfrac{n!}{(n-3)!3!} = \dfrac{n(n-1)(n-2)}{6}$.

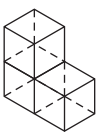
25. a. 18 **b.** 25 does not divide 360°; such a regular polygon does not exist. **c.** Does not exist; the sum is always 360°. **d.** Does not exist; the equation $\dfrac{n(n-3)}{2} = 4860$ has no natural number solution.

27. a. Line and turn **b.** Line, turn, and point **c.** Line.

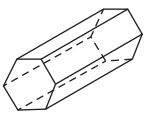
28. a.



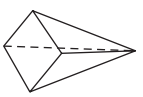
b.



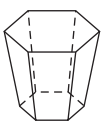
c.



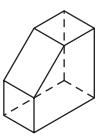
d.



e.

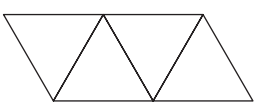


f.

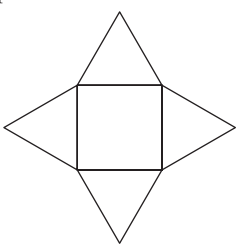


31. Answers vary. There are several possibilities.

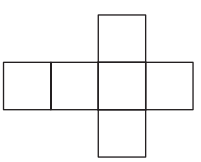
a.



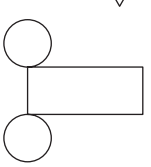
b.



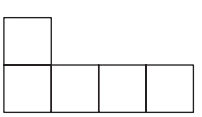
c.



d.



32. Answers vary. **a.** One possibility is:

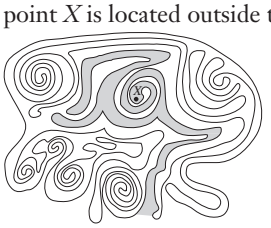


b. Eight different nets. That is, the bottom square may be placed in any of eight positions. **33. d.** $5 + 5 - 8 = 2$ **e.** $12 + 8 - 18 = 2$ **f.** $10 + 7 - 15 = 2$

Answers to Now Try This

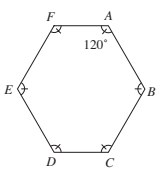
1. a. They do not have to be. Consider any point D such that $AD + DB > AB$. Point A, B, and D cannot be collinear. **b.** No, the points must be collinear. **c. i.** $\overleftrightarrow{AB}$ **ii.** $\overline{AB}$ **iii.** $\overleftrightarrow{AB}$ **iv.** $\overrightarrow{AB}$ **v.** $\overleftrightarrow{AB}$ **vi.** $\overrightarrow{AB}$ **3. a.** No. If skew lines had a point in common, there would be a single plane that contains the lines which would contradict the definition of skew lines. **b.** Skew lines cannot be parallel. By definition, parallel lines are in the same plane. Skew lines are not. **c.** Lines n and o are parallel; each of lines m, n, o, and ℓ is parallel to itself; lines n, m, and l are skew as are lines o and m. **4.** $8.42° = 8° + 0.42(60) = 8°25.2' = 8°25' + (0.2)(60) = 8°25'12''$ **5. a.** By folding a crease onto itself, we have created two angles that are both supplementary and congruent. They must have measure 90°, making the creases perpendicular. **b.** Construction **c.** One obtains a 45° angle by bisecting a 90° angle. The 135° angle is 90° + 45°. The 22°30′ angle is obtained by bisecting the 45° angle. **d.** The least angle measures 90°. **e.** The least angle measures 45°. **f.** The least angle measures $\dfrac{360°}{2^n}$. **6. a.** It is possible for a line intersecting a plane to be perpendicular to only one line in the plane. Drawings will vary. **b.** It is not possible for a line intersecting a plane to be

perpendicular to two distinct lines lying in the plane and not be perpendicular to the plane. **c.** Yes. If a line intersects a plane in point P and is perpendicular to two lines in the plane through P, then it is perpendicular to every line in the plane through P. **d.** ℓ is perpendicular to α. **8.** One approach to this problem is to start shading the area surrounding point X. If we stay between the lines, we should be able to decide whether the shaded area is inside or outside the curve. The shaded part of the following figure indicates that point X is located outside the curve.

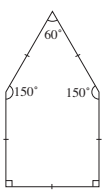


9. Answers vary.

a.



b.



12. a. The sum of the measures of the interior and exterior angles is $n \cdot 180°$. Therefore, the sum of the measures of the interior angles is $n \cdot 180° - 360°$ or $(n-2)180°$. **b.** $\dfrac{(n-2)\cdot 180°}{n}$ or $180° - \dfrac{360°}{n}$

13. Euler's formula is $V + F - E = 2; 8 + 6 - 12 = 2;$ $10 + 7 - 15 = 2; 12 + 8 - 18 = 2$

Answer to the Preliminary Problem

Walking around the star can solve the problem, but a simpler approach is to notice that by extending the sides of a regular pentagon we can obtain the vertices of the star. The sum of the measures of the interior angles of the pentagon is $3 \cdot 180°$, so the measure of each interior angle of the pentagon is $\dfrac{3 \cdot 180°}{5} = 108°$. We can choose any side of the pentagon and the isosceles triangle formed adjacent to the side to find that the base angles of the isosceles triangles each measure 72° (supplementary angles). From this we deduce that the measure of each shaded angle is $180° - 2 \cdot 72° = 36°$.

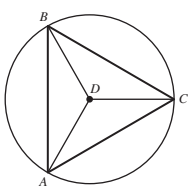
CHAPTER 12

Assessment 12-1A

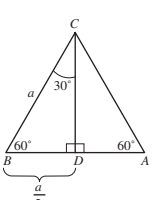
1. a. True; CPCTC. **b.** False; these are not corresponding angles. **c.** False, these are not corresponding parts. **3.** $\triangle ABD \cong \triangle CDB; \angle A \cong \angle C, \angle D \cong \angle B, \angle B \cong \angle D,$ $\overline{AB} \cong \overline{CD}, \overline{AD} \cong \overline{CB}, \overline{BD} \cong \overline{DB}$ **6.** We know that $\overline{AC} \cong \overline{EC}$ and $\overline{BC} \cong \overline{DC}$. $\angle ACB \cong \angle ECD$ because of vertical angles. Therefore, $\triangle ACB \cong \triangle ECD$ by SAS. Hence, $\angle DEC \cong \angle BAC$ by CPCTC and $\overline{AB} \parallel \overline{DE}$ since they have congruent alternate interior angles. **8.** We know $\overline{AB} \cong \overline{BC}$ and $\overline{AD} \cong \overline{CD}$. Also, $\overline{BD} \cong \overline{BD}$. Therefore, $\triangle ABD \cong \triangle CBD$ by SSS and $\angle ABD \cong \angle CBD$ by CPCTC. **9.** It is not possible if the three points are collinear. If the three points are not collinear, then the point is the center of the circumcircle around the triangle determined by the three points. **11. b.** The side of greatest length is opposite the angle of greatest measure. **12.** $\triangle ADB \cong \triangle CBD$ by HL because $\overline{AB} \cong \overline{CD}$, $\overline{DB} \cong \overline{BD}$, and $\angle ADB$ and $\angle CBD$ are right angles. Thus, $\angle ABD \cong \angle CDB$. Because these angles are alternate interior angles created by lines $\overleftrightarrow{AB}$, $\overleftrightarrow{DC}$, and the transversal $\overline{BD}$, it follows that $\overline{AB} \parallel \overline{DC}$. Also $\overline{BC} \parallel \overline{AD}$ because $\angle ADB \cong \angle CBD$ (alternate interior angles). Therefore, $ABCD$ is a parallelogram since opposite sides are parallel. **13.** Answers vary. **a.** Construct a triangle congruent to $\triangle CAB$ or an isosceles triangle congruent to any isosceles triangle with vertex angle at A. **b.** First construct a segment $\overline{AB}$ of length 4 cm and then find point C as the intersection of circles centered at A and B with respective radii of 2 cm and 3 cm. **f.** One way is to use a protractor to draw the 75° angle and then mark the 6 cm and 7 cm sides on the sides of the angle. **g.** Use a protractor to draw the 40° angle and mark its vertex as A. Then on one of the sides mark point B such that $AB = 7$ cm. Then let C be the intersection of the other side of the angle with a circle centered at B with radius 6 cm. Two triangles are possible. **h.** Either proceed as in part (g) or construct a triangle with two sides of 6 cm each and included angle of 100°. **i.** Construct a right angle and then mark on its sides legs of 4 cm and 8 cm. Join the endpoints of these segments for the remaining side. **j.** (b) Yes, by SSS; (c) Yes, by SSS; (d) There is no triangle because $4 + 5 < 10$; (e) Yes, by SSS; (f) Yes, by SAS; (g) The triangle is not unique; (h) Yes, by SAS; (i) Yes, by SAS. **14.** Construct a right isosceles triangle and the perpendicular bisector of the hypotenuse. **15. a.** Construct an angle congruent to $\angle A$. Construct an angle congruent to $\angle B$ with the same vertex as the first construction and sharing a side with that triangle and the other side in its exterior. **b.** Construct an angle with vertex B congruent to $\angle A$ such that one side is a side of $\angle B$ and the other side is in the exterior of $\angle B$. **16.** No, if a triangle is equilateral all the angles measure 60°. **17. a.** Construction. The triangles $\triangle ABO$, $\triangle BCO$, $\triangle CDO$, and $\angle DAO$ are congruent (SAS) isosceles right triangles. Therefore, the congruent angles in each triangle measure 45°. Consequently all the angles in $ABCD$ are 90°, and all the sides are congruent. **b.** Because the arcs BE and EC each measures 45°, the chords $\overline{\overset{\frown}{BE}}$ and $\overline{\overset{\frown}{EC}}$ are congruent. Bisecting each of the right central angles, we get the vertices of the regular octagon. **c.** Answers vary. Construct F such that $CF = BE$. Then extend $\overline{OE}$ to intersect the circle at G. Next extend $\overline{OF}$ to intersect the circle at H. Now connect E, C, F, D, G, A, H, B to obtain a regular octagon. **18.** Answers vary. Fold the square in half so that one pair of opposite sides fall on each other. Unfold and repeat for the other pair of opposite sides. Where the two folds intersect is the required center. The segment connecting the center to one of the vertices of the square is the radius. **19.** Point C can be found as the intersection of the line and the perpendicular bisector of $\overline{AB}$. **20.** The center of the circle is the intersection of the perpendicular bisectors of any two adjacent sides of the square. The radius of the circle is the distance from the center to any of the vertices of the square. Alternatively the center can be found as the intersection of the diagonals. **22. a.** Construct point C as the intersection of two circles with centers at A and B and radius AB. **b.** Any of the angles of the triangle in part (a) will measure 60°. **c.** Any of the exterior angles in part (a) will measure 120°. **23. b.** From part (a), $\angle A \cong \angle B$ by CPCTC.

Assessment 12-1B

1. a. True, CPCTC **b.** True, CPCTC **c.** False, these are not corresponding parts **3.** $\triangle ABC \cong \triangle DEC$; $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle BCA \cong \angle ECD$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EC}$, $\overline{CA} \cong \overline{CD}$ **5.** Draw $\overline{BD}$. We know that $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{BC}$, and $\overline{BD} \cong \overline{BD}$. Therefore, $\triangle ABD \cong \triangle CDB$ by SSS and $\angle ABD \cong \angle CDB$ by CPCTC. Hence, $\overline{AB} \parallel \overline{CD}$ because we have congruent alternate interior angles. $\angle ADB \cong \angle CBD$ by CPCTC and therefore, $\overline{AD} \parallel \overline{BC}$ because we have congruent alternate interior angles. **10.** No; for example, consider the 4 vertices of a parallelogram that is not a rectangle. The perpendicular bisectors of the 4 sides are not concurrent. **11.** No. Answers vary. If four such points A, B, C, D existed in the plane, any three would have to be the vertices of an equilateral triangle. Then D would be equidistant from the vertices of the equilateral $\triangle ABC$ and hence the center of the circle that circumscribes $\triangle ABC$. Because the three triangles with the common vertex D are congruent, $m(\angle ADB) = 120°$ and hence $AD \neq AB$ and the four points cannot be equidistant from each other.



12. b. The side opposite the greatest angle is longer than the side opposite the smallest angle. **c.** The hypotenuse is longer than any of the legs. This is so because in a right triangle one angle measures 90° and the other angles are acute. The hypotenuse is the side opposite the greatest angle. **13. a.** The lengths must be the same because they are corresponding parts of congruent triangles. **b.** You cannot make the same claim if the plane containing the stakes is not perpendicular to the antenna. **14.** Construct an equilateral triangle and then the perpendicular bisector of a side. **16. a.** By HL. Alternatively by ASA because the angles in each triangle measure 30°, 60°, 90°. **b.** Let a be the length of each of the sides of the equilateral triangle $\triangle ABC$. Because $BD = DA$, $BD = \frac{a}{2}$. Thus in $\triangle BCD$, $\overline{BD}$ is opposite the 30° angle and $BD = \frac{a}{2} = \frac{BC}{2}$.



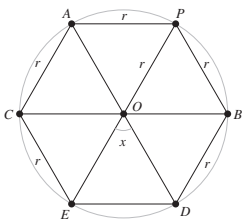
17. a. The 12-gon is regualr because it has congruent interior angles. The arcs BG and GC each measure 30°. Because $12 \cdot 30° = 360°$, 12 congruent 30° arcs can be constructed. The corresponding 12 chords are congruent to form a 12-gon. **b.** Connect O with each of the vertices. Each angle with vertex O is $\frac{360°}{6}$, or 60°. Because each triangle like $\triangle BCO$ is isosceles, its base angles are congruent and hence each measures $\frac{(180° - 60°)}{2}$, or 60°. Thus each triangle is equilateral and therefore each side of the hexagon is congruent to the radius. **18. a.** F is the midpoint of both diagonals ($\triangle ABC \cong \triangle ADC$ by SSS and $\angle CAB \cong \angle CAD$ by *CPCTC* and $\triangle BAF \cong \triangle DAF$ by SAS implies $\overline{BF} \cong \overline{DF}$, so F is a midpoint; a similar argument will show $\overline{AF} \cong \overline{CF}$). **19.** Answers vary. Pick any three points A, B, and C on the circle and construct the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$. The intersection of the perpendicular bisectors is the center. **20.** The center of the circle is the intersection of the perpendicular bisectors of any two adjacent sides of the rectangle. The radius of the circle is the distance from the center to any of the vertices of the rectangle. Alternatively the center can be found as the intersection of the diagonals. **21. a.** Not possible unless the rhombus is a square. **b.** Possible. Answers vary. The center O of the circle is the intersection of any two perpendicular bisectors of two adjacent sides. Alternatively the center is the intersection of two segments joining opposite vertices. The radius is the segment from O to any of the vertices. **22. b.** Use Theorem 12-5(b) to bisect a 60° angle and take the supplement of the resulting 30° angle. **23. a.** The three sides are congruent in any order; that is, $\triangle ABC \cong \triangle ABC$, $\triangle ABC \cong \triangle BAC$, $\triangle ABC \cong \triangle CAB$, $\triangle ABC \cong \triangle ACB$, $\triangle ABC \cong \triangle BCA$, $\triangle ABC \cong \triangle CBA$. **b.** If the rectangle is $ABCD$, then $ABCD \cong ABCD$, $ABCD \cong BADC$, $ABCD \cong DCBA$, $ABCD \cong CDAB$. **24. b.** The center is on the hypotenuse. **c.** Because the center of the circle is on the hypotenuse, the radius is half the length of the hypotenuse.

Mathematical Connections 12-1

Reasoning

1. *GHEF* **2.** No, because 5 cm + 4 cm < 12 cm no triangle can be formed. **3.** The third side must be less than 24 cm and greater than 4 cm. **4. a.** $\overline{CD}$ and $\overline{CE}$ **b.** $\angle CAB$ and $\angle CBA$ **c.** No, no congruence criteria are satisfied. **d.** Yes, SAS **5.** P is not on the circle because of the definition of a circle. **6.** Let $\angle A$ be the vertex angle formed by two congruent sides. Both parts of Theorem 12-5 can be proved by drawing the angle bisector of $\angle A$ and proving that the two triangles created are congruent by SAS. **7. a.** $\triangle ABC \cong \triangle ADC$ by SSS. Hence, $\angle BAC \cong \angle DAC$ and $\angle BCA \cong \angle DCA$ by CPCTC. Therefore, $\overleftrightarrow{AC}$ bisects $\angle A$ and $\angle C$. **b.** The angles formed are right angles. By part (a), $\angle BAM \cong \angle DAM$. Hence, $\triangle ABM \cong \triangle ADM$ by SAS. $\angle BMA \cong \angle DMA$ by CPCTC. Since $\angle BMA$ and $\angle DMA$ are adjacent congruent angles, each must be a right angle. Since vertical angles formed are congruent, all four angles formed by the diagonals are right angles. **c.** By part (a) and SAS, $\triangle BAM \cong \triangle DAM$. Hence, $\overline{BM} \cong \overline{MD}$ by CPCTC. **8. a.** If point C lies in the interior of $\triangle ABD$, part (a) is consistent. **b.** For part (b) to hold, the diagonals must be extended in order to meet. By extending and allowing one diagonal to be outside the figure, the result holds. **c.** See parts (b) and (a). With these conditions, the result is true. **9.** No. If the diagonals of a rectangle are perpendicular it is a square. If they are not perpendicular, the rectangle is not a square. In each case the diagonals can be 19 in. long. **10.** $\overline{CB} \cong \overline{CD}$ since each is 2 units long. $\overline{AB} \cong \overline{AD}$ because they can be viewed as hypotenuses of two congruent right triangles with legs 1 and 3 units long. **11.** The construction will work by moving the drafting triangle as before until the other leg passes through the given point. **12. a.** Following the directions in the problem after points A and B are constructed, we construct points C and D so that $AC = BD = r$, where r is the radius of the circle. We next construct point E so that $CE = r$. If we now draw

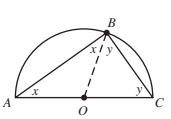
an arc with center D and radius r, we find that the arc intersects the circle at the existing point E. Thus only six points are determined.



b. The figure is a regular hexagon. To see why, notice that the fact that the arc with center D and radius r intersects the circle at the existing point E is an experimental observation. However, rather than drawing that arc, we can just connect E with D and prove that $ED = r$. From the above construction it follows that the five triangles: $\triangle OBP$, $\triangle OPA$, $\triangle OAC$, $\triangle OCE$, and $\triangle OBD$ are equilateral and therefore all their angles measure 60°. Consequently $x = m(\angle EOD) = 360° - 5 \cdot 60° = 60°$. Because $\triangle OED$ is isosceles, $m(\angle OED) = m(\angle ODE)$ and therefore each of these angles measures $\frac{(180° - 60°)}{2}$ or 60°. Thus $\triangle EOD$ is equilateral and therefore $ED = r$. Because the angles at the vertices of the hexagon are each 60° + 60°, or 120°, we obtain a regular hexagon.

Open-Ended

13. Constructions vary. **14.** Constructions vary. **15.** Answers vary. For example, draw radius $\overline{OB}$. Now we have two isosceles triangles, $\triangle ABO$ and $\triangle OCB$ as shown below (the sides are congruent because they are radii). The base angles in each triangle are congruent. The measures, x, of the base angles of $\triangle ABO$ are equal. Similarly the measures, y, of the base angles of $\triangle OCB$ are equal. The sum of the measures of the angles in $\triangle ACB$ is 180°. Therefore, $x + x + y + y = 2x + 2y = 2(x + y) = 180°$ and $x + y = 90°$. Since the $m(\angle B) = x + y = 90°$, we have a right angle, and the proof of Thales' Theorem is complete.



16. a. Answers vary. Some objects are tins of food and floor tiles. **b.** Answers vary. Photographs and their enlargements, an original and its projected image in an overhead projector, and a slide and its image are possible examples. **17.** Answers vary. One possible pattern is a square stamp-block quilt, which is a quilt constructed entirely of squares. **18. a.** $\triangle AFB \cong \triangle CED$ by SAS because $\overline{AF} \cong \overline{CE}$, $\overline{FB} \cong \overline{ED}$, and $\angle F \cong \angle E$ (each is a right angle). Hence, $\overline{AB} \cong \overline{CD}$ (corresponding parts in congruent triangles). **b.** Because $\triangle AFB$ is a right triangle, we have $m(\angle A) + m(\angle B) = 90°$. The congruence of triangles in part (a) implies that $\angle A \cong \angle C$. Hence, $m(\angle C) + m(\angle B) = 90°$. This implies that $\angle G$ in $\triangle BCG$ is a right angle and hence that $\overline{AB}$ and $\overline{CD}$ are perpendicular. **19. a.** The right angle is the greatest angle in a right triangle. **b.** The distance is the length of the perpendicular segment from the point to the line. Any other segment will be a hypotenuse of a right triangle in which the perpendicular segment is a leg and hence shorter than the hypotenuse.

Cooperative Learning

20. Answers vary.

Connecting Mathematics to the Classroom

21. For two figures to be congruent, they must be the same shape and the same size. All squares are the same shape but not the same size. A tracing of a small square can be placed on top of a large square to show that they are not congruent. **22.** In the triangles, there are two sides and an angle in one triangle congruent to two sides and an angle of the other. However, in order to use SAS the angle must be the included angle between the two sides in each triangle. In these triangles, the angles are not the included angles. **23.** The symbol $\cong$ is used only for congruent objects. Because AB and CD designate lengths of segments and not the segments themselves, it is not true that $AB \cong CD$. Notice that if segments are congruent, then they are of the same length; hence it is correct to write $AB = CD$. **24.** Perhaps the "best" definition relies on transformational geometry, discussed in Chapter 13. Two figures can be defined to be congruent if, and only if, one figure can be mapped onto the other by successively applying translation, reflection, rotation, or glide reflection. **25.** She is correct; if $AC = BC$ all corresponding parts are congruently SSS.

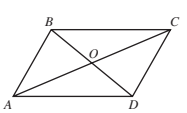
GeoGebra Activities

26. See online appendix. **27.** See online appendix. **28.** See online appendix. **29.** See online appendix.

Assessment 12-2A

2. a. No, by ASA, the triangle is unique. **b.** No, by AAS, the triangle is unique. **c.** Yes, the sides can be of any length. **4.** When the parallel ruler is open at any setting, the distance $BC = BC$; it is given that $AB = DC$ and $AC = BD$, so $\triangle ABC \cong \triangle DCB$ by SSS. Hence, $\angle ABC \cong \angle DCB$ by CPCTC. Because these are alternate interior angles formed by lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ with transversal line $\overleftrightarrow{BC}$, $AB \parallel DC$. **5. a.** $\triangle ABD \cong \triangle CBD$ by ASA **b.** There is not enough information for congruence. **6.** $\angle C \cong \angle F$ is needed to have congruence by ASA or $\overline{AB} \cong \overline{DE}$ to use SAS. **8.** We know that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$. Therefore, $\triangle ABC \cong \triangle DEF$ by SAS and $\angle BAC \cong \angle EDF$ by CPCTC. **9.** $\overline{BD} \cong \overline{BC}$, $\angle ABD \cong \angle EBC$ (vertical angles), and $\angle A \cong \angle E$ (alternate interior angles of parallel lines). Therefore, $\triangle ABD \cong \triangle EBC$ by AAS. **12.** An isosceles trapezoid is formed, and the angles formed with each base are congruent. The congruent angles are sometimes referred to as the base angles of the isosceles trapezoid. Other pairs of angles are supplementary. **13. d.** False. A trapezoid may have only one pair of opposite sides parallel. **14. b.** No. If the quadrilateral has three right angles, then the fourth must also be a right angle because the sum of the measures of the four angles is 360°. **c.** No; any parallelogram with a pair of right angles must have right angles as its other pair and hence it must be a rectangle. **15.** Because $AB = BC$, B is equidistant from A and C. By Theorem 12-4, point B is on the perpendicular bisector of $\overline{AC}$. Similarly, D is on the perpendicular bisector of $\overline{AC}$. Because two points determine a unique line, $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$. **18.** Answers vary. Either the arcs or the central angles must have the same measure (radii are the same since the sectors are part of the same circle). An alternative condition is that the segments joining endpoints of each arc are congruent. **19. a.** Kite. $\overline{DX} \cong \overline{AX}$; $\angle D$ and

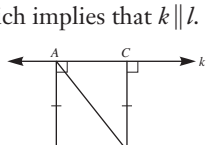
$\angle A$ are right angles; $\overline{DP} \cong \overline{AQ}$. Thus, $\triangle PDX \cong \triangle QAX$ by SAS. Hence, $\overline{PX} \cong \overline{QX}$ by CPCTC. $\overline{DC} \cong \overline{BA}$ as opposite sides of a rectangle. $\overline{PD} \cong \overline{QA}$. Hence, $\overline{CP} \cong \overline{BQ}$. $\overline{CY} \cong \overline{BY}$, given, and $\angle C$ and $\angle B$ are right angles. Therefore, $\triangle CYP \cong \triangle BYQ$ by SAS, so $\overline{QY} \cong \overline{PY}$ and $PXQY$ is a kite. **b.** The answer does not change. However, when P and Q are midpoints of $\overline{DC}$ and $\overline{AB}$, respectively, $PXQY$ is a rhombus. **20.** Make one of the quadrilaterals a square and the other a rectangle. **21.** Construct the first kite by constructing a segment to become a diagonal and then construct two isosceles triangles with the segment as a common base. Construct the second kite starting with a segment not congruent to the first and construct two isosceles triangles with that segment as their common base but the sides congruent to the corresponding sides of the isosceles triangles in the first construction. **23. b.** The corresponding lengths of the two perpendicular sides of the rectangles must be equal. **c.** Answers vary; one solution is that two corresponding adjacent sides must have equal lengths and the included angle of one must be congruent to the other. **24. a.** If in the accompanying figure $\overline{AD} \parallel \overline{BC}$ and $\overline{AD} \cong \overline{BC}$, then the pairs of alternate interior angles are congruent. That is, $\angle OBC \cong \angle ADO$ and $\angle BCO \cong \angle DAO$. Thus, $\triangle CBO \cong \triangle ADO$ by ASA. Consequently, $\overline{BO} \cong \overline{OD}$ and $\overline{AO} \cong \overline{OC}$ (by CPCTC). Now $\triangle ABO \cong \triangle DCO$ by SAS as the angles at O are vertical. Hence, $\angle BAO \cong \angle DCO$. These angles are alternate interior angles created by $\overline{AB}$, $\overline{CD}$, and the transversal $\overleftrightarrow{AC}$. Thus, $\overline{AB} \parallel \overline{CD}$ and by definition $ABCD$ is a parallelogram ($\overline{BC} \parallel \overline{AD}$ is given). **b.** In this figure, $AO = OC$ and $BO = OD$. In addition, $\angle BOC \cong \angle DOA$ are vertical angles. Thus by SAS, $\triangle BOC \cong \triangle DOA$. Consequently, $\angle ADO \cong \angle CBO$. Because these are alternate interior angles created by lines $\overleftrightarrow{BC}$, $\overleftrightarrow{AD}$, and transversal $\overleftrightarrow{BD}$, $\overline{BC} \parallel \overline{AD}$. From the congruence of the triangles, it also follows that $\overline{BC} \cong \overline{DA}$. Thus, $ABCD$ is a quadrilateral with pair of opposite sides parallel and congruent and therefore a parallelogram. (Use part (a).)



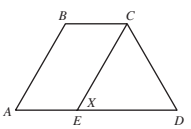
25. Answers vary; for example, the polygons must have the same number of sides with one pair of sides congruent. All regular polygons with the same number of sides are similar, so if they have the same number of sides with one pair congruent, they are congruent.

Assessment 12-2B

1. b. Infinitely many are possible. **2.** (a) No; by ASA, the triangle is unique. (b) Yes; AAA determines a unique shape, but not size. **3. b.** No; AAA does not ensure congruence. **4.** Because the angles at A and at C are corresponding congruent angles (they are right angles), it follows that $\overline{AB} \parallel \overline{CD}$. We have $\triangle ABD \cong \triangle DCA$ by SAS since $\overline{AB} \cong \overline{BC}$ (given), $\overline{AD} \cong \overline{DA}$, and $\angle BAD \cong \angle CDA$ (alternate interior angles created by the parallels $\overline{AB}$ and $\overline{CD}$ and transversal $\overleftrightarrow{AD}$). Hence by CPCTC, $\angle ADB \cong \angle DAC$, which implies that $k \parallel l$.

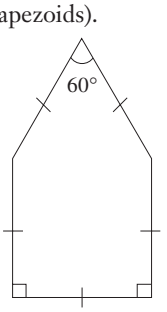


5. a. $\triangle ADC \cong \triangle ABC$ by ASA **b.** Not enough information for congruence. **6.** If $\overline{AB} \cong \overline{DE}$, we have congruence by ASA or since $\angle BCA$ must be congruent to $\angle ECD$ (third angles in the triangle), then to use AAS we need $\overline{BC} \cong \overline{EC}$ or $\overline{AC} \cong \overline{DC}$. **7.** $m(\angle 1) = 120°$, $m(\angle 2) = 60°$, $m(\angle 3) = 60°$, $m(\angle 4) = 60°$, $m(\angle 5) = 30°$ **8.** $\angle P \cong R$, $\angle PQS \cong \angle RSQ$ (alternate interior angles of parallel lines), $\overline{SQ} \cong \overline{QS}$ (reflexive property), and so $\triangle PSQ \cong \triangle RQS$ by AAS. Therefore, $\overline{PQ} \cong \overline{RS}$ by CPCTC. **9.** Because $\triangle ISO$ is isosceles, the base angles must be congruent and sum to $180° - 110° = 70°$. Therefore, each base angle in $\triangle ISO$ measures 35°. Since $\triangle ANG$ is isosceles, each base angle has the same measure and so $m(\angle G) = 35°$, and hence the measure of $\angle N$ is 110°. Therefore, $\triangle TRI \cong \triangle ANG$ by SAS. Other angles and sides could have also been used to complete the proof. **12.** Construct $\overline{CE}$ such that $\overline{CE} \parallel \overline{AB}$. Thus, $ABCE$ is a parallelogram and consequently $\overline{AB} \cong \overline{CE}$, which makes $\triangle DEC$ isosceles. This implies that $\angle x \cong \angle D$. Because $\angle x \cong \angle D$ (corresponding angles created by the parallels $\overline{AB}$ and $\overline{CE}$) and $\angle x \cong \angle A$ we have $\angle A \cong \angle D$.

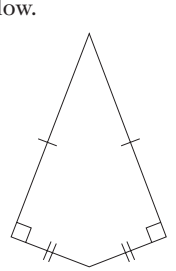


13. a. False, a square is a rectangle and a rhombus. **c.** True (in fact, all squares are trapezoids).

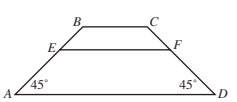
14. a. Answers vary.



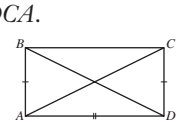
b. No, because the sum of the measures of four acute angles totals less than 360°. **c.** Yes. To construct such a kite, construct a right triangle in which no angle measures 45° and reflect it along the hypotenuse as shown below.



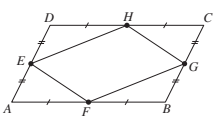
18. Answers vary. For example, two isosceles trapezoids $AEFD$ and $ABCD$, where $\overline{EF} \parallel \overline{AD}$ as shown.



19. a. Because the diagonals bisect each other we know from 24(b) in Assessment 12-2A, that the quadrilateral is a parallelogram and that opposite sides are congruent. We need to prove that one of the angles of the quadrilateral is a right angle. In the figure below, because the diagonals are congruent, we know by SSS that $\triangle ABD \cong \triangle DCA$.



The fact $\triangle ABC \cong \triangle DCA$ follows from SSS by observing that $\overline{AB} \cong \overline{DC}$ (opposite sides of a parallelogram), $\overline{AD} = \overline{DA}$, and $\overline{BD} = \overline{CA}$ (given). Thus, $\angle BAD \cong \angle CDA$. These angles are also supplementary since $\overline{AB} \parallel \overline{CD}$. Thus, each must be a right angle. **b.** Construct two congruent line segments that bisect each other. The endpoints of these line segments will be the vertices of a rectangle. **c.** If the angles created by the diagonals are not congruent, the corresponding sides of the two rectangles will not be congruent. **20. b.** Suppose *ADCB* in part (a) is a parallelogram. By *SAS*, $\triangle EDH \cong \triangle GBF$, thus $\overline{EH} \cong \overline{GF}$. Similarly, $\triangle EAF \cong \triangle GCH$ so $\overline{EF} \cong \overline{GH}$. If opposite sides of a quadrilateral are congruent, it is a parallelogram.



21. a. Answers vary. Two adjacent sides of one must be congruent to the corresponding adjacent sides of the other. **b.** Possibilities include: (i) The sides and one angle are congruent to the corresponding parts in the other kite, or (ii) corresponding diagonals are congruent. **c.** Answers vary. The vertex (where the congruent sides intersect) angles are congruent, and one of the two congruent sides in one triangle is congruent to the corresponding side in the other. **22. a.** $\angle ABD \cong \angle CDB$ and $\angle ADB \cong \angle CBD$ (alternate interior angles with respect to parallel lines) $\overline{BD} \cong \overline{DB}$. Then $\triangle ABD \cong \triangle CDB$ (by *ASA*). Therefore, $\angle BAD \cong \angle DCB$ (CPCTC). Similarly, $\angle ABC \cong \angle CDA$. **b.** Because $\triangle ABD \cong \triangle CDB$, we have $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$ (CPCTC). **c.** $\angle BAC \cong \angle DCA$ and $\angle ABD \cong \angle CDB$ (alternate interior angles), and $\overline{AB} \cong \overline{DC}$ [from part (b)]. Then $\triangle BAF \cong \triangle DCF$, so $\overline{AF} \cong \overline{CF}$ and $\overline{BF} \cong \overline{DF}$ (CPCTC). **d.** In $\triangle ABD$, $m(\angle BAD) + m(\angle ABD) + m(\angle ADB) = 180°$. From part (a), $\angle CBD \cong \angle ADB$. Substitution yields $m(\angle BAD) + m(\angle ABD) + m(\angle CBD) = 180°$. Since $m(\angle ABC) = m(\angle ABD) + m(\angle CBD)$, then $m(\angle BAD) + m(\angle ABC) = 180°$. $\angle ABC$ and $\angle BAD$, therefore they are supplementary. **23.** Answers vary. Two rhombuses are congruent if a side of one is congruent to a side of the other and an angle of one is congruent to an angle of the other. To see this, note that in a rhombus all four sides are congruent. Therefore, showing that the two rhombuses have one pair of congruent sides is sufficient to conclude all the sides are congruent.

Mathematical Connections 12-2

Reasoning

1. a. Yes, if you know four parts then you have enough to use one of SSS, SAS, ASA, or AAS. **b.** No, for example, AAA is not enough to establish congruence. **2.** SSS, SAS, ASA, AAS, and HL (for right triangles) **3.** Show $\triangle ABC \cong \triangle EDC$ by ASA and then $\overline{AB} \cong \overline{DE}$ by CPCTC. The bridge is 92 m long. **4.** Each of the fabric triangles is congruent to the others by SAS. **5. a.** $\angle GHE$ **b.** $\overline{GH}$ **c.** Draw diagonals $\overline{BD}$ and $\overline{FH}$, then $\triangle BCD \cong \triangle FGH$ by SAS. **6.** The triangle formed by Stan's head, Stan's feet, and the opposite bank is congruent to the triangle formed by Stan's head, Stan's feet, and the spot just obscured by the bill of his cap. These triangles are congruent by ASA since the angle at Stan's feet is 90° in both triangles, Stan's height is the same in both triangles, and the angle formed by the bill of his cap is the same in both triangles. The distance across the river is approximately equal to the distance he paced off because these distances are corresponding parts of congruent triangles.

7. One way is to make both legs of the ironing board the same length and fasten them together with a hinge at their midpoints. If one of these legs is attached to the board at a fixed spot and the other leg can be attached at various spots, then the height of the ironing board can be adjusted. Since the legs form the diagonals of a rectangle, the board will always be parallel to the floor. (It can be shown that a quadrilateral whose diagonals are the same length and bisect each other is a rectangle. (See Exercise 19(a) in Section 12-2B.) In most commercially available ironing boards, the legs are designed to form the diagonals of a trapezoid. **8.** Given $\angle BAC$, construct a circle with center *A* and any radius. Let *D* and *E* be the points where the circle intersects the sides of the angle. Next construct the perpendicular bisector of $\overline{DE}$. **9.** No, because the congruent sides are not corresponding sides. **10. a.** The distances are equal. **b.** The distances from every point on the angle bisector of an angle to the sides of the angle are equal. **c.** $\triangle APC \cong \triangle APB$ by AAS; hence $\overline{PC} \cong \overline{PB}$. **d.** If a point *P* in the interior of an angle is equidistant from the sides of an angle, then it is on the angle bisector of the angle. To prove this statement, we assume that $\overline{PC} \cong \overline{PB}$ and prove that $\overrightarrow{AP}$ bisects $\angle A$. We have $\triangle APC \cong \triangle APB$ by HL. Thus, $\angle CAP \cong \angle PAB$ by CPCTC **11. a.** Yes **b.** No, for example, neither 4 angles nor 4 sides determine congruence.

Open-Ended

12. Answers vary. Possible questions and answers concerning figures with vertices on the nails (or dots of the dot paper) are: **a.** How many noncongruent right isosceles triangles are possible? Answer: 5 **b.** How many noncongruent isosceles trapezoids that are not rectangles are there? Answer: 6 **c.** How many noncongruent squares are there? Answer: 5

Cooperative Learning

13. a. Answers vary. In many secondary-school and middle-school geometry books, kites are not defined. For many of these books, a trapezoid is a quadrilateral with exactly one set of parallel sides. **b.** Answers vary. **c.** The definition used in this text allows a structure among the quadrilaterals that does not exist with the definition in part (a), namely, the subset relationships among the quadrilaterals. The one in this text does cause some difficulty in the definition of an isosceles trapezoid. Therefore it is defined in terms of its base angles, not its sides.

Connecting Mathematics to the Classroom

14. The student is wrong. $\angle 1 \cong \angle 2$ implies that $\overline{AD}$ and $\overline{BC}$ are parallel, but does not imply that the other two sides are parallel. **15.** The symbol $=$ is used for identical objects. Two triangles are equal if they represent the same set of points. Congruent triangles are not necessarily identical because their positions may be different. **16.** The SSS congruency condition assures that triangles are rigid and hence contribute to supporting the structure. **17.** Extend the shorter side and construct the perpendicular from a vertex not on the side to the line containing the side. **18.** This is true because the distance from *A* or from *A′* to the line $\overleftrightarrow{BC}$ is the distance between the two parallel lines. **19.** $\triangle ABC \cong \triangle EDC$ by SAS, so $\angle B \cong \angle D$. But in the picture the angles are not congruent, so the figure must not be accurate.

GeoGebra Activities

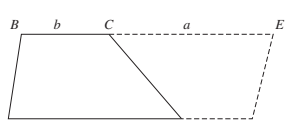
20. See online appendix.

Review Problems

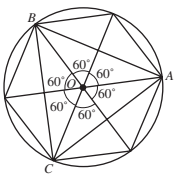
21. The triangles that are congruent to triangle $\triangle ABC$ are triangles $\triangle BCD$, $\triangle CDE$, $\triangle DEA$, and $\triangle EAB$, as well as triangles $\triangle CBA$, $\triangle DCB$, $\triangle EDC$, $\triangle AED$, and $\triangle BAE$. They are all congruent by SAS. **22.** Construction **23.** Construction **24. a.** Yes, SAS **b.** Yes, SSS **c.** Yes, HL

Assessment 12-3A

6. b. The altitude is the extension of the cable from vertex A to the ground. **7. a.** The perpendicular bisectors of the sides of an acute triangle meet inside the triangle. **b.** The perpendicular bisectors of the sides of a right triangle meet at the midpoint of the hypotenuse. **c.** The perpendicular bisectors of the sides of an obtuse triangle meet outside the triangle. **8. a.** The point is equidistant from all vertices because it is on all three perpendicular bisectors. Being at the intersection of two of the perpendicular bisectors forces the point to be equidistant from all three vertices. **9. a.** If the rectangle is not a square, it is impossible to construct an inscribed circle. The angle bisectors of a rectangle do not intersect in a single point. **b.** Possible. The center of the circle is the intersection of the diagonals (which are also the angle bisectors of the vertices), and the radius of the circle is the distance from the center to any of the sides. **c.** Possible. The intersection of the three longest diagonals is the center of the circle. **12.** The figure is a rhombus. The diagonals bisect each other, so the figure is a parallelogram. Because in addition the diagonals are perpendicular to each other, it is a rhombus. **15.** The center is where the angle bisectors of the angles meet (the incenter). The radius is the perpendicular distance from the incenter to a side. **17.** Answers vary. **a.** If the parallelogram is not a rectangle, cut along an altitude whose foot lies on the opposite side of the parallelogram. If the parallelogram is a rectangle, cut along any line through the point where the diagonals meet such that the line is not a diagonal and is not parallel to any side. **b.** Make a copy of a given trapezoid $ABCD$ and put it upside down next to $\overline{CD}$ as shown. More precisely, extend $\overline{BC}$ so that $\overline{CE} = a$ and extend $\overline{AD}$ so that $DF = b$. Because $\overline{BE} \parallel \overline{AF}$ and $BE = AF$ (the length of each is $a + b$), $ABEF$ is a parallelogram.

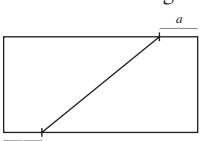


19. b. Infinitely many noncongruent rectangles can be constructed. The endpoints of two segments bisecting each other and congruent to the given diagonal determine a rectangle, but since the segments may intersect at any angle, there are infinitely many such rectangles. **c.** Not possible because the sum of the measurements of the angles would be greater than 180°. **d.** There is no parallelogram because the fourth angle must also be a right angle. **20. a.** Construct an equilateral triangle and bisect one of its angles. **b.** Bisect a 30° angle, then add the 30° and 15° angles or bisect a right angle. **c.** Combine 60° and 15° angles or combine 45° and 30° angles. **22. a.** The point is determined by the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Because the point is on the angle bisectors of $\angle BAD$, it is equidistant from its sides. Because it is on the perpendicular bisector of $\overline{BC}$, it is equidistant from B and C. **b.** The point is determined by the intersection of the angle bisectors of $\angle A$ and $\angle B$. **23.** Answers vary. One way is to construct six 60° central angles with common vertex at the center O. Then construct three 120° central angles with vertex O. The sides of these angles intersect the circle at points A, B, and C, which are vertices of an equilateral triangle. $\triangle ABC$ is equilateral because $\triangle AOB$, $\triangle BOC$, and $\triangle AOC$ are congruent (by SAS). Another approach is to construct a regular inscribed hexagon and then choose any vertex of the hexagon and two other nonconsecutive vertices.



Assessment 12-3B

1. a. Construct line n through P perpendicular to ℓ, then a line through P perpendicular to n. **b.** Through P draw any line intersecting ℓ. Let Q be the point of intersection. Find the midpoint M of $\overline{PQ}$. Through M draw any line intersecting ℓ at Q'. Find P' so that M is the midpoint of $\overline{P'Q'}$. Then $\overleftrightarrow{P'P'}$ is the required line. **8.** Construction. The center of the circle is on each angle bisector, therefore equidistant from the sides of each angle, that is, from the sides of the triangle. **9.** Yes, because the angle bisectors are concurrent. One of the diagonals is on the angle bisectors of opposite angles. The other angle bisectors intersect on that diagonal. **10.** Answers vary. One way is to construct the perpendicular bisector n of $\overline{AB}$ and label M its point of intersection with $\overline{AB}$. Next find two points C and D on n such that $MC = MD = MA$. The required square is $ACBD$. **12.** Construction. Use the fact that the diagonals of a rectangle are congruent and bisect each other. **13.** Construction. Draw $\overline{BE}$. Extend $\overline{BE}$ to a point D such that $BE = ED$. Now $\overline{BD}$ is one diagonal of the square. Next construct a perpendicular to $\overline{BD}$ at E. Then find points A and C such that $BE = AE = CE$. $ABCD$ is the desired square. **15.** The center is the intersection of any two angle bisectors or any two of the longest diagonals. The radius is the distance from the incenter to a side. **17.** The following figure shows two such congruent trapezoids. For different values of a we get different trapezoids.



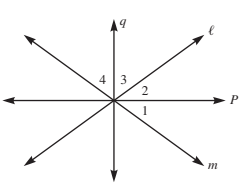
18. a. $\overleftrightarrow{PQ}$ is the perpendicular bisector $\overline{AB}$. **b.** Q is on the perpendicular bisector of $\overline{AB}$ because $\overline{AQ} \cong \overline{QB}$; similarly, P is on the perpendicular bisector of $\overline{AB}$. There is unique line through two points so the perpendicular bisector is $\overleftrightarrow{PQ}$. **c.** $\overleftrightarrow{PQ}$ is the angle bisector of $\angle APB$; $\overleftrightarrow{QC}$ is the angle bisector of $\angle AQB$. **d.** Because $\overleftrightarrow{PC}$ and $\overleftrightarrow{QC}$ are on perpendicular bisectors of the bases of isosceles triangles $\triangle APB$ and $\triangle AQB$, each is an angle bisector. **19. a.** Without the angle between the sides, there are infinitely many parallelograms that could be constructed from the two given sides. **b.** Possible. Construct two perpendicular segments bisecting each other and congruent to the given diagonals. **c.** There is no unique parallelogram because the other pair of congruent sides can be of any length. **20.** Answers vary. **a.** Bisect a 60° angle from an equilateral triangle and then bisect it again. **b.** $105° = 90° + 15°$; thus construct adjacent angles of 90° and 15°. **c.** $120° = 180° - 60°$; thus a 120° angle is a supplement of a 60° angle. **21.** Make the edge of the straightedge coincide with ℓ; let one of the legs of the right triangle slide along the leg of the straightedge until the other leg goes through P. The line along the leg containing P is the required

perpendicular. **22. a.** The point is determined by the intersection of the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$. **b.** Not possible. If such a point existed, it would have to be on the perpendicular bisectors of all the sides, but because the perpendicular bisectors of the four sides of a quadrilateral do not necessarily intersect in a single point (this is the case for the given quadrilateral), such a point does not exist. **c.** This is possible only if the angle bisectors are concurrent, which is not the case for a general quadrilateral. **23.** Answers vary. Referring to the figure and answer to exercise 23 in Assessment 12-3A, construct an equilateral triangle *ABC* inscribed in the circle. Next construct tangents to the circle at points *A*, *B*, and *C*. The points of intersection of the tangents are the vertices of a required triangle. To prove that the triangle is equilateral, let *D* be the intersection of the perpendiculars at *A* and *B*, then the quadrilateral *AOBD*, where *O* is the center of the circle, has two right angles at *A* and *B* and a 120° angle at *O*. Hence the angle at *D* measures $360° - (90° + 90° + 120°)$ or 60°. Similarly the other angles measure 60° and hence the triangle is equilateral.

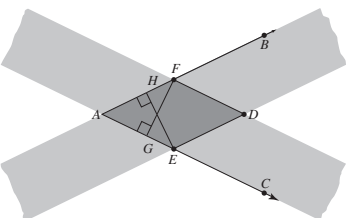
Mathematical Connections 12-3

Reasoning

1. Lines ℓ and n are parallel because congruent corresponding angles (right angles) are formed. **2.** No. If *B* were the midpoint, then $(3x + 5) + (5x + 1) = 30$. Solving for x we have $x = 3$. If $x = 3$, then the two parts would have measures 14 and 16, which cannot happen if *B* is the midpoint. **3.** Answers vary. Construct a perpendicular to $\overleftrightarrow{AC}$ at *A*. The new angle formed is the complement of $\angle A$ and has measure 40°. If we bisect this 40° angle, we have a 20° angle. **4. a.** The diagonals of *ABCD* are congruent and bisect each other, so the figure is a rectangle. **b.** Construct a rectangle *ABCD* as in part (a). To construct a rectangle that is not congruent to *ABCD*, choose different diameters that intersect at different angles than the diameters in *ABCD*. **5.** Connect the three dots to form a triangle. Construct the perpendicular bisectors of the three sides of the triangle. The point where they meet will be the site of the airport. **6.** In the figure below, $m(\angle 1) + m(\angle 2) + m(\angle 3) + m(\angle 4) = 180°$ (straight angle). Because $m(\angle 1) = m(\angle 2)$ and $m(\angle 3) = m(\angle 4)$, we have $m(\angle 2) + m(\angle 3) = \frac{180°}{2} = 90°$ and therefore lines *p* and *q* are perpendicular.



7. Answers vary. Given $\angle BAC$, place one strip of the tape so that an edge of the tape is along $\overline{AB}$ and another strip of the tape so that one of its edges is along $\overline{AC}$, as shown. Two edges of the strips of tape intersect in the interior of the angle at *D*. Connect *A* with *D*. $\overline{AD}$ is the angle bisector. Because the diagonals of a rhombus bisect its angles, this construction can be justified by showing that *AEDF* is a rhombus. (*E* and *F* are the points of intersection of the tops of the tape pieces and their bottoms.) *AEDF* is a parallelogram. It remains to be shown that $\overline{AF} \cong \overline{AE}$. For that purpose, we show that $\triangle FAG \cong \triangle EAH$. We have $\overline{FG} \cong \overline{EH}$ because the two strips of tape have the same width. $\angle A \cong \angle A$, and the angles at *H* and *G* are right angles. Thus, the triangles are congruent by AAS.



8. Rhombus. For proof, see the solution to problem 7.
9. Notice that the vertices *E*, *F*, *G*, *H* are on the angle bisectors of the four angles each with vertex at *O*. Because each pair of vertical angles at *O* are congruent, the triples *F*, *O*, *H* and *G*, *O*, *E*, are collinear. Since $\overleftrightarrow{OF}$ and $\overleftrightarrow{OG}$ bisect adjacent supplementary angles, $m(\angle FOG) = 90°$. Similarly, $m(\angle GOH) = 90°$. Because $\triangle ABO \cong \triangle CDO$ and $\triangle BCO \cong \triangle DAO$, it follows that $OF = OH$ and $OG = OE$. Consequently, $\overline{FH}$ and $\overline{GE}$ are perpendicular bisectors of each other, which makes *EFGH* a rhombus.

Open-Ended

10. Answers vary. Most students will probably lobby to have other tools included as construction tools. **11.** Answers vary. **12.** Answers vary. Construct a circle with center *O*. Use a protractor or a Geometry Utility (GSP or GeoGebra) to construct five 72° central angles that are adjacent, each to the next, with vertices at *O*. At the intersections of the angles with the circle, construct tangents to the circle. The five tangents intersect at points that are the vertices of the required pentagon. (The construction can be achieved with only compass and straightedge but is beyond the scope of this text.)

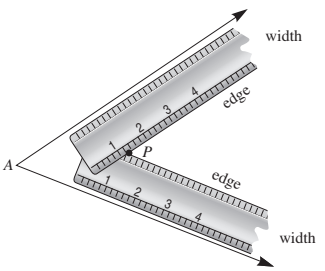
Cooperative Learning

13. a. Answers vary. When we unfold the paper, we get two creases, $\overline{AB}$ and $\overline{PQ}$, each of which is a perpendicular bisector of the other. This is so because *M* is the midpoint of each segment and all the angles formed at *M* are right angles (two congruent angles are formed at *M* whose measures add up to 180°, and hence each must measure 90°). Every point on the perpendicular bisector of $\overline{PQ}$ is equidistant from the endpoints *P* and *Q* (Theorem 12-3). **b.** Because *A* and *B* are on the perpendicular bisectors of $\overline{PQ}$, $AP = AQ$ and $PB = BQ$. Because *Q* is on the perpendicular bisector of $\overline{AB}$, $AQ = QB$. Hence $AP = AQ = QB = PB$. A quadrilateral with all sides congruent is a rhombus. **14. a.** Yes **b.** The measures are the same. **c.** Answers vary. **15.** Answers vary. Some are discussed in the text.

Connecting Mathematics to the Classroom

16. Answers vary. One might argue that if you choose any point *P* on a line and let *A* and *B* be points such that *P* is between *A* and *B*, then *P* bisects the line because the two half-line $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are congruent. **17.** *E* is not on $\overline{DF}$. Therefore it is not a midpoint. **18.** No. Perpendiculars must be constructed from the incenter to the sides in order for the segments to have equal measure. **19.** The width of the ruler can be used to find points in the interior of the angle at a distance from a side that is the width of the ruler. When this is done at both sides of the angle, the point *P* where the edges intersect is on the angle bisector

because it is the same distance from the sides of the angle. $\overleftrightarrow{AP}$ is the angle bisector.



20. $\overline{OD}$ is the radius if, and only if, it is perpendicular to $\overline{AC}$. If $AB \neq BC$, then $\overline{OD}$ is not perpendicular to $\overline{AC}$.

GeoGebra Activities

21. a. In an acute triangle, the altitudes are concurrent at a point in the interior of the triangle and in a right triangle at the vertex of the right angle. In an obtuse triangle, the lines containing the altitudes are concurrent at a point in the exterior of the triangle. **b.** Answers vary, but the distances are equal. **c.** The distances from every point on the angle bisector to the sides of the angle are equal. **22.** See online appendix.

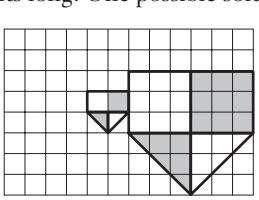
Review Problems

23. $\triangle ABC \cong \triangle DEC$ by ASA ($\overline{BC} \cong \overline{EC}$, $\angle ACB \cong \angle DCE$ as vertical angles, and $\angle B \cong \angle E$ as alternate interior angles formed by the parallels $\overline{AB}$ and $\overline{ED}$ and the transversal $\overleftrightarrow{EB}$). $\overline{AB} \cong \overline{DE}$ by CPCTC. **24.** Constructions **25.** Because $\angle B \cong \angle E$ and $\angle A \cong \angle D$, then $\angle C \cong \angle F$. Then because $\overline{AC} \cong \overline{DF}$, we have $\triangle ABC \cong \triangle DEF$ by SAS. **26.** 50°

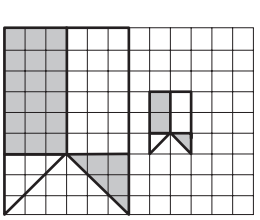
Assessment 12-4A

3. Yes. Corresponding angles are congruent and sides are proportional. **4.** Yes, they are similar. It is given that $\angle BAD \cong \angle B_1A_1D_1$. Because the pairs $\angle ABC$ and $\angle BAD$ as well as $\angle A_1B_1C_1$ and $\angle B_1A_1D_1$ are supplementary, it follows that $\angle ABC \cong \angle A_1B_1C_1$. Thus the rhombi have congruent corresponding angles. Because in a rhombus adjacent sides are congruent, $\frac{AD}{AB} = 1 = \frac{A_1D_1}{A_1B_1}$. This implies $\frac{AD}{A_1D_1} = \frac{AB}{A_1B_1}$ and therefore ratios of corresponding sides are proportional.
5. b. Yes; sides are proportional and angles are congruent. **c.** No; two rectangles that are not squares are similar if, and only if, the corresponding sides are proportional.
d. Always similar because the ratio of the corresponding longer sides equals the ratio of the corresponding shorter sides and the angles are congruent. **6.** Make all dimensions three times as long; for example, in part (c) each side would be three diagonal units long. One possible solution set is shown here:

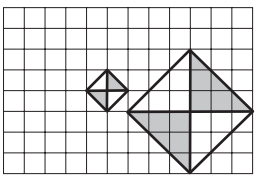
a.



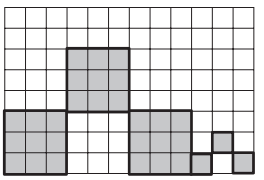
b.



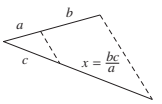
c.



d.



7. b. i. $\frac{2}{3}$ **ii.** $\frac{1}{2}$ **iii.** $\frac{4}{3}$ **iv.** $\frac{2}{3}$ or the reciprocals **12.** Construction as follows:



13. a. Yes, because the angles stay the same, and the ratios of corresponding sides are proportional. **15.** $133\frac{1}{3}\%$, but most copy machines will not allow this setting. **19.** $\angle BCA \cong \angle DEC$ (alternate interior angles), $\angle A \cong \angle ECD$ (alternate interior angles), $\triangle ABC \sim \triangle CDE$ (AA similarity). **20. a.** The ratio of the perimeters is the same as the ratio of the sides. **b.** If a, b, c, d are the sides of one quadrilateral and a_1, b_1, c_1, d_1 the corresponding sides of a similar quadrilateral then

$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = \frac{d}{d_1} = r \text{ (the scale factor).}$$

$$\text{Now, } \frac{a + b + c + d}{a_1 + b_1 + c_1 + d_1} = \frac{a_1r + b_1r + c_1r + d_1r}{a_1 + b_1 + c_1 + d_1}$$

$$= \frac{(a_1 + b_1 + c_1 + d_1)r}{a_1 + b_1 + c_1 + d_1} = r.$$

Hence the ratio of the perimeters is r. An analogous proof works for any two similar n-gons. **21. a.** From the Midsegment Theorem, we know that each midsegment is half the measure of the opposite side of $\triangle ABC$. Thus the sides of each of the smaller triangles are half as long as the corresponding sides of $\triangle ABC$. Hence by SSS they are congruent to each other. **b.** Yes; the ratio of the corresponding sides is $\frac{1}{2}$, and therefore by SSS similarity the triangles are similar. **23.** They are circles, so have exactly the same shape, and hence are similar.

Assessment 12-4B

1. No. The corresponding sides are not proportional.

2. $1{:}35\frac{3}{16}$ or 1:35.1875 **4.** They are not necessarily similar. As a counterexample, consider a square and a non-square rectangle.
5. e. Yes **6.** Answers vary. For example, $1 \times 1, 2 \times 2, 3 \times 3$ right isosceles triangles are all similar. **7. a.** $\triangle ABC \sim \triangle DEF$ by SSS similarity since $\frac{4}{6} = \frac{4}{6} = \frac{2}{3}$; scale factor $\frac{2}{3}$ **b.** $\triangle ABC \sim \triangle EDA$ by SAS similarity since $\frac{3}{6} = \frac{4}{8}$ and each has an included right angle; scale factor $\frac{1}{2}$ **c.** $\triangle ABE \sim \triangle ACD$ by AA similarity; scale factor $\frac{6}{6+2}$ or $\frac{3}{4}$ **d.** $\triangle ABE \sim \triangle DBC$ by AA similarity;

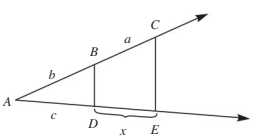
scale factor $\frac{6}{8}$ or $\frac{3}{4}$ **8. a.** $\triangle ABC \sim \triangle ADE$ by SAS similarity because $\frac{5}{5+5} = \frac{7}{7+7} = \frac{1}{2}$; $x = 4.5$

b. $\triangle ABC \sim \triangle EDF$ by AA similarity; $x = \frac{20}{3}$

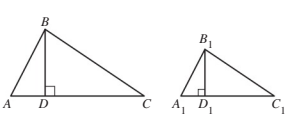
c. $\triangle QPT \sim \triangle RPS$ by SAS similarity since $\frac{3.2}{7.2} = \frac{4}{9}$; $x = \frac{27}{4}$

d. $\triangle ABC \sim \triangle BED$ by AA similarity since each has a right angle and $m(\angle CBA) = 90° - m(\angle DBE)$ and in $\triangle BDE$, $m(\angle E) = 90° - m(\angle DBE)$, so $m(\angle CBA) = m(\angle E)$; $x = \frac{8}{3}, y = \frac{10}{3}$ **10. a.** Both triangles share $\angle B$. In addition $\angle C \cong \angle E$ because they are corresponding angles. Thus $\triangle ABC \sim \triangle DBE$ by AA similarity. **b.** $2\frac{2}{9}$

12. Let segments b and a be on a side of an arbitrary angle as shown. On the other side of the angle mark point D so that $DA = c$. Connect B with D, and through C draw $\overline{CE} \parallel \overline{BD}$. Then $DE = x$.



13. b. $\frac{75}{32}$ **14.** Make one pentagon regular and the other with two right angles that looks like a house and has congruent sides. **20. a.** The ratio of corresponding heights is the same as the ratio of corresponding sides. **b.** Suppose $\triangle ABC \sim \triangle A_1B_1C_1$; then $\triangle ABD \sim \triangle A_1B_1D_1$ (see figure) by AA similarity. Hence $\frac{BD}{B_1D_1} = \frac{AB}{A_1B_1}$. A similar argument works if the altitudes are not in the interior of the triangles.



21. a. Answers vary. (i) Connect B with D and apply Theorem 12-16 to triangles ABD and BCD.

(ii) $MN = \frac{1}{2}a + \frac{1}{2}b = \frac{1}{2}(a + b)$. **b.** From part (a), $c = \frac{1}{2}(a + b)$ or $2c = a + b$.

$$\begin{aligned} c + c &= a + b \\ c + c - b &= a \\ c - b &= a - c \end{aligned}$$

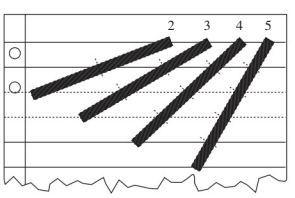
This shows that b, c, and a form an arithmetic sequence having difference $c - b$, which is to $a - c$. **c.** Part (b) proved that the lengths of any 3 consecutive segments form an arithmetic sequence, so the lengths of all 10 segments form an arithmetic sequence. Use the formula for the sum of an arithmetic sequence to find $S_{10} = \frac{10}{2}(b_1 + b_{10}) = 5(a + b)$.

22. a. Because $\overline{NP}$ and $\overline{MN}$ are midsegments, $\overline{NP} \parallel \overline{BD}$ and $\overline{MN} \parallel \overline{AC}$. Thus, $NSTV$ is a parallelogram. **b. i.** Perpendicular. Then, and only then, the parallelogram will have a right angle and therefore be a rectangle. **ii.** Perpendicular and congruent. Then, and only then, $MN = NP$ (each is half as long as a diagonal) and $\overline{MN}$ and $\overline{NP}$ will be perpendicular.

Mathematical Connections 12-4

Reasoning

1. Any two cubes are similar because they have the same shape. **2.** Answers vary. For example, the two prisms must be the same shape but not necessarily the same size. The prisms are similar if their bases are similar and all lengths of corresponding sides are proportional. **3.** Yes, each has a non-base angle of 40° and since they are isosceles each must have congruent base angles of $\frac{(180° - 40°)}{2} = 70°$. Now the triangles are similar by AA similarity. **4.** No, consider one right triangle to be a 3-4-5 triangle and the other triangle to be an isosceles right triangle with hypotenuse equal to 5. **5.** All three angles are congruent so the triangle must be equilateral. **6.** Lay the licorice diagonally on the paper so that it spans a number of spaces equal to the number of children. (See the figure.) Cut on the lines. Equidistant parallel lines will divide any transversal into congruent segments.



7. a. (i) $\triangle ABC \sim \triangle ACD$ by AA similarity because $\angle ACB$ and $\angle ADC$ are right angles and $\angle A$ is common to both. (ii) $\triangle ABC \sim \triangle CBD$ by AA similarity because $\angle ACB$ and $\angle CDB$ are right angles and $\angle B$ is common to both. (iii) Using (i) and (ii) $\triangle ACD \sim \triangle CBD$ by the transitive property of the similarity relation. **b.** (i) $\frac{AB}{AC} = \frac{BC}{CD} = \frac{AC}{AD}$ (ii) $\frac{AB}{CB} = \frac{BC}{BD} = \frac{AC}{CD}$ (iii) $\frac{AC}{CB} = \frac{AD}{CD} = \frac{CD}{BD}$ **c.** From part (b): $\frac{a}{x} = \frac{c}{a}$; so $a^2 = cx$. Similarly, $\frac{b}{y} = \frac{c}{b}$; so $b^2 = yc$.

d. From part (c): $\begin{aligned} a^2 + b^2 &= cx + cy \\ &= c(x + y) \\ &= c \cdot c \\ &= c^2 \end{aligned}$

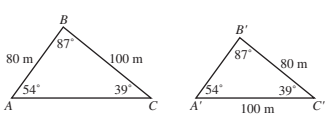
In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. **e.** $\triangle ADC \sim \triangle CDB$ implies $\frac{AD}{CD} = \frac{DC}{DB}$ or $\frac{y}{h} = \frac{h}{x}$. Hence, $h^2 = xy$. **f.** From (e), $\frac{h}{x} = \frac{y}{h}$ and hence x, h, y are in a geometric sequence. **8.** Since $m(\angle A) = 36°, m(\angle C) + m(\angle B) = 144°$, and $m(\angle C) = m(\angle B) = 72°$. Since $\triangle BCD$ is isosceles, $m(\angle CDB) = 72°$. Hence, $m(\angle DBC) = 180° - 144° = 36°$, and therefore $\triangle BCD$ is a golden triangle. **9. a.** $x = \sqrt{5} - 1 \approx 1.236$ **b.** The smaller rectangle is a golden rectangle if , and only if, $\frac{2}{\sqrt{5} - 1} = \frac{1 + \sqrt{5}}{2}$. This is true if $4 = (\sqrt{5} - 1)(\sqrt{5} + 1)$ which is true if $4 = 5 - 1$. Since $4 = 4$, the smaller rectangle is a golden rectangle and because the ratios of the side lengths are all $\frac{1 + \sqrt{5}}{2}$, it is similar to the original rectangle.

Open-Ended

10. Answers vary. For example, the figures have the same shape but not the same size. The corresponding angles are congruent and the corresponding sides are not equal in measure. The sides are proportional. **11.** Answers vary. For example, both the SSS theorems relate the lengths of the sides in one triangle to the lengths of the sides in the other triangle. The congruence theorem requires the sides to be congruent, and the similarity theorem requires the sides to be proportional. Both SAS theorems relate congruent angles and the sides forming these angles. The congruence theorem requires the corresponding sides to be congruent, and the similarity theorem requires the corresponding sides to be proportional. **12. a.** Answers vary depending upon the construction. **b.** The perimeters of the bases should be in the same ratio as the heights in part (a). **13.** Points lie along the same line because similar triangles are formed. The line goes through the origin.

Cooperative Learning

14. Answers vary. **15. a.** Answers vary. **b.** The following are two different-size triangles with the given data. The given measures of the angles were only approximate; such triangles with these exact measures do not exist. The triangles are similar but not congruent. (The ratio of the corresponding sides is $\frac{80}{100}$, or $\frac{4}{5}$.) Hence, the surveyor and the architect could both have been correct in their conclusions.



c. Answers vary.

Connecting Mathematics to the Classroom

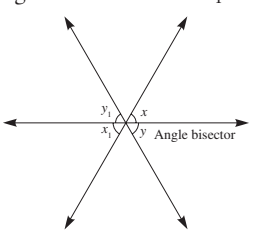
16. Al is correct. When Betty listed the similarity statement for the triangles, she did not list the corresponding vertices in the correct order for each triangle. **17.** Bob is correct and you should discuss why he is correct. Each right triangle has a right angle and these are congruent. In addition, there are two congruent acute angles so the triangles are similar by AA similarity.
18. The student is not correct and should be encouraged to try to find a counterexample. For example, suppose one triangle has angle measures of 40°, 70°, and 70° and another triangle has angle measures of 40°, 40°, and 100°. Two angles are congruent, but the triangles are not similar. **19.** No, they are not necessarily similar. For example, a rectangle that is not a square and a square have all angles congruent but are not similar. **20.** The student is incorrect. The corresponding sides are not proportional.
21. No, such a condition is not meaningful since at least two sides in one triangle and corresponding sides in the other are needed for a proportion. But ASA similarity would not be needed even if it made sense, because having two corresponding angles congruent already guarantees similarity.

GeoGebra Activities

22. See online appendix. **23.** See online appendix. **24.** See online appendix.

Review Problems

25. Construction. For example, start with the given base and construct the perpendicular bisector of the base. The vertex of the required triangle must be on that perpendicular bisector. Starting at the point where the perpendicular bisector intersects the base, mark on the perpendicular bisector a segment congruent to the given altitude. The endpoint of the segment not on the base is the vertex of the required isosceles triangle. **26.** Construction. For example, construct an equilateral triangle with the given side, then construct the perpendicular from any vertex to the opposite side. **27.** Answers vary. Students may suggest that angles of measure 45° be constructed with the endpoints of the hypotenuse as vertices of the 45° angles and the hypotenuse as one of the sides of the angles. Both angles need to be constructed on the same side of the hypotenuse. **28.** Because vertical angles are congruent, referring to the figure we have $x = x_1$ and $y = y_1$. Since $x = y$, we get $x_1 = y_1$. **29.** SASAS or ASASA



Chapter 12 Review

2. Parallelogram; $\triangle ADE \cong \triangle CBF$ by SAS, so $\angle DEA \cong \angle CFB$; $\angle DEA \cong \angle EAF$ (alternate interior angles between the parallels $\overline{DC}$ and $\overline{AB}$ and the transversal $\overline{AE}$), so $\angle EAF \cong \angle CFB$; and therefore $\overline{AE} \parallel FC$, $\overline{EC} \parallel \overline{AF}$ (parallel sides of the square); two pairs of parallel opposite sides implies a parallelogram. **7.** Find point O, the intersection of the perpendicular bisector of $\overline{AB}$ and line ℓ. The required circle has center O and radius OA.
11. a. $\frac{48}{32} = \frac{24}{16} = \frac{3}{2}$, so the rectangles are similar.
12. The triangles are the same shape, but they cannot be congruent since they are not the same size. The student is confusing congruence and similarity. Examples of both congruence and similarity should be given. **19. b.** Answers vary. $\overline{AD}$ and $\overline{BE}$ are diameters that bisect each other (at the center). Hence $ABDE$ is a rectangle and so $\angle ABD$ is a right angle. **20.** No. The statement is true if, and only if, the diagonals bisect each other.
21. Congruent triangles have two corresponding angles that are congruent. This makes them similar by AA similarity.
22. a. $\triangle BCN \cong \triangle DEN$ by ASA because $BN = ND$ (given), $\angle CBN \cong \angle EDN$ (alternate interior angles created by the parallels $\overline{BC}$ and $\overline{AD}$) and transversal $\overline{BD}$, and the angles with vertex N are vertical angles. **b.** From the congruence in part (a), it follows that $CN = EN$, and $AM = CM$ (given).
c. Because $\overline{MN}$ is a midsegment in $\triangle ACE$, $MN = \frac{1}{2}AE$.
Now $AE = AD - ED = AD - BC$ because $ED = BC$ (from the congruence in part (a)). Thus $AE = a - b$, and therefore $MN = \frac{1}{2}AE = \frac{1}{2}(a - b)$. **23.** $\angle 1 \cong \angle 2, \angle 3 \cong \angle 4$ (given), $\overline{AC} \cong \overline{AC}$, (reflexive), $\triangle ABC \cong \triangle ADC$ by ASA. Therefore $\overline{BC} \cong \overline{DC}$ by CPCTC. $\overline{CE} \cong \overline{CE}$ (reflexive) and therefore $\triangle BCE \cong \triangle DCE$ by SAS.

Answers to Now Try This

1. a. $\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F, \overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}, \overline{AC} \cong \overline{DF}$. **b.** $\triangle ABC \cong \triangle DEF, \triangle ACB \cong \triangle DFE, \triangle BAC \cong \triangle EDF, \triangle BCA \cong \triangle EFD, \triangle CAB \cong \triangle FDE, \triangle CBA \cong \triangle FED$. **2.** Construct a circle with center at P and any radius. Mark A and B as the points of intersection of the arc with ℓ. Now construct the perpendicular bisector of $\overline{AB}$. **4.** Answers vary for **(a)** and **(b)**, for example, the opposite angles are not supplementary. **5.** Yes. If $ABCD$ is a quadrilateral whose diagonals intersect at M, then congruent vertical angles with vertex at M are formed. Because the diagonals bisect each other, it follows from SAS that $\triangle AMD \cong \triangle CMB$. Hence, $\angle MAD \cong \angle MCB$ and therefore $\overline{BC}$ is parallel to $\overline{AD}$. In a similar way, we can show that $\overline{AB}$ is parallel to $\overline{DC}$ and hence $ABCD$ is a parallelogram. **6. a.** One way to accomplish the construction is to place the hypotenuse of the triangle on the given line and to place the ruler so that one of the legs of the triangle will be on the ruler. Now, keeping the ruler fixed, slide the triangle so that the side of the triangle on the ruler touches the ruler all the time. Slide the triangle on the ruler until the given point is on the hypotenuse. The line containing the hypotenuse is parallel to the given line. **b.** The angles formed by the hypotenuses and the ruler are congruent. Because they are corresponding angles, the hypotenuses are parallel. **7.** In Figure 12-50, if $PD = PE$ then by HL (Theorem 12-3), $\triangle APD \cong \triangle APE$ and hence $\angle PAD \cong \angle PAE$. **8.** Construct the diagonals of the square. Use the intersection of the diagonals as the center of the inscribed circle. The radius is the perpendicular distance from the center to any side. **9.** Using the result that the median to the hypotenuse in a right triangle is half as long as the hypotenuse (Example 9), we can prove that one of the angles in all such triangles is 60°. Thus, all the triangles are similar by AA similarity.

Answer to the Preliminary Problem

The distance along Ocean Drive bordering the lots is $40\,\text{m} + 34\,\text{m} + 30\,\text{m} = 104\,\text{m}$. Let x, y, and z be the ocean frontage for lots A, B, and C respectively. Using Theorem 12-15 we can set up the following proportions:

Lot A: $\dfrac{104}{130} = \dfrac{40}{x}$ and $x = 50\,\text{m}$.

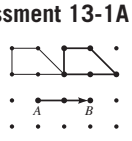
Lot B: $\dfrac{104}{130} = \dfrac{34}{y}$ and $y = 42.5\,\text{m}$.

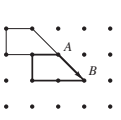
Lot C: $\dfrac{104}{130} = \dfrac{30}{z}$ and $z = 37.5\,\text{m}$.

Therefore Lot A has the most ocean frontage and hence the highest selling price.

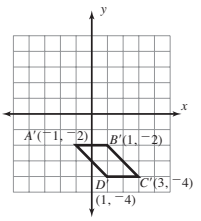
CHAPTER 13

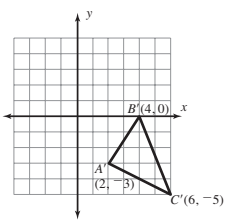
Assessment 13-1A

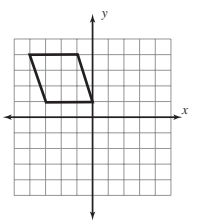
1. a. 

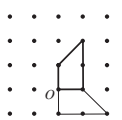
b. 

2. a. Construction **b.** Construct the images B' and C' of B and C respectively. $\overline{B'C'}$ is the required image.

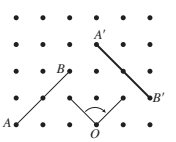
5. a. 

b. 

6. 

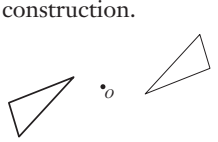
7. 

10. Choose any two points P and Q on ℓ, and construct their images P' and Q'. $\overrightarrow{P'Q'}$ is the desired image. **11.** *Hint:* Reverse the rotation (to the counterclockwise direction) to locate $\overline{AB}$, that is, the preimage.



12. Answers vary, but H, I, N, O, S, X, or Z could appear in such rotational words. Examples include SOS. Variations could use M and W in rotational images, for example, MOW.

13. c. ℓ and ℓ' are parallel. If P and Q are any points on ℓ and P' and Q' their respective images, then from the definition of a translation $\overline{PP'}$ and $\overline{QQ'}$ are parallel and congruent to $\overline{AB}$. Hence, $\overline{PP'}$ and $\overline{QQ'}$ are parallel and congruent. Thus, $PP'Q'Q$ is a parallelogram and therefore $\ell' \parallel \ell$. **14.** *Hint:* An angle whose measure is 45° can be constructed by bisecting a right angle. An angle whose measure is 60° can be constructed by first constructing an equilateral triangle. These angles can be used to complete the construction.

16. 

18. a. *Hint:* First rotate $\triangle ABC$ by angle α to obtain $\triangle A'B'C'$, and then rotate $\triangle A'B'C'$ by angle β to obtain $\triangle A''B''C''$. **c.** Yes, by rotation about O by angle $|\alpha - \beta|$ in the direction of the larger of α and β. **19. b.** The image of the point with coordinates (a, b) under a half-turn with the origin as center must be on a circle with center at $(0, 0)$ and passing through (a, b). It must also be on the line connecting $(0, 0)$ with (a, b). The equation of that line is $y = \dfrac{b}{a}x$. The only point that satisfies this equation and is at the same distance from the origin as (a, b) is $(^-a, ^-b)$. **21. a.** $B(h + a, k)$. **b.** Applying the Pythagorean Theorem in the triangle with hypotenuse $OA = a$ and legs of length h and k we get $h^2 + k^2 = a^2$. Then the product of the slopes of $\overline{OB}$ and $\overline{AC}$ is

$$\frac{k}{h + a} \cdot \frac{k}{h - a} = \frac{k^2}{h^2 - a^2} = \frac{k^2}{^-k^2} = {}^-1.$$

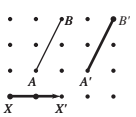
c. There is a turn symmetry of 180° about the point of intersection of the two diagonals. **22. b.** Through B construct $\overrightarrow{BE}$ so that $\overline{BE} \parallel \overline{CD}$, $BE = CD$, and $\overrightarrow{BE}$ points in the same direction as

$\overrightarrow{CD}$. Then $\overrightarrow{AE}$ is a slide arrow that corresponds to the required translation. **23. a.** Rotation by 90° or a multiple of 90° about the point of intersection of the diagonals. **b.** Rotation by 72° or a multiple of 72° about O, where O is the point of intersection of the perpendicular bisectors of the sides. **c.** Rotation by $\frac{360°}{n}$ and multiples of $\frac{360°}{n}$ about O, where O is the intersection of the perpendicular bisectors of the sides. **d.** Rotation about the center by any turn angle. **e.** No rotational symmetries **24. a.** *Hint:* Half-turn about the point of intersection of the diagonals. A rectangle has half-turn symmetry because the diagonals bisect each other. **b.** Same as (a) **25. a.** *Hint:* Find the image of the center and construct the circle with that image as the center and the same radius. **b.** The circles have a single point in common.

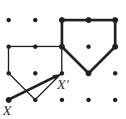
Assessment 13-1B

1. Reverse the translation so that the image completes a slide from X' to X (to its preimage). Then check by carrying out the given motion in the "forward" direction; that is, see if $\overline{AB}$ goes to $\overline{A'B'}$.

a.

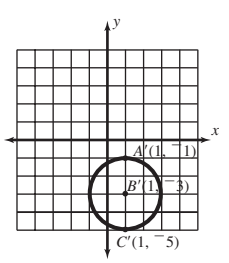


b.

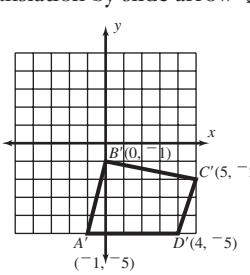


2. a. Construct as suggested by the following. Trace $\overline{B'C'}$ and the line containing the slide arrow on the tracing paper and label the trace of B' as B and the trace of C' as C. Mark on the original paper and on the tracing paper the initial point of the arrow by P and the head of the arrow by Q. Slide the tracing paper along the line $\overleftrightarrow{PQ}$ so that Q will fall on P. Trace $\overline{BC}$. The segment $\overline{BC}$ is the preimage of $\overline{B'C'}$ under the translation. **b.** *Hint:* Construct the images of B' and C' under the translation by slide arrow $\overrightarrow{QP}$.

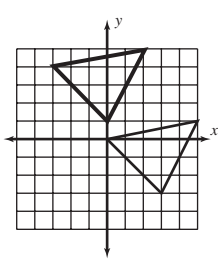
5. a.



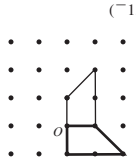
b.



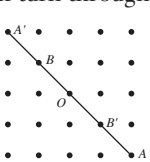
6.



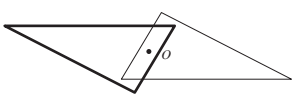
7.



9. P', P, and O are collinear because the measure of angle POP' must be 180°, and $OP = OP'$. **10.** Choose points P and Q on ℓ. Construct P' ($P' \neq P$) on $\overrightarrow{OP}$ such that $OP = OP'$. Similarly find Q'. $\overleftrightarrow{P'Q'}$ is the desired preimage. **11.** Locate $\overline{AB}$, the preimage, by completing a half turn through O.



12. MOM does not have this property. There is no half-turn that will make MOM be the image of MOM. **13. a.** When the figure is creased and folded along the perpendicular through O to $\overline{PP'}$, the point P falls on P'. This shows that the perpendicular also bisects $\overline{PP'}$. Alternatively, by the definition of rotation, $PO = P'O$. This implies that O is equidistant from P and P', and hence, it is on the perpendicular bisector of $\overline{PP'}$. **b.** From (a), O is on the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$ as well as $\overline{CC'}$. Consequently, O can be found by finding the point at which any two of the perpendicular bisectors intersect. $\angle AOA'$ is the angle of rotation which is 180°. **c.** The perpendicular bisectors of the segments $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are not concurrent. **14.** Construction method, vary. *Hint:* An angle whose measure is 30° can be constructed by first constructing an equilateral triangle and then bisecting one of its angles. **16.**

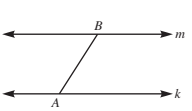


17. a. $\ell' \neq \ell$ and $\ell' \parallel \ell$ **b.** ℓ' and ℓ intersect at a 60° angle. **18. b.** The vertices A and B trace an identical path if, and only if, $OA = OB$; that is, if and only if, O is on the perpendicular bisector of $\overline{AB}$. Thus all points O for which two vertices trace an identical path are the points on the perpendicular bisectors of the sides of the triangle. **c.** Yes, the intersection of the perpendicular bisectors of the sides (center of the circumscribed circle) **21.** O must be the midpoint of the segments $\overline{AA'}$ and $\overline{BB'}$. A' must lie on the x-axis and have coordinates $(a, 0)$; B' must lie on the y-axis and have coordinates $(0, {}^{-}b)$. **22. b.** A translation from A to C **23. b.** Turn symmetry about the intersection of perpendicular bisectors by 60° or a multiple of 60° in either direction **c.** Turn symmetry by 90° or a multiple of 90° about the intersection of the diagonals in either direction **e.** Half-turn about the center of the circle **25.** The circles have the same radius and only one point P in common. **26.** Translating the figure by $\overrightarrow{AB}$ or by $\overrightarrow{BA}$ results in the same figure.

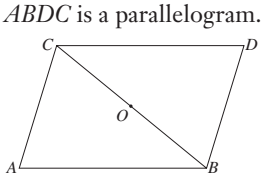
Mathematical Connections 13-1

Reasoning

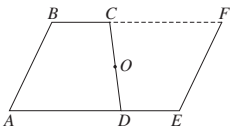
1. If the congruent segments are $\overline{AB}$ and $\overline{CD}$, find $\overline{C'D'}$, the image of $\overline{CD}$ under the translation from C to A. Then rotate $\overline{C'D'}$ about A by $\angle BAD'$ in an appropriate direction to make the image of D' be B. **2.** Answers vary. **a.** Apply a half-turn about any point on the line that is halfway between the given lines. Alternatively, the center can be determined by the midpoint of a segment whose endpoints are on the lines. **b.** If $k \parallel m$, A is any point on k and B is on m, then the image of k under the translation by $\overrightarrow{AB}$ is m.



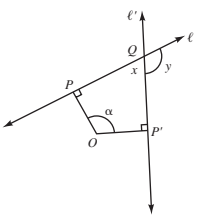
3. $(2 - h, {}^{-}k)$ One way to see why, is to notice that the y-coordinate of the image point is ${}^{-}k$. Substituting $y = {}^{-}k$ in the equation of the line through $(1, 0)$ and (h, k) we get $x = 2 - h$. **4. a.** A parallelogram with one diagonal; under a half-turn, the image of a line is parallel to the line. Thus, $\overline{AB} \parallel \overline{CD}$ and $\overline{AC} \parallel \overline{DB}$; therefore, $ABDC$ is a parallelogram.



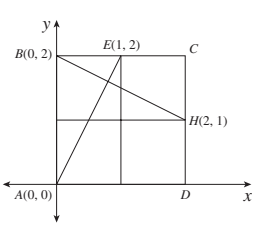
b. A parallelogram with an extra segment; the image of $\overline{AB}$ is $\overline{FE}$ and thus $\overline{AB} \parallel \overline{EF}$. Consequently, $\overline{BF} \parallel \overline{AE}$ and $ABFE$ is a parallelogram.



c. A square; because the image of A is C and the image of B is D, the image of $ABCD$ is $CDAB$. **5.** Since P is on ℓ, its image P' is on ℓ'. Rotation preserves angle measure. Because the image of $\overline{OP}$ is $\overline{OP'}$ and $\overline{OP}$ is perpendicular to ℓ, the image of ℓ must be perpendicular to $\overline{OP'}$. Since the line k is perpendicular to ℓ, by uniqueness of perpendicular to $\overline{OP'}$ at P', k must be the image of ℓ. **6. a.** *Hint:* Use the figure below for $\propto = 90°$. Then $x = 90°$ and $y = 90°$. **b.** Because $OPQP'$ has two right angles we have $x + \alpha = 180°$. Also $x + y = 180°$, and hence $y = \alpha$.



7. Rotate ΔCAF by 90° counterclockwise about A. The image of C is E, the image of F is B, and the image of A is A. Hence the image of $\overline{CF}$ is $\overline{EB}$. Because a rotation preserves distance $CF = EB$. From problem 6 we know that the angle between $\overline{CF}$ and its image equals the angle of rotation, which is 90°. Thus, the segments are perpendicular. **8. a.** Rotate $\overline{BH}$ about O by 90° counterclockwise. The image of H is E since the angles at O are 90°. The image of B is A because $\angle BOA$ is a right angle. Hence the image of $\overline{BH}$ is $\overline{EA}$. Consequently, the segments are congruent and by the same argument as in the answer to problem 7, they are perpendicular. **b.** We can create a coordinate system with $\overleftrightarrow{AD}$ as the x-axis, and $\overleftrightarrow{AB}$ as the y-axis and the origin at A. If BE is a unit then $A(0, 0)$, $E(1, 2)$ $B(0, 2)$ and $H(2, 1)$. Hence $m_{AE} = \dfrac{2 - 0}{1 - 0} = 2$, and $m_{BH} = \dfrac{1 - 2}{2 - 0} = \dfrac{^{-}1}{2}$. Since the product of the slopes is $^{-}1$, the segments are perpendicular.

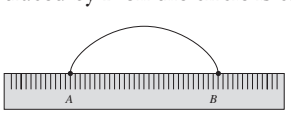


Open-Ended

9. Answers vary. **10.** Answers vary. **11.** Answers vary. If the rotation is by 30°, there will be $\dfrac{360}{30}$, or 12 images.

Cooperative Learning

12. a. The path will look like the one shown in the following figure. Such a path traced by P on the circle is called a *cycloid*.



b. The path is not an arc of a circle. The perpendicular bisectors of all the chords (segments connecting two points on the arc) do not intersect in a single point. **c.** The length of $\overline{AB}$ is the circumference of the circle.

Connecting Mathematics to the Classroom

13. Yes, the student is correct because the two described components could be constructed as components of the given translation. An analogy is the vertical and horizontal component of a vector. **14.** The result of a 180° rotation clockwise or counterclockwise is the same, so no direction is needed. **15.** This is true for circles but not for all figures. For example, the image of $\triangle ABC$ with a right angle at C, under half turn about C, cannot be obtained by a translation. **16.** Alicia needs to find the image of all the points on $\overrightarrow{AC}$, which means that she needs to find the image of $\overrightarrow{AC}$ under a half-turn in P. The image of a "good" point on $\overrightarrow{AC}$ must be on $\overrightarrow{AB}$ and on the image of $\overrightarrow{AC}$. Thus X is the intersection of $\overrightarrow{AB}$ with the image of $\overrightarrow{AC}$. Point Y can be found as the preimage of X under half-turn about P or by simply intersecting $\overrightarrow{XP}$ with $\overrightarrow{AC}$.

GeoGebra Activities

17. (6) Construction **(7) (a)** Congruent **(b)** Yes **(c)** No **(8) (a)** The segments are congruent. **(b)** Measurements **(c)** Yes
18. 9–11 Constructions;
- The object and its rotation image have the same orientation.
- The object and image are congruent and the object and its image are the same figure.

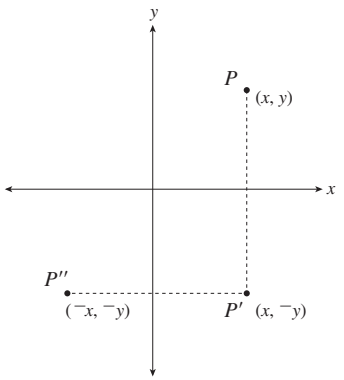
19. An equilateral triangle has 120° rotational symmetry. Other triangles do not have this property.

Assessment 13-2A

1. *Construction. Hint:* Locate the images of vertices directly across (perpendicular to) ℓ on the geoboard. **2. b.** Perpendicular bisector of the segment or the line containing the segment **d.** Perpendicular bisectors of the sides and lines containing the diagonals **h.** If the kite is not a rhombus, the line containing the diagonal determined by vertices of the noncongruent angles is a line of symmetry. If a rhombus then each line containing a diagonal is a line of symmetry **i.** Perpendicular bisectors of parallel sides and three lines containing diameters determined by vertices on the circumscribed circle **4.** The image is a circle congruent to the original circle with a different center. **5. a.** The final images are congruent but in different locations, and hence not the same. **b.** A translation by a slide arrow from P to R is determined as follows: Let P be any point on ℓ and Q on m such that $\overline{PQ} \perp \ell$. Point R is on $\overrightarrow{PQ}$ such that $PQ = QR$. **6.** Constructions vary, but final image is a translation of the original. **7.** Answers vary. The line of reflection is the perpendicular bisector of $\overline{BB'}$. **8. a.** Examples include MOM, WOW, TOOT, and HAH. **b.** Examples include BOX, HIKE, CODE, etc. B, C, D, E, H, I, K (depending on the font), O, and X may be used. **c.** 1, 8, 11, 88, 101, 111, 181, 808, 818, 888, 1001, 1111, 1881 **10. a.** The images are the same. **b.** They commute. **11.** None of the images has a reverse orientation, so there are no reflections or glide reflections involved. Thus,

1 to 2 is a rotation.
1 to 3 is a rotation.
1 to 4 is a translation down.
1 to 5 is a rotation (with an exterior point as the center of rotation).
1 to 6 is a translation.
1 to 7 is a translation.

13. b. $(^{-}x, ^{-}y)$; the following figure:



16. a. The line through P perpendicular to $\overleftrightarrow{O_1O_2}$ or the perpendicular bisector of QQ', where Q and Q' are the points of intersection other than P of $\overleftrightarrow{O_1O_2}$ with circle O_2 and circle O_1, respectively **19. a.** Glide reflection determined by slide arrow $\overrightarrow{BA}$ and line ℓ. **b.** Translation by a $\overrightarrow{AC}$, where A, B, and C are collinear and B is the midpoint of $\overline{AC}$.

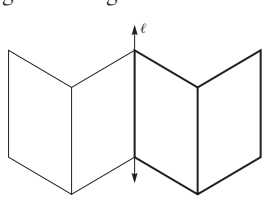
Assessment 13-2B

1. Locate the images of the vertices directly across (perpendicular to) ℓ on the geoboard.



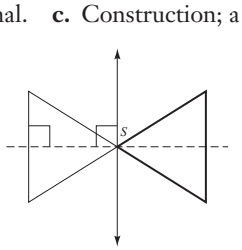
2. a. The perpendicular bisector of the chord connecting the endpoints of the arc **b.** The line containing the ray **c.** The lines bisecting the vertical angles formed by the perpendicular lines; and the lines themselves **d.** Perpendicular bisectors of pairs of parallel sides; and diagonals if a square **e.** If not equilateral, perpendicular bisector of the side that is not congruent to the other two; if equilateral, the perpendicular bisectors of each side **f.** Perpendicular bisector of parallel sides if there is exactly one pair of parallel sides and they are not congruent **g.** The lines containing the diagonals and lines bisecting parallel sides if a square **h.** There will be n reflecting lines in all. If n is even, the lines contain the perpendicular bisectors of the parallel sides or the lines containing the diameters of the circumscribed circle containing the vertices. If n is odd, the lines are the ones containing the perpendicular bisectors of the sides. **3.** The image is the original triangle.

4.



5. a. Constructions vary, but the final image is a rotation of this original. **c.** Construction; a half-turn about O **6.** Construction

7. a.



b. The answer in part (a) uses a line perpendicular to an altitude of the original triangle and contains the vertex of that altitude. Depending on the type of triangle, the "bow-tie" may be skewed, but the two "bases" are parallel. **c.** There are three reflecting lines possible. Two (other than the one used in part (a) could be similarly found at the other vertices. **8.** The reflecting line is shown below.

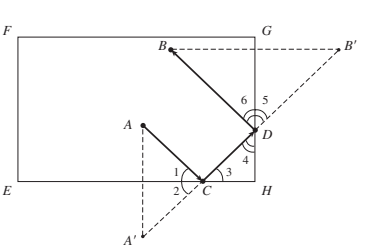


10. a. Answers may vary but no more than 3 are needed. **b.** It cannot be a translation or rotation because the orientation is reversed. It cannot be a reflection because there is no single reflecting line; the perpendicular bisectors of $\overline{YY'}$ and $\overline{ZZ'}$ are not the same. **11. a.** Answers vary. Part (a) could be moved to part (b) by a reflection in the line through the common point of the parts and containing a diagonal of the large square. Also a rotation about the center of the large square by 90° counterclockwise can be used. **b.** Answers vary. A rotation of 90° clockwise about the center of the large square will work. **13. a.** $(^{-}x, y)$ **b.** $(^{-}y, ^{-}x)$ **15. a.** $y = ^{-}3x$ **b.** $y = x$ **c.** $x = 0$ **17.** The diameter of the circle must be congruent to the side of the square. The line of reflection is the perpendicular bisector of the segment connecting the center of the circle with the point of intersection of the diagonals of the square. **19. a.** Glide reflection determined by slide arrow $\overrightarrow{BA}$ and line $\overleftrightarrow{AB}$ **b.** Translation by a $\overrightarrow{AC}$, where A, B, and C are collinear and B is the midpoint of $\overline{AC}$.

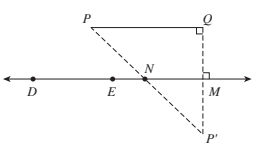
Mathematical Connections 13-2

Reasoning

1. a. If $\overline{AB} \cong \overline{BC}$, then the perpendicular bisector of $\overline{AC}$ is the required line. The image of B is B, the image of A is C, and the image of C is A. Hence the image of $\triangle ABC$ is $\triangle CBA$. **b.** Equilateral triangles; each side could be considered a base. **c.** No **d.** All lines containing diameters will satisfy this situation. Diameters divide a circle into two congruent semicircles. **2.** Answers vary. Find A', the image of A under reflection in $\overline{EH}$, and B', the image of B under reflection in $\overline{GH}$. Mark the intersections of $\overline{A'B'}$ with $\overline{EH}$ and $\overline{GH}$, respectively, by C and D. The player should aim the ball at A toward the point C. The ball will hit C, bounce off D, then bounce off and hit B. To justify the answer, we need to show that the path A–C–D–B is such that $\angle 1 \cong \angle 3$ and $\angle 4 \cong \angle 6$. Notice that $\angle 1 \cong \angle 2$ and $\angle 6 \cong \angle 5$ because the image of an angle under reflection is congruent to the original angle. Also, $\angle 2 \cong \angle 3$ and $\angle 4 \cong \angle 5$, as each pair constitutes vertical angles. Consequently, $\angle 1 \cong \angle 3$ and $\angle 4 \cong \angle 6$.



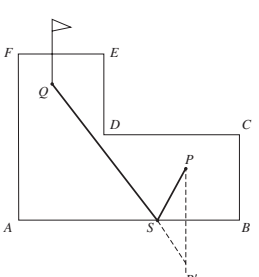
3. a. By the midsegment theorem $\overline{MN} \parallel \overline{BC}$. Hence $\overline{MN}$ bisects the altitude of $\triangle ABC$ from A to $\overline{BC}$ and is perpendicular to $\overline{MN}$. **b.** Since $\overline{AA'}$ is perpendicular to $\overline{BC}$, $\overline{MD}$ is a $\overleftrightarrow{MD}$, its image is A'. Similar argument holds when B is reflected in $\overleftrightarrow{NE}$. **c.** Reflections are isometries that preserve angle measure. **d.** The sum of the measures of the interior angles in a triangle is 180°. The reason is that the angles at A' add up to a straight angle. **4.** Let M be the midpoint of $\overline{AB}$ and N the midpoint of $\overline{DC}$. The image of $\overline{AC}$ under reflection in $\overleftrightarrow{MN}$ is $\overline{BD}$ because the image of A is B and the image of C is D. **5.** The angle of incidence is the same as the angle of reflection. With the mirrors tilted 45°, the object's image reflects to 90° down the tube and then 90° to the eyepiece. The two reflections "counteract" each other, leaving the image upright. **6. a.** This can be checked with geometry utility or manually. **b.** Because the angles at Q and M are right angles, $\overline{MN} \parallel \overline{PQ}$. By the midsegment theorem, since M is the midpoint of $\overline{QP'}$, N is the midpoint of $\overline{PP'}$ (see figure below). **c.** Connect A with A' and B with B'. Mark the respective midpoints as F and H. Then $\overleftrightarrow{FH}$ is the line of reflection. Next construct A_1 the image of A in the line of reflection. $A_1\overrightarrow{A'}$ is the slide arrow.



7. a. $X'P = X'P'$ and $Y'P = Y'P''$ because a reflection preserves distance. **b.** $XP = XP'$ and $YP = YP''$ because a reflection preserves distance. P', X, Y and P'' are collinear by construction. **c.** From part (b) it follows that the perimeter of $\triangle PXY$ equals $P'P''$. Similarly the perimeter of $\triangle X'PY'$ equals $P'X' + X'Y' + Y'P''$. Because the shortest path connecting P' with P'' is $\overline{P'P''}$, $P'P'' < P'X' + X'Y' + Y'P''$. Hence the perimeter of $\triangle PXY$ is less than the perimeter of $\triangle X'PY'$.

Open-Ended

8. Answers vary. **a.** Reflect P, the location of the ball in $\overline{AB}$. The intersection of $\overline{P'Q}$ with $\overline{AB}$ is the point at which the ball should be aimed.



b. Reflect P in $\overline{CB}$ and Q in $\overline{AB}$. Connect the images P' and Q' by a straight line. The intersection of $\overline{P'Q'}$ with $\overline{BC}$ is the point at which the ball should be aimed. **9.** Answers vary. **10.** Answers vary.

Cooperative Learning

11. a. Constructions will vary. **b.** The experiment will work for rectangular tables in which the length is twice the width and for any position of B.

Connecting Mathematics to the Classroom

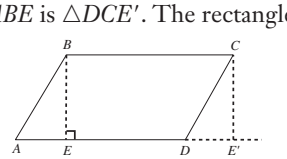
12. No, with a single point and its image, any of the described motions could work. **13.** Having only a segment and its image is not enough to determine the transformation. **14.** Sammi is incorrect. To determine if the motions are equivalent, one has to consider the entire plane. The orientations are different and therefore the isometries cannot be equal. **15.** The figures drawn have additional reflection symmetries in lines perpendicular to the y-axis through A and through N, respectively. Figures that don't have such lines of symmetry will not have a half-turn symmetry. A scalene triangle is an example.

GeoGebra Activities

16. (5) (a) The measure of an angle and the measure of its reflected angle are equal. **(b)** The length of a segment and the length of its reflected segment are equal. **(c)** A polygon and its reflection are congruent. **(d)** The orientations of an object and its reflection are opposite. **(e)** The reflecting line contains the midpoint of each segment joining a point and its reflection. **17. (1)** (**i**) It might be called a "slide" reflection because it is a composition of a "slide" and a reflection. Glide is used instead of "slide." (**j**) They lie on the reflecting line. **18. a.** Construction **b.** The line desired, q, is such that the angle formed by lines m and n in that order is the same as the angle formed by the lines q and p in that order.

Review Problems

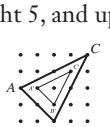
19. Depending on how they are drawn: 0, 1, and 8 **20.** $(^{-}a, ^{-}b)$ **21.** A rotation by any angle about the center of the circle will result in the same circle. **22.** Construct $\overline{BE}$ perpendicular to $\overline{AD}$, as shown. Next translate $\triangle ABE$ by the slide arrow from B to C. The image of $\triangle ABE$ is $\triangle DCE'$. The rectangle $BCE'E$ is the required rectangle.



Assessment 13-3A

1. Answers vary. **a.** Slide the smaller triangle down 3 units (translation). Then complete a dilation with scale factor 2 using the top-right vertex as the center. **b.** Slide the smaller triangle right 5, and up 1. Then complete the dilation as in (a).

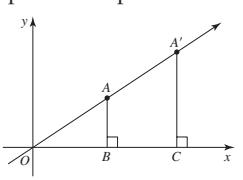
2.



3. Answers vary. **a.** Translation taking B to B' followed by a dilation with center B' (and scale factor 2) **b.** Dilation with scale factor $\frac{1}{2}$ and center A followed by a half-turn with the midpoint of $\overline{C'B'}$ as center. **4.** $x = 6$; $y = 5.2$; scale factor $\frac{2}{5}$

7. Dilation with center O and scale factor $\frac{1}{r}$

8. a. Answers vary. A possible explanation follows:



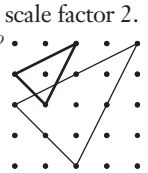
Let $A'(x', y')$ be the image of $A(x, y)$ under the dilation. From the definition of a dilation, we have $\frac{OA'}{OA} = r$. Since $\triangle OA'C \sim \triangle OAB$, $\frac{OC}{OB} = r$, which implies that $\frac{x'}{x} = r$ and hence $x' = rx$. Similarly, $y' = ry$. **b.** Answers vary; the dilation with center at the origin and scale factor 2 followed by a half-turn about the origin. **9.** Answers vary; a translation taking O_1 to O_2 followed by a dilation with center at O_2 and scale factor $\frac{3}{2}$.

10. The set of images of the set of integers on a number line is the set of points where coordinates are multiples of 3, that is, $\{\ldots, {}^{-}6, {}^{-}3, 0, 3, 6, 9, \ldots\}$. **11.** $x = \frac{98}{15}$; $y = \frac{90}{7}$; scale factor $= \frac{7}{15}$ **12.** The enlargement of a $2'' \times 3''$ photograph to a $4'' \times 6''$ photograph can be achieved by a dilation with scale factor 2 and any center.

Assessment 13-3B

1. Answers vary. **a.** Rotate the smaller triangle 90° counterclockwise with the lower-right vertex of the smaller triangle as the center of rotation. Then complete a dilation with scale factor 2 using the same point as center. **b.** Assume the lower-left corner of the grid has coordinates $(0, 0)$. Translate the smaller triangle using $(x, y) \rightarrow (x - 5, y - 1)$. Next use a dilation with $(1, 2)$ as center and scale factor 2.

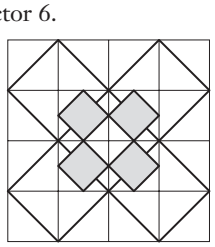
2.



3. Answers vary. **a.** Rotate the larger triangle 90° counterclockwise using center B; translate to take B to B'; then complete a dilation with scale factor $\frac{1}{2}$ and center B'. **b.** Half-turn about C followed by a dilation with center C and scale factor $\frac{3}{2}$

6. c. Answers depend on the order. **7.** Dilation with center at the origin and scale factor $\frac{3}{4}$ **8.** Answers vary; a translation taking O_1 to O_2 followed by a dilation with center O_2 and scale factor 6.

9.

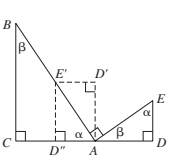


10. The scale factor $= \frac{A'B'}{AB} = \frac{3}{4}$. The center must be the intersection of $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ **12.** This is not possible because $\frac{4.5}{2} \neq \frac{6.5}{3}$.

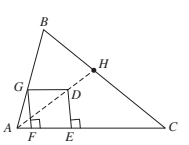
Mathematical Connections 13-3

Reasoning

1. a. It does change unless the scale factor is 1. For example, consider the segment whose endpoints are $(0, 0)$ and $(1, 1)$, which has length $\sqrt{2}$. Under the dilation with center at $(0, 0)$ and scale factor 2, the image of the segment is a segment whose endpoints are $(0, 0)$ and $(2, 2)$. That segment has length $2\sqrt{2}$. **b.** It does not change. Under a dilation, the image of a triangle is a similar triangle and the corresponding angles of two similar triangles are congruent. **c.** It does not change. Given two parallel lines, draw a transversal that intersects each line. Because the lines are parallel, the corresponding angles are congruent. From (b), the images of the angles will also be congruent and hence the image lines will be parallel. **2.** The transformation can be achieved by a reflection followed by a dilation. Indeed $(x, y) \rightarrow (x, {}^{-}y)$; a reflection in the x-axis followed by $(x, {}^{-}y) \rightarrow (3x, {}^{-}3y)$; a dilation with center at the origin and scale factor 3, take (x, y) to $(3x, {}^{-}3y)$. Thus, the image of any figure will be similar to the figure. **3. a.** A single dilation with center O and scale factor $\frac{1}{2} \cdot \frac{1}{3}$ or $\frac{1}{6}$. Let P be any point and P' its image under the first dilation and P'' the image of P' under the second dilation. Then $\frac{OP'}{OP} = \frac{1}{2}$ and $\frac{OP''}{OP'} = \frac{1}{3}$. Consequently, $\left(\frac{OP'}{OP}\right) \cdot \left(\frac{OP''}{OP'}\right) = \frac{1}{2} \cdot \frac{1}{3}$, or $\frac{OP''}{OP} = \frac{1}{6}$. Thus, P' can be obtained from P by a dilation with center O and scale factor $\frac{1}{2} \cdot \frac{1}{3}$ or $\frac{1}{6}$. **b.** Dilation with center O and scale factor $r_1 r_2$ **4. a.** Yes. Suppose the dilation with center O has a scale factor r. The image of any point P on the circle with radius x is P' such that $\frac{OP'}{OP} = r$. Thus, $OP' = r(OP)$ or $OP' = rx$. This means that the image of every point on the circle is at the same distance rx from O and hence on a circle with radius rx. **b.** Let C be the center of the circle with radius x and let P be any point on the circle. Let C' and P' be the respective images of C and P under a dilation with center O and scale factor r. Because the image of $\overline{CP}$ is $\overline{C'P'}$, it follows that $C'P' = rCP = rx$. This means that all the images of points on the circle are at the same distance rx from the point C'. Hence, the image of the circle is a circle with center C' and radius rx. **5. a.** Answers vary. One possibility is to rotate $\triangle AED$ about A by 90° counterclockwise and get the image as $\triangle AE'D'$. Then find the image $AE'D''$ of $\triangle AE'D'$ under a half-turn in m, the midpoint of $\overline{AE'}$. Finally apply the dilation with center A and scale factor $\frac{CA}{D''A}$

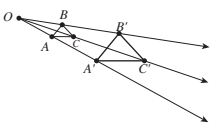


b. Answers vary. For example, dilation with center A and scale factor AB, followed by a rotation by 90° counterclockwise $\overline{AE}$ about A, followed by a half-turn about the midpoint of $\overline{AB}$.
6. Construct a square like the one shown. Construct $\overrightarrow{AD}$. The intersection of $\overrightarrow{AD}$ with $\overline{BC}$ is vertex H of the required square. Construct the other vertices by drawing lines parallel and perpendicular to $\overline{AC}$.



7. Let the center of the smaller circle be O and its radius r. The center of the second circle m and radius R. Translate the smaller circle by slide arrow $\overrightarrow{OM}$ and then apply the dilation with

center m and scale factor $\frac{R}{r}$. The image of the smaller circle will be the larger one. **8.** The original triangle cannot be uniquely determined because the scale factor is not known. There are infinitely many possible preimages of $\triangle A'B'C'$. We can choose point A anywhere on $\overrightarrow{OA'}$, construct through A a line parallel to $\overline{A'B'}$, and mark B as the intersection of that parallel with $\overrightarrow{OB'}$. Similarly, point C is determined as the intersection of $\overrightarrow{OC'}$ with a line through B parallel to $\overline{B'C'}$. The dilation with center O and scale factor $\frac{OB'}{OB}$ will take ΔABC onto $\Delta A'B'C'$.



Open-Ended

9. Answers vary; for example, a 50% reduction in size when making a photocopy. **10.** Answers vary.

Cooperative Learning

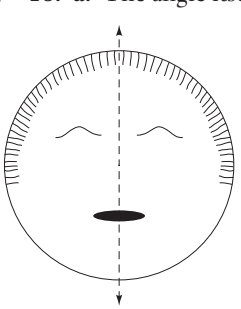
11. a. If p is the perimeter of the figure and p' the perimeter of the image, then $p' = 3p$. **b.** $p' = rp$ **c.** Answers vary.

Connecting Mathematics to the Classroom

12. If two triangles are not similar, then they do not have the same shape and never would, even if the size is changed. It is not possible to transform one onto the other using isometries and a dilation.
13. The student is incorrect. In the original grid, a square might have area 1 sq. unit but in the image grid, that same square has area 4 sq. units. The student is correct that the image is a grid, but it is not the same size.

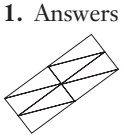
Review Problems

14. a. The translation determined by slide arrow from N to M **b.** A counterclockwise rotation of 75° about O **c.** A clockwise rotation of 45° about A **d.** A reflection in m and translation from B to A **e.** A second reflection in n **15. a.** $(4, 3)$ reflects about m to $(4, 1)$; $(4, 1)$ reflects about n to $(2, 1)$.
b. $(0, 1) \rightarrow (0, 3) \rightarrow (6, 3)$ **c.** $(^{-}1, 0) \rightarrow (^{-}1, 4) \rightarrow (7, 4)$
d. $(0, 0) \rightarrow (0, 4) \rightarrow (6, 4)$ **16. a.** The angle itself
b. The square itself **17.**

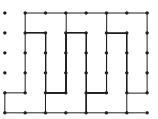


Assessment 13-4A

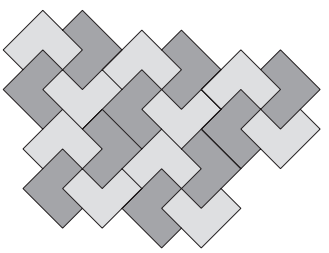
1. Answers vary.



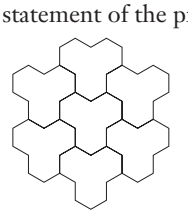
2. a. *Hint:* Perform half-turns about the midpoints of all sides.
b. Yes. Successive 180° turns of a quadrilateral about the midpoints of its sides will produce four congruent quadrilaterals around a common vertex, with each of the quadrilateral's angles being represented at each vertex.
3. Answers vary.



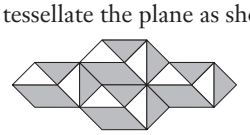
4. Answers vary. *Hint:* Consider figures like a pentagon formed by combining a square and an equilateral triangle. **5.** The shape will tessellate a plane as seen below:



7. a. The dual is another tessellation of squares (congruent to those given). **b.** The tessellation of equilateral triangles; it is illustrated in the statement of the problem.
8. Answers vary.

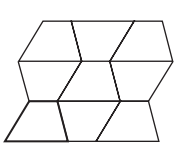


9. The hexagon will tessellate the plane as shown:

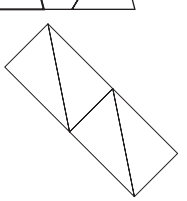


Assessment 13-4B

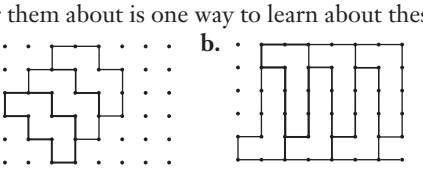
1. Answers vary.



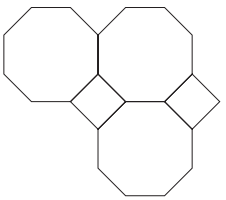
2. Answers vary. **a.**



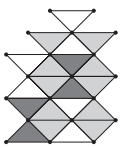
b. Yes; any triangle and one congruent to it can form a parallelogram and any parallelogram tessellates a plane.
3. Answers vary. Experimentation by cutting out shapes and moving them about is one way to learn about these types of problems.
a. **b.**



4. Answers vary.

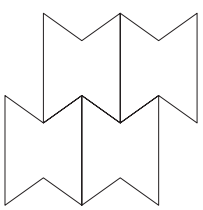


5. The bow-tie figure can tessellate the plane as shown:



6. Answers vary. **7. (17)** A translation moves A to B. **(18)** A translation moves C to D; a rotation moves C to E.

8.

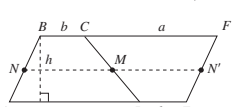


9. The arrow cannot tessellate the plane; the "head" fits the "tail," but there are gaps elsewhere.

Mathematical Connections 13-4

Reasoning

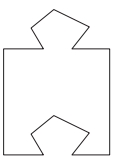
1. a. To tessellate the plane with $\triangle ABC$, we find $\triangle A'CB$—the image of $\triangle ABC$ under a half-turn about M, the midpoint of $\overline{BC}$. If N is the midpoint of $\overline{AB}$, then the image of N is N'. It can be shown that N, M, and N' are collinear and hence that $ANN'C$ is a parallelogram. Because the image of $\overline{NM}$ is $\overline{MN'}$, it follows that $NM = MN'$. Hence, $NM = \frac{1}{2}NN' = \frac{1}{2}AC$. Thus, $NM = \frac{1}{2}AC$. Also $\overline{NM} \parallel \overline{AC}$. **b.** The image of $ABCD$ under a half-turn about M (the midpoint of CD) is the trapezoid $FEDC$. Because the trapezoids are congruent, $ABFE$ is a parallelogram. If N is the midpoint of $\overline{AB}$, then its image N' is the midpoint of $\overline{EF}$. It can be shown that N, M, N' are collinear. Because $NBFN'$ is a parallelogram ($\overline{BN}$ and $\overline{FN'}$ are congruent and parallel), $NN' = BF = a + b$. Since $NM = N'M$, $NM = \frac{a+b}{2}$.



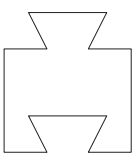
2. See page 799. **3.** First tessellate the plane with regular hexagons and then divide each hexagon into congruent pentagons by three segments connecting the center of a hexagon to the midpoint of every other side. The segments inside one hexagon should not have points in common with segments inside a neighboring hexagon. Actually, one could draw the Y-shape segments in one hexagon and then apply a translation, taking the center of one hexagon to the center of a neighboring one. **4. a.** Under a half-turn the image of a line is a parallel line. **b.** Answers vary. For example, half-turns.

Open-Ended

5. Answers vary. In the given design, the tessellation can be obtained by half-turns followed by translations. **6.** Answers vary. **7.** Answers vary. For example, a right rectangular prism will tesselate space, but a cylinder or a tetrahedron will not. **8.** Answers vary. For example, we can cut out part of a square from the square

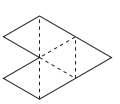


9. Answers vary. For example, an equilateral triangle using half-turns in a midpoint of a side and translation by a vector parallel to a side and congruent to the side. **10.** Answers vary. For example,



Cooperative Learning

11. Answers vary. **12. a.** Two such figures can be put together to form a parallelogram. Because parallelograms tessellate the plane, the original figure tessellates the plane. **b.** There are 12 such figures. Two figures are different if they are not similar. **c.** Answers vary. For example the following figure is a rep-tile:



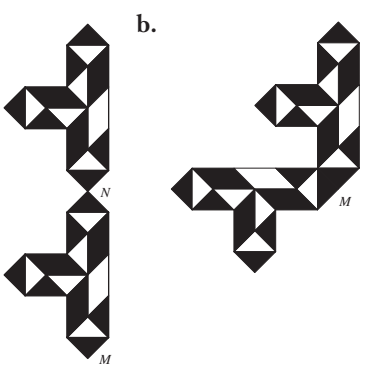
13. All will tessellate. Answers vary. **14.** See page 800.

Connecting Mathematics to the Classroom

15. Isaiah is basically correct. The mortar joints may alter the appearance, but this is comparable to a rectangle tiling a plane. **16.** Answers vary. Whether the image tessellates a plane depends on the kaleidoscope. **17.** Each interior angle of a regular pentagon is 108°. If one puts three pentagons around a vertex, one gets 324°. What is left over is 36° and cannot be an angle of another pentagon.

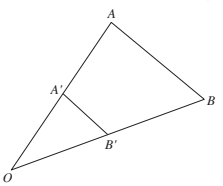
Review Questions

18. a. **b.**



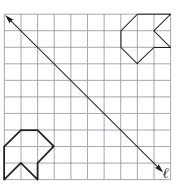
19. a. 90°, 180°, 270° rotation about center **b.** 180° rotation about the center (half-turn about that center); reflections across vertical and horizontal lines through the center

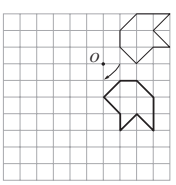
20. 180° rotation about the center of the dark-colored square **21. a.** It is a square. **b.** The respective images are $(0, a), (a, 0), (0, {}^-a), ({}^-a, 0)$. **22.** If O is on $\overleftrightarrow{AB}$, then $\overline{A'B'}$ is on $\overleftrightarrow{AB}$ and the segments are parallel. If O is not on $\overline{AB}$ as shown, then $OA' = rOA$, $OB' = rOB$, and $\angle O \cong \angle O$, making $\triangle OAB \sim \triangle OA'B'$. Thus, $\angle OA'B' \cong \angle OAB$. These are corresponding angles of $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ when cut by transversal $\overleftrightarrow{OA}$. Thus $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$, implying $\overline{AB} \parallel \overline{A'B'}$.

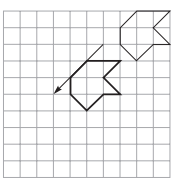


23. Such dilation does not exist because the image of a line is a parallel line (problem 22) but the lines intersect at $(0, 0)$.

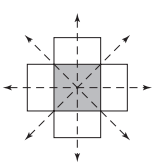
Chapter 13 Review

1. a.


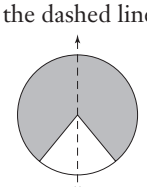
b.


c.


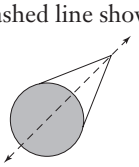
3. a. 4 reflections through dashed lines as shown; also rotations of 90°, 180°, and 270° about a center at the intersection of the dashed lines.



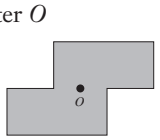
b. One reflection in the dashed line shown



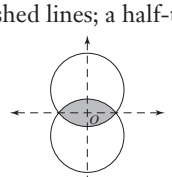
c. One reflection in the dashed line shown



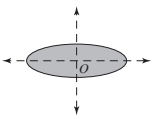
d. A half-turn about center O



e. Two reflections in dashed lines; a half-turn about center O

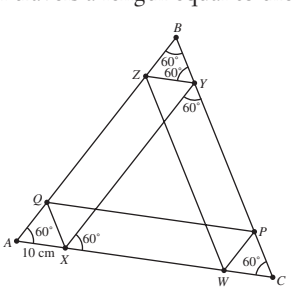


f. Two reflections in dashed lines; a half-turn about center O



8. Answers vary. Let $\triangle H'O'R'$ be the image of $\triangle HOR$ under a half-turn about R. Then $\triangle SER$ is the image of $\triangle H'O'R'$ under a dilation with center R and scale factor $\frac{2}{3}$. Thus, $\triangle SER$ is the image of $\triangle HOR$ under the half-turn about R followed by the dilation described above. **9.** Answers vary. Rotate $\triangle PIG$ 180° (half-turn) about the midpoint of $\overline{PT}$, then perform a dilation with scale factor 2 and center $P' (= T)$. **11.** $(x, y) \rightarrow (x, y)$. It is the identity transformation with every point its own image.
13. b. A rotation about O by 60° counterclockwise.
c. i. and **ii.** A dilation with center O and scale factor 6.
17. An interior angle of a regular octagon has measure 135°. Two together have measure 270°; three together have measure more than 360°. Octagons cannot fit together at a point without overlapping or leaving a gap. **19. a.** It will tessellate. **b.** It will tessellate. **c.** It will not tessellate. **20. a.** $({}^-5, 0), (6, 0)$, or $\left(\frac{25}{6}, 0\right)$ **b.** $({}^-2, 4), (3, {}^-4)$, or $\left(\frac{{}^-7}{6}, 4\right)$ **c.** Answers vary. $(x, y) \rightarrow (x + a, y)$; $(x, y) \rightarrow (x - a, y)$; $(x, y) \rightarrow (x - 3, y - 4)$
d. The product of the slopes of respective diagonals is ${}^-1$.
21. Find the image O' of the center of the circle O under the dilation with center A and scale factor $\frac{1}{2}$. Then construct the circle with center O' and radius $\frac{1}{2}$ of the radius of the circle. **22.** As the path is traced, the length of $XY + YZ$ is the same as the length AC. We know this because of 60° angles formed as the ball bounces at Y making $\angle BYZ$ have a measure of 60°. (*Hint:* Remember congruent angles of incidence and angles of reflection.) Thus $AXYZ$ is a parallelogram making $\overline{AX} \cong \overline{ZY}$ and $\overline{XY} \cong \overline{AZ}$. Thus $XY + YZ = AX + AZ$.

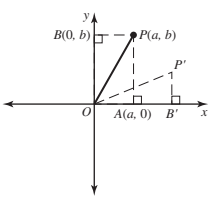
Also $\triangle BYZ$ is equilateral so $\overline{BZ} \cong \overline{ZY}$. Thus $XY + YZ = AZ + ZB$ or AB. Hence $XY + YZ = 1$ m. Similarly $ZW + WP = BC = 1$ m, and $PQ + QX = AC = 1$ m. Thus the length of the path is $3(1\text{ m}) = 3$ m. The perimeter of $\triangle ABC$ is also 3 m, so the ball travels a length equal to the perimeter.



Answers to Now Try This

1. a. One way to do this is to connect A and M. Then construct point C so that $AC = MN$ and $NC = AM$. Then C is the required image of A. **b.** Connect A and N.

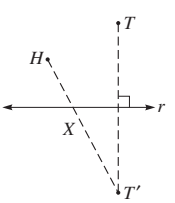
Find point D so that $AD = MN$ and $DM = AN$. Then D is the image of A under the translation that takes N to M. **2.** Use repeatedly a translation defined by the slide arrow from the lower left corner of a square to the lower right corner. **4. a.** Answers will vary. **b.** A figure with point symmetry is its own image under a half-turn. **c.** ℓ is its own image ℓ'. **d.** Connect P to any point X on ℓ. Find the point X' on $\overrightarrow{XP}$ so that $\overline{XP} \cong \overline{X'P}$. Construct ℓ' through X' parallel to ℓ. ℓ' is the desired image. **6.** Find the line such that the figure folds onto the image. The fold line is the reflecting line. **7.** The accompanying figure suggests the reason why the image of $P(a, b)$ is $P'(b, a)$. The line $y = x$ bisects the right angle in the first quadrant formed by the x- and y-axes. Let P' be the reflection image of P in the line $y = x$. Hence $\triangle OPB \cong \triangle OP'B'$. Thus $OB = OB' = b$ and $PB = P'B' = a$. The coordinates of P' are (b, a).



8. $MB = MB'$ and $BP = B'P$. Points A, P, and B' make a triangle. Therefore, they are not all on the same line and by the Triangle Inequality we know that $AP + PB'$ must be longer than AB'. A, M, and B' are collinear. Therefore, $AM + MB < AP + PB$.

Answer to the Preliminary Problem

Let H represent one house, X the roadside spot and T the other house. Let T' be the reflection of T in the road r as shown below. The intersection of $\overline{HT'}$ and r determines the point on the road at which X should be placed.



CHAPTER 14

Assessment 14-1A

7. Answers vary. **a.** Inches **b.** Inches **c.** Feet **8. a.** 0.35; 350 **b.** 163; 1630 **c.** 0.035; 3.5 **d.** 0.1; 10 **e.** 200; 2000 **9. a.** Decidollar **b.** Centidollar **c.** Dekadollar **d.** Hectodollar **e.** Kilodollar **11.** 6m; 5218 mm; 245 cm; 700 mm; 91 mm; 8 cm **14.** Answers vary. **a.** Approximately 7 cm **b.** Approximately 11 cm **23. b.** About Mach 4 **25. b.** The arc has to be longer because 8 cm is greater than 6 cm, and the radius is in the numerator of the fraction in the computation.

Assessment 14-1B

4. Answers vary. **7.** Answers vary. **a.** Inches **b.** Inches **c.** Feet **8. a.** 0.2; 200 **b.** 152; 1520 **c.** 0.09; 9 **d.** 110; 1100 **14.** Answers vary. **a.** 9 cm **b.** 17 cm **16.** Answers vary. **19. a.** Answers vary. For example, add 4 squares to the right end of the top row. **23. a.** About $9.5 \cdot 10^{12}$ km **25. b.** It is shorter because the ratio of the radii is 3:4, so the arc lengths must be in the same ratio.

Mathematical Connections 14-1

Reasoning

1. Answers vary. **a.** The metric system is a base-ten system and works "almost like" our money system. Conversions are much easier because of the decimal relationship between certain units of measure. **b.** Metric relationships are easier to remember. Also, because it is a base-ten system, conversions are easier. **c.** Original property descriptions and some building materials will probably not change. **2.** The outer curve has a greater radius and a correspondingly greater distance to run. To compensate for the extra distance, the runner in the outer lane is given an apparent head start. **3. a.** The square inscribed in the circle has perimeter $2r\sqrt{2}$. The square circumscribed about the circle has perimeter $4r$. **b.** $2.8 \approx 2\sqrt{2} < \pi < 4$ **4.** Answers vary. Possibly the centimeter measurement would be more accurate because the unit of measure is smaller.

Open-ended

5. a. Answers vary. For example: 1-1-1 and 2-2-1 work; 2-3-6 and 1-2-3 do not work. **b.** Using the Triangle Inequality, a triangle can be constructed if, and only if, $a + b > c$ and $a + c > b$ and $b + c > a$. **6.** Answers vary.

Cooperative Learning

7. a. Answers vary. **b.** Answers vary. **c.** Answers vary. Student should conjecture that the length of the outstretched arm is about the same as a person's height **d.** Answers vary. They should all agree with the conjecture in part (c). **8. a.** Answers vary. **b.** Regardless of the radius of each circle, the amount of wire needed is the same. **c.** Let the radius of each circle be r cm. The circumference of each circle is $2\pi r$. The number of circles is $\frac{60}{2r}$. (Why?) Hence the amount of wire, which is the total circumference of all the circles is $\left(\frac{60}{2r}\right)(2\pi r) = 60\pi$ cm. Because this number does not depend on the value of r, the amount of wire needed for the chain is the same regardless of the radius of each circle.

Connecting Mathematics to the Classroom

9. Though the question may seem confusing, some students may think that the circumference measures both the "outside" and "inside" of the curve instead of the curve itself. One typically discusses the length of the semicircle. **10.** The height is $3d$, where d is the diameter of a tennis ball. The perimeter of the can is πd or about $3.14d$, which is greater then $3d$. **11.** $\frac{22}{7}$ is an approximation. If more accuracy is needed, this approximation may not be accurate enough. **12.** The teacher should agree with her. This may not seem accurate but the height of the wire above Earth would be over 3 m.

Assessment 14-2A

4. a. 0.0588; 58,800 **b.** 0.000192; 1.92 **c.** 15,000; 1,500,000 **d.** 0.01; 10,000 **6. a.** 3 sq. units **b.** 3 sq. units **c.** 2 sq. units **d.** 5 sq. units **12. c.** Answers vary. For example, the metric

system is easier because you only have to move the decimal point to convert units within the system. **13. d.** Cannot tell because the height is unknown. The area is less than or equal to 60 cm^2.

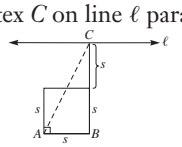
19. a. 16π cm^2 **20. e.** $1:1; 1:\left(\frac{\pi}{2}+1\right); 1:1; 1:(\pi-2):(4-\pi)$

31. a. 24 cm^2 **b.** 28 cm^2 **c.** 16 cm^2 **d.** 12 cm^2

32. $\frac{bh}{2}$ square units **33. b.** Answers vary. The graph does not lie along a line but rather along a curve known as a hyperbola.

Assessment 14-2B

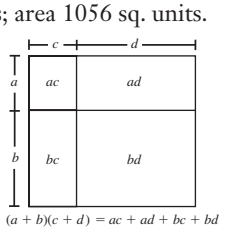
6. a. 6 sq. units **b.** 4.5 sq. units **9.** Answers vary. One possibility is to have vertex C on line ℓ parallel to $\overline{AB}$.



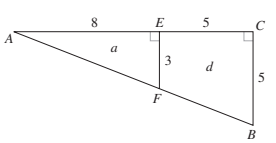
13. a. The shape of the rectangle becomes less like a square; that is, it becomes elongated. **b.** The shape of the rectangle becomes more like a square. The maximum area for minimum perimeters for a rectangle is when you have a square. **19.** The circle in the square has less wasted space. **20. a.** $\frac{\pi r^2}{4}$ **b.** $\frac{\pi r^2}{8}$ **c.** $\frac{\pi r^2}{16}$

d. 1:1; 3:1; 7:1 **21.** The area of the outside shaded portion is the same as the sum of the areas of the three inside shaded regions, 9π $in.^2$. **23. b.** The area is 1.21 times as great or the area is increased by 21%. **24.** $MA = 32$ units and $AT = 33$ units; area 1056 sq. units.

25.



26. Although the pieces look like they should fit together, they do not really fit. To see this, assume the pieces do fit. We then obtain the following figure:



Since $\triangle AEF \sim \triangle ACB$, we have $\frac{8}{13} = \frac{3}{5}$, which is a contradiction. This implies that pieces like those in the figure cannot fit together to form a triangle. In order for the pieces to fit together, the measure of $\overline{EF}$ must satisfy $\frac{8}{13} = \frac{EF}{5}$; hence, $EF = \frac{40}{13} = 3\frac{1}{13}$. Since $3\frac{1}{13}$ is close to 3, the discrepancy is so small that the pieces appear to fit. **27.** The first is a better buy. **29.** Triangle BDE has $\frac{1}{9}$ the area of triangle ABC. **30. a.** $\frac{1}{3}$ **b.** $\frac{1}{5}$ **c.** $\frac{3}{5}$ **d.** $\frac{1}{3}$ **e.** $\frac{5}{9}$ **f.** $\frac{9}{8}$ **36. b.** Answers vary. The graph does not lie along a line but rather along a curve known as a hyperbola.

Mathematical Connections 14-2

Reasoning

1. Answers vary. The triangle may be drawn on graph paper and its area estimated. Students may look up Heron of Alexandria to find the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is one-half of the perimeter of the triangle with sides a, b, and c. In this case, the area is $\sqrt{15(15-6)(15-11)(15-13)}$, or $\sqrt{1080}$, approximately 32.9 $in.^2$. **2.** John is correct if they were thinking about area. John's garden has area 900 ft^2 and Al's garden has area 450 ft^2, so John has twice the area that Al has for a garden. **3. a.** The area of the 10-in. pizza is 25π $in.^2$. The area of the 20-in. pizza is 100π $in.^2$. Because the area of the 20-in. pizza is 4 times as great, this pizza might cost 4 times as much, or \$40. However, this is not the case because other factors are considered rather than just the area of the pizza. **b.** If the price is based only on the area of pizza, then the ratio between prices should be $1:k^2$. **4.** Rotate the shaded region 180° about point E. The area of the triangle is the same as the area of the parallelogram that is formed. The area of the parallelogram is $\left(\frac{h}{2}\right)b$. **5.** After the rotations, a rectangle is formed. The area of the rectangle is length times width, which in the case of the parallelogram is the same as the base times the height. **6.** The original area is lw. The new area is $(1.1 \cdot l)(0.9w) = 0.99\,(lw)$. Therefore, the area is decreased by a factor of 0.01 or 1%. **7.** The new figure is a parallelogram with base $b_1 + b_2$ and height h, where b_1 and b_2 are the bases of the original trapezoid. The area of the parallelogram is $A = h(b_1 + b_2)$. Because this is twice the area of the original trapezoid, we divide by 2 yielding the formula for the area of the original trapezoid.

Open-ended

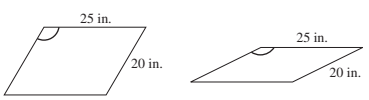
8. a. Answers vary depending on the size of the hand. **b.** Answers vary. For example, many people will trace their hands on square-centimeter paper and then count the number of squares that are entirely contained in the outline. Next, they count the number of squares that are partially contained in the hand outline and multiply this number by $\frac{1}{2}$. The final estimate is the sum of the two numbers. **9. a.** Answers vary; for example, a square that is 4 cm on a side and a rectangle that is 7 cm × 1 cm. **b.** Answers vary. For example, a square that is 4 cm on a side and a rectangle that is 8 cm × 2 cm. **10. a.** Construction **b.** Construction **c.** The length is approximately equal to half the circumference. The height is approximately equal to the radius r. Therefore $A = \frac{1}{2}Cr$. As the circle is cut into more and more sectors and put back together, the shape becomes more and more like a parallelogram.

Cooperative Learning

11. a. 5 sq. units **b.** 12 units **c. i.** 3 **ii.** 15 **iii.** 20 **d.** Answers vary depending on the chosen shape. **e.** Answers depend on the chosen shape. **12.** Answers vary.

Connecting Mathematics to the Classroom

13. Answers vary. For example, you could show her that with a given length of string you could enclose many different areas. For example, with a 16-in. string you could have a 4 in. × 4 in. square with area 16 in.^{2}. But you could also have a 1 in. × 7 in. rectangle with area 7 in.^{2}. If you formed a circle, you would have area over 20 in.^{2}. **14.** No. An angle is a union of two rays. The student probably means the area of the interior of an angle. However, because the area of the interior of an angle is infinite, it does not have a measurable area. **15.** We use square centimeters or square inches to indicate the area of a square 1 cm or 1 in. on a side. However, we cannot have a square with 1 *are* or 1 *hectare* on a side because *are* and *hectare* are not linear measures. The area of a square 10 m on a side is 1 a, and 1 ha is the area of a square 100 m on a side. **16.** The area of a garden does not depend on the perimeter of the garden. For example, a garden that is 2 m by 6 m has a perimeter of 16 m and an area of 12 m^2. A garden that is 4 m by 4 m also has a perimeter of 16 m but has an area of 16 m^2. **17.** If the parallelogram is also a rectangle, then what Jimmy says is correct, but if it is not a rectangle, the area becomes smaller as the marked angle becomes greater, as seen in the following figures. He needs to multiply the base times the height of the parallelogram to find the area.



18. a. Yes, because if $A = s^2$, then $s = \sqrt{A}$ and $p = 4s = 4\sqrt{A}$. **b.** No, for example, if $A = 20$, then the sides could be 4 and 5 and $P = 18$; or the sides could be 2 and 10 and $P = 24$. **c.** No, for example, if $A = 20$, then the side could be 5, the height 4 and $P = 4 \cdot 5 = 20$; or the side could be 10, and height 2, and $P = 4 \cdot 10 = 40$. **d.** Yes. $P = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{A\pi}$.

GeoGebra Activities

19. See Appendix answers.

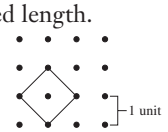
Review Problems

20. $\left(8 + \frac{\pi}{2}\right)$ ft, or approximately 9.57 ft **21. a.** Approximately 6326 km **b.** 9937.5 km **22.** $(2\pi - 6)r$, or approximately $0.28r$, is the difference. **23. a.** 1:3 **b.** 12πcm **24.** $(12 + 2\pi)$ cm **25. a.** 200 **b.** $\frac{1}{6}$ **c.** 0.0025 **d.** About 1.42

Assessment 14-3A

3. Answers vary. **a.** Make a right triangle with legs of length 2 and 3, and then the hypotenuse has the required length. **b.** Make a right triangle with legs of length 1 and 2, and then the hypotenuse has the required length.

4. Answers vary, for example,



16. The area of the trapezoid is equal to the sum of the areas of the three triangles. Thus,

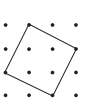
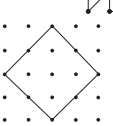
$$\frac{1}{2}(a + b)(a + b) = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$$
$$\frac{1}{2}(a^2 + 2ab + b^2) = ab + \frac{1}{2}c^2$$
$$\frac{a^2}{2} + ab + \frac{b^2}{2} = ab + \frac{c^2}{2}.$$

Subtracting ab from both sides and multiplying both sides by 2, we have $a^2 + b^2 = c^2$. The angle formed by the two sides of length c has measure 90°. (Why?) **17.** That the small quadrilateral is a square that is found on pages 848–849. The area of the large square is equal to the sum of the area of the small square and the areas of the four right triangles. Therefore, $(a + b)^2 = c^2 + 4\left(\frac{ab}{2}\right)$. Thus, $a^2 + 2ab + b^2 = c^2 + 2ab$, which in turn implies that $c^2 = a^2 + b^2$. **18. b.** The area of the largest triangle is equal to the sum of the areas of the other two triangles. **21.** $\frac{c\sqrt{3}}{4}$ using the 30°-60°-90° triangle relationship **27.** Both sprinkler systems cover the same percentage of the square field and it does not matter which system is used if the only selection criterion is the amount of land covered by the system. **28. a.** $90\sqrt{2}$ ft, or approximately 127 ft **b.** $\sqrt{10125} = 45\sqrt{5}$, or approximately 101 ft **c.** $\sqrt{10125} = 45\sqrt{5}$, or approximately 101 ft **30.** Any similar triangle will have sides of lengths $3k$, $4k$, and $5k$, and $(3k)^2 + (4k)^2 = (5k)^2$. **31.** The area is $\frac{1}{2}AC \cdot BC + \frac{1}{2}\left(\pi\left(\frac{1}{2}AC\right)^2 + \pi\left(\frac{1}{2}BC\right)^2\right) - \frac{1}{2}\pi\left(\left(\frac{1}{2}AB\right)^2 = \frac{1}{2}AC \cdot BC + \frac{1}{8}\pi(AC^2 + BC^2 - AB^2) = \frac{1}{2}AC \cdot BC$.

Assessment 14-3B

1. a. $\sqrt{8}$, or approximately 2.8 units **b.** $\sqrt{13}$, or approximately 3.6 units **3.** Answers vary. **a.** Make a right triangle with legs of length 1 and 3, and then the hypotenuse has the required length. **b.** Make a right triangle with legs of length 1 and 4, and then the hypotenuse has the required length. **4.** Answers vary. The perimeter of the figure shown is $13 + 2\sqrt{2} + \sqrt{5}$, which is certainly greater than 12.



5. a.  **c.** 

8. The longest piece of straight spaghetti will be $\sqrt{116} = 2\sqrt{29}$, or approximately 10.8 in. **10.** $\sqrt{10}$, or approximately 3.2 m **12. a.** $\frac{s^2\sqrt{3}}{4}$ **b.** $\frac{s^2}{2}$ **13. a.** $4\sqrt{21}$ cm^2 **b.** $2\sqrt{24}$ cm^2 or $4\sqrt{6}$ cm^2 **16.** $\triangle ACD \sim \triangle ABC$; $\frac{AC}{AB} = \frac{AD}{AC}$ implies

$\frac{b}{c} = \frac{x}{b}$, which implies $b^2 = cx$; $\triangle CBD \sim \triangle ABC$;
$\frac{AB}{CB} = \frac{CB}{DB}$ implies $\frac{c}{a} = \frac{a}{y}$, which implies
$a^2 = cx$; $a^2 + b^2 = cx + cy = c(x + y) = cc = c^2$.
17. It can be shown that the outside quadrilateral is a square of area c^2. The inside quadrilateral is a square with area $(b - a)^2$, or $b^2 - 2ab + a^2$. Each of the four right triangles has area $\frac{ab}{2}$. Therefore,

$$\begin{aligned} c^2 &= (b^2 - 2ab + a^2) + 4\left(\frac{ab}{2}\right) \\ &= b^2 - 2ab + a^2 + 2ab \\ &= b^2 + a^2. \end{aligned}$$

Therefore, $c^2 = a^2 + b^2$.
18. Yes; the areas are $\frac{\pi b^2}{8}$, $\frac{\pi a^2}{8}$, and $\frac{\pi c^2}{8}$ but $b^2 + a^2 = c^2$, so $\frac{\pi b^2}{8} + \frac{\pi a^2}{8} = \frac{\pi c^2}{8}$. **20.** The side lengths are $5, 7\sqrt{2}$, and 5, so the triangle is isosceles. **27. a.** $90\sqrt{2}$ ft, or approximately 127 ft **b.** $\sqrt{10125}$, or approximately 101 ft **c.** $\sqrt{10125} = 45\sqrt{5}$, or approximately 101 ft **29.** Any similar triangle will have sides of lengths $5k$, $12k$, and $13k$, and $(5k)^2 + (12k)^2 = (13k)^2$. **30.** $OC = x\sqrt{2}$, so $OB = x\sqrt{2}$; the areas of the respective squares are x^2 and $2x^2$. **32.** Draw altitudes $\overline{BE}$ and $\overline{DF}$ of triangles BCP and DCP, respectively. $\triangle ABE \cong \triangle CDF$ by AAS. Thus, $\overline{BE} \cong \overline{DF}$. Because $\overline{CP}$ is a base of $\triangle BCP$ and $\triangle DCP$, and because the heights are the same, the areas must be equal.

Mathematical Connections 14-3

Reasoning

1. Answers vary. **a.** Let the length of the side of the square be s. Draw the diagonal of the square and make the new square have side lengths equal to the diagonal. Then the area is $\left(\sqrt{2}s\right)^2 = 2s^2$. **b.** Make the lengths of the sides of the new square $\frac{1}{2}$ the length of the diagonal, or $\left(\frac{\sqrt{2}s}{2}\right)$; then the area of the new square is $\left(\frac{\sqrt{2}s}{2}\right)^2 = \frac{s^2}{2}$. **2.** Draw a perpendicular segment from Y to the segment representing 200 m. Draw segment $\overline{XY}$. A right triangle is formed with side lengths 100 m and 150 m. Using the Pythagorean theorem, we find that $XY = \sqrt{100^2 + 150^2} = \sqrt{32{,}500} \approx 180$ m. **3.** If a and b represent the lengths of the legs and c represents the length of the hypotenuse in the right triangle, then $c^2 = a^2 + b^2$. We are given that $c^2 + a^2 + b^2 = 200$. Substituting, we have $c^2 + c^2 = 200$ and $2c^2 = 200$ and $c^2 = 100$. Solving for c, we have $c = 10$.
4. Answers vary. Because $13 = 2^2 + 3^2$, we can construct a right triangle with legs 2 cm and 3 cm. **5. a.** Answers vary. For example, 6-8-10, 7-24-25, and 12-16-20. **b.** Yes; we know that if a-b-c is a Pythagorean triple, then $a^2 + b^2 = c^2$. This implies that $4(a^2 + b^2) = 4c^2$, and that $(2a)^2 + (2b)^2 = (2c)^2$.
c. No. For example, consider adding 2 to 3-4-5, which results in 5-6-7, not a Pythagorean triple.
d.

$$\begin{aligned} a^2 + b^2 &= (2uv)^2 + (u^2 - v^2)^2 \\ &= 4u^2v^2 + u^4 - 2u^2v^2 + v^4 \\ &= u^4 + 2u^2v^2 + v^4 \\ &= (u^2 + v^2)^2 \\ &= c^2 \end{aligned}$$

Therefore a-b-c is a Pythagorean triple. **6. a.** For the interior of the circle, the inequality should be $(x - h)^2 + (y - k)^2 < r^2$.
b. For the exterior of the circle, the inequality should be $(x - h)^2 + (y - k)^2 > r^2$.

Open-ended

7. Answers vary depending on the triangles drawn, but the Pythagorean theorem works only for the right triangles.
8. Answers vary.

Cooperative Learning

9. Answers vary depending on the triangles formed.

Connecting Mathematics to the Classroom

10. Yes, the same type of relationship does hold. For a proof and discussion, see G. Polya, *Mathematics and Plausible Reasoning*, Vol. 1 (Princeton, N.J.: Princeton University Press, 1954, pp. 15–17). Several examples were given in the Assessment sections. **11.** This can't be the equation of a circle because of the $^-16$ value. The radius squared is always a positive number.
12. If x is the length of the missing leg, then $3^2 + x^2 = 5^2$. Therefore, $x^2 = 5^2 - 3^2$ and $x = \sqrt{5^2 - 3^2}$, not $\sqrt{3^2 + 5^2}$.
13. The points lie in the interior of the circle with center at $(0, 0)$ and having radius equal to 1. **14.** What Sarah describes is what may happen, but is a misuse of measures for area. The clerk should have talked about square yards of carpet. **15.** A square rod is the area of a square with sides one rod in length, where a rod is a unit of length equal to 16.5 feet.

GeoGebra Activities

16. See Appendix answers. **17.** See Appendix answers.

Review Problems

18. 0.032 km^2, 3.2 m^2, 322 cm^2, 3020 mm^2 **19. a.** 33.25 cm^2 **b.** 30 cm^2 **c.** 32 m^2 **20. a.** 10 cm, 10π cm, 25π cm^2 **b.** 12 cm, 24π cm, 144π cm^2 **c.** $\sqrt{17}$ m, $2\sqrt{17}$ m, $2\pi\sqrt{17}$ m **d.** 10 cm, 20 cm, 100π cm^2 **21.** $\frac{25}{\pi}$ m^2 **22.** Samuel got the better buy by \$0.05 per yd. **23.** 6 cm

Assessment 14-4A

1. a. No **b.** Yes **c.** No **d.** Yes **2. a.** 96 cm^2 **b.** 216π cm^2 **c.** 236 cm^2 **d.** 64π cm^2 **e.** 96π cm^2 **8.** 10.5π in.^{2}, or approximately 33 in.^{2} **16.** $118{,}300{,}000\pi$ km^2, or approximately 370,000,000 km^2 **18. a.** $\sqrt{\frac{10648}{6}}$, or approximately 42 cm **b.** Approximately $42\sqrt{2} \approx 58.8$ cm

Assessment 14-4B

1. a. Yes **b.** Yes **c.** No **2. a.** 90 cm^2 **b.** 1500π ft^2 **c.** $\left(32\pi + 16\pi\sqrt{5}\right)$ cm^2 **3.** 3.25 L, so buy 4 L
4. a. They have equal lateral surface areas. **b.** The one with radius 6 m **6. a.** The surface area is multiplied by 4. **b.** The surface area is multiplied by 9. **c.** The surface area is multiplied by k^2. **7.** $45 + 6\sqrt{61}$, or approximately 92 ft^2

9. c. Yes; for example, place five cubes in the shape of a *C*. Then adding a cube to the center of the *C* would add no surface area. **11.** 507 cm^2 **12. a.** The figure will be a right circular cone, $100\pi\left(1 + \sqrt{5}\right)$ cm^2. **b.** The figure will be a right circular cylinder, 1350π cm^2. **c.** The figure will be a truncated right cone with large end radius 25 cm and small end with radius 15 cm, surface area 2250 cm^2. **13. a.** $100\left(1 + \sqrt{5}\right)$ in.^{2}, or approximately 323.6 in.^{2} **b.** $25\pi\left(1 + \sqrt{5}\right)$, or approximately 254.2 in.^{2} **14.** Yes; SA $= 66\pi$ ft^2, or approximately 207 ft^2, so he will have enough paint. **16.** The surface area of Jupiter is 121 times as great as that of Earth.

Mathematical Connections 14-4

Reasoning

1. a. Estimates vary. **b.** The height of the can is $3d$ or $6r$, where d is the diameter of the ball. The circumference of the can is $2\pi r$. Therefore, the *LSA* is $(2\pi r)6r = 12\pi r^2$. The surface area of three balls is $3(4\pi r^2) = 12\pi r^2$. Therefore, the surface area of the three balls is the same as the lateral surface area of the can. **2.** Doubling the radius **3.** No effect as 12 cm^2 were lost and 12 cm^2 were gained. **4.** $r = \ell$ because if $\pi r^2 = \pi r\ell$ then $r = \ell$. **5. a.** About 5720 mm^2 **b.** Answers vary depending on the ball.

Open-Ended

6. a. 2% **b.** Answers vary. **c.** Estimates vary depending on the size of the person and the desk. **d.** Answers vary. **7.** Answers vary. For example, draw a net for a right rectangular prism that is 4 cm × 4 cm × 3 cm.

Cooperative Learning

8. a.

Number of Faces Painted	Number of Pieces
6	0
5	0
4	0
3	8
2	12
1	6
0	1

b.

Number of Faces Painted	Number of Pieces
6	0
5	0
4	0
3	8
2	24
1	24
0	8

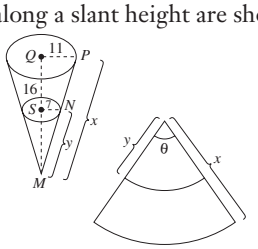
c. There could never be more than three faces painted. The number of pieces with three faces painted is always 8. The number of pieces with two faces painted is always a multiple of 12. The number of pieces with two faces painted is $12(n - 2)$. The number of pieces with one face painted is the number of pieces in the center square of each face. The length of the square is $(n - 2)$, so there are $(n - 2)^2$ pieces on each of the six faces. Therefore, there are $6(n - 2)^2$ pieces. The number of pieces with no faces painted is the number of interior cubes, which is a cube whose edge length is 2 less than the edge length of the original cube, or $(n - 2)^3$. **9.** The peeling of an orange can be used to demonstrate the formula for the surface area of a sphere, $SA = 4\pi r^2$.

Connecting Mathematics to the Classroom

10. She would need 9 times as much cardboard. If each measurement is tripled, then the area of each face is increased by a factor of 9; that is, $A_1 = \ell w$ and $A_2 = (3\ell)(3w) = 9\ell w = 9A_1$. Because this is true for all faces, the surface area is multiplied by 9. **11.** Jodi is incorrect. Because the surface area of a cone is $SA = \pi r^2 + \pi r\ell$, if she doubles the radius and halves the slant height she will have $SA = \pi(2r)^2 + \pi(2r)\frac{\ell}{2}$, which reduces to $SA = 4\pi r^2 + \pi r\ell$, which is not the same as the original surface area. **12.** Abi is correct. She is using the factored form of the formula for surface area of a cone. She should use the distributive property of multiplication over addition to confirm that the two forms are equivalent. **13.** Jan is incorrect. You could point out that if you double each side of a rectangle, you increase the area by a factor of 4. For example, $A_1 = \ell \cdot w$ and $A_2 = (2\ell)(2w) = 4(\ell \cdot w) = 4A_1$. You might also just look at a cube with side s. Then $SA = 6s^2$. If you double each side, then you have $SA = 6(2s)^2 = 6 \cdot 4s^2 = 4(6s^2)$, which is 4 times as great as the original surface area. **14.** She is not correct. For example, the surface area can be made as large as one wants by making a thinner and thinner pancake shape.

Review

15. a. 100,000 **b.** 1.3680 **c.** 500 **d.** 2,000,000 **e.** 1 **f.** 1,000,000 **16.** $10\sqrt{5}$ cm **17.** $20\sqrt{5}$ cm **18. a.** 240 cm, 2400 cm^2 **b.** $\left(10\sqrt{2} + 30\right)$ cm, 75 cm^2 **19.** Length of each side is 25 cm; diagonal is 30 cm long. **20.** $\sqrt{3}$ units **21.** The cone and the flattened region obtained by slitting the cone along a slant height are shown.



To construct the flattened ring we need to find x, y, and θ. Because $\triangle MQP \sim \triangle MSN$, we have $\frac{16 + MS}{MS} = \frac{11}{7}$. Hence, $MS = 28$ cm. In $\triangle MSN$, we have $28^2 + 7^2 = y^2$, or $y \approx 28.86$. In $\triangle PQM$: $x^2 = 11^2 + 44^2$, or $x \approx 45.35$ cm. To find θ, we roll the sector with radius y and central angle θ into the cone whose base is 7 cm and whose slant height is y. Hence, $2\pi y \cdot \frac{\theta}{360} = 2\pi \cdot 7$, or $\theta \approx \frac{7 \cdot 360}{28.86} \approx 87^\circ 19'$.

22. a. 0.15 **b.** 0.02 **c.** 1866.24 **d.** 2.54

Assessment 14-5A

14. The Great Pyramid has the greater volume. It is approximately 25.7 times as great. **16.** It is multiplied by 2.197; 119.7% increase. **17.** $66\frac{2}{3}\%$ occupied by the balls, so $33\frac{1}{3}\%$ occupied outside the balls. **20. a.** Answers vary. For example, a square base with sides 5 m and a height of 12 m. **b.** Infinitely many. Because $V = 100 = \frac{a^2 h}{3}$, where a is a side of the square base, then $300 = a^2 h$. This equation has infinitely many solutions. **28.** Answers vary. **a.** Draw any segment through the intersection point of the two diagonals that has endpoints on the parallel sides. There are infinitely many ways to do this. **b.** Consider any plane through the intersection line that contains the point of intersection of the two diagonal planes.

Assessment 14-5B

5. a. 6000, 6, 6000 **b.** 0.2, 0.2, 200 **c.** 1200, 1.2, 1.2 **d.** 3000, 3, 3000 **e.** 0.202, 0.202, 202 **f.** 6.5, 6.5, 6500 **9. a.** 1000, 1, 1 **b.** 19,200, 19.2, 19.2 **c.** 50 cm, 10,000,10 **d.** 40 cm, 12,000, 12 **10.** The larger melon is the better buy. The volume of the larger melon is 1.728 times the volume of the smaller melon, but the larger melon is 1.5 times as expensive. **14.** No; it is only $\frac{1}{3}$ of the volume for $\frac{1}{2}$ the price. **15.** $\frac{125}{18\pi}$, or approximately 2.2 cm **19.** $\sqrt[3]{\frac{1}{2}}$, or approximately 0.8 times the original radius **25. b.** 200,000 kg, or 200 t **28.** Answers vary. **a.** Draw any segment through the intersection point of the two diagonals that has endpoints on the parallel sides. There are infinitely many ways to do this. **b.** Use the plane determined by 2 parallel diagonals in opposite faces.

Mathematical Connections 14-5

Reasoning

1. a. Doubling the height will only double the volume. Doubling the radius will multiply the volume by 4. This happens because the value of the radius is squared when volume is computed. **b.** Yes. Doubling the height doubles the volume; doubling the radius multiplies the volume by 4. **2.** Answers vary. For example, fill the shape with sand and dump the sand into a container that can be used to measure the sand. **3.** The metric problem is much easier to work because the conversions are much easier. They just involve moving the decimal point. The relationships among length, volume, capacity, and mass are much easier than in the English system. **4.** Answers vary. An assumption is that comparable-sized rooms hold a comparable amount of furniture. **5. a.** We know that the height of each figure is $2r$ because the height of the sphere is $2r$. The volume of the cylinder is $\pi r^2 2r = 2\pi r^3$. The volume of the cone is $\frac{1}{3}\pi r^2 \cdot 2r = \frac{2}{3}\pi r^3$. The volume of the sphere is $\frac{4}{3}\pi r^3$. **b.** Using a common denominator of 3, the three formulas are $\frac{6}{3}\pi r^3$, $\frac{2}{3}\pi r^3$, and $\frac{4}{3}\pi r^3$. The ratio is 6:2:4, which simplifies to 3:1:2. **6. a.** The stack on the right has the greater surface area because the surface area on the left is only a part of that showing on the right. Each shifted coin adds additional surface area. **b.** The volumes are the same using Cavalieri's principle.

Open-Ended

7. Answers vary, but should have volumes close to 24π or 75.4 in.^{3}; for example, a rectangular prism that is 6 in. long, 4 in. wide, and 3.14 in. high. **8.** If the cookies are stood up so that the 6-cm diameter is toward the front of the box, then the minimum surface area is obtained by having two cookies across (12 cm), two cookies high (12 cm), and 12 cookies deep (12 cm). **9.** Answers vary, but the volume must be 1000 cm^3. Some students will worry about shelf space while others will worry about what shape is easiest to hold. **10.** Answers vary. For example, consider the same size cube of water and mercury. The mercury has the greater weight.

Cooperative Learning

11. a.–c. Answers vary. **12.** Answers vary. There are people who dispute Bergmann's rule.

Connecting Mathematics to the Classroom

13. Consider a cube with side s cm. We need $s^3 = 6s^2$. This happens when $s = 6$. The volume is 216 cm^3 and its surface area is 216 cm^2. **14.** The given box has a volume of 125 cm^3 and not 5 cm^3. The student most likely thinks 5 cm^3 means (5 cm)^3. **15.** For each degree change in Celsius, there is a $\frac{9}{5}$ degree change in Fahrenheit. When a person's temperature is 2 degrees above normal Celsius, it is $\left(\frac{9}{5}\right)^2$ or 3.6 degrees above normal Fahrenheit. Therefore, being 2 degrees above normal Celsius could be more serious than being 2 degrees above normal Fahrenheit. **16.** The volume of the original rectangular prism is given by $V_1 = \ell w h$. The volume of the new prism is $V_2 = (2\ell)(2w)(3h) = 2 \cdot 2 \cdot 3\,(\ell w h) = 12\ell w h = 12\,V_1$. Therefore, Andrea is correct. **17.** Suppose the volume of each of the small containers is $\pi r^2 h$. Then the total volume is $2(\pi r^2 h)$ for two containers. The volume of the large container is $\pi(2r)^2 h = \pi 4r^2 h = 4\pi r^2 h$. Therefore, the volume of the large container is twice that of the two small containers combined. **18.** Two figures are similar if they have the same shape, but may be different in size. Therefore, all cubes are similar and all spheres are similar.

Review Problems

19. a. $20 + 6\pi$ cm; $48 + 18\pi$ cm^2 **b.** 40π cm; 100π cm^2 **20. a.** 35 **b.** 0.16 **c.** 400,000 **d.** 0.00052 **21. a.** Yes **b.** No **22. a.** 2400π cm^2 **b.** $\left(6065 + 40\sqrt{5314}\right)$ cm^2 **23.** $6\sqrt{3}$ in. **24.** The diagonal from exercise 23 is the hypotenuse of a right triangle containing the diagonal of a face as a leg.

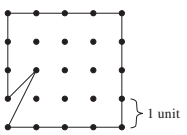
Chapter 14 Review

1. a. $16\frac{2}{3}$ **b.** $\frac{947}{1760}$, or approximately 0.538 **4.** $\frac{1.5}{\pi}$ m

5. She is incorrect. A circle with radius 6 cm has circumference 12π cm. She may be confusing area and circumference. **7.** The pieces of the trapezoid are rearranged to form a rectangle with width $\frac{h}{2}$ and length $(b_2 + b_1)$. The area is $A = \frac{h(b_1 + b_2)}{2}$, which is the area of the initial trapezoid. **8.** Area($\triangle ABC$) < Area($\triangle ABD$) = Area($\triangle ABE$) < Area($\triangle ABF$). All the triangles have the same base, so the ordering is just by height and $\triangle ABD$ and $\triangle ABE$ have the same height.

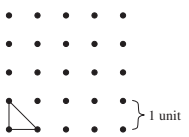
15. a. $SA = \left(64 + 32\sqrt{13}\right)\text{ cm}^2; V = 128\text{ cm}^3$
b. $SA = 96\pi\text{ cm}^2; V = 96\pi\text{ cm}^3$
c. $SA = 100\pi\text{ m}^2; V = \dfrac{500\pi}{3}\text{ cm}^3$
d. $SA = 54\pi\text{ cm}^2; V = 54\pi\text{ cm}^3$
e. $SA = 304\text{ m}^2; V = 320\text{ cm}^3$ **17.** The graph on the right has 8 times the volume of the figure on the left, rather than double as it should be.

23. a. Answers vary.



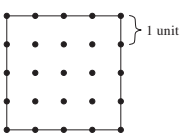
(a) Perimeter = $15 + \sqrt{2} + \sqrt{5} > 16$

b. Answers vary.



(b) Perimeter is $2 + \sqrt{2}$

c.



(c) Area = 16 units^2

Answers to Now Try This

4. a. $\dfrac{b}{2}\cdot\dfrac{h}{2} = \dfrac{bh}{4}$ **b.** The triangle was twice as large as the folded rectangle, so multiply by 2. Thus, the area of the triangle is $2\cdot\dfrac{bh}{4} = \dfrac{1}{2}bh$. **5. a.** You could build the triangle and then measure the angles to see whether there was a right angle. If the angle is a right angle, then the triangle is a right triangle. You could measure the three sides and use the Pythagorean theorem and its converse to see whether a right triangle is formed.
b. If the three lengths of a right triangle are multiplied by a fixed number, then the resulting lengths determine a right triangle; for example, if the right triangle lengths are 3-4-5, and the fixed number is 5, then 15-20-25 is a right triangle. **c.** If the three lengths of a right triangle are multiplied by a fixed number, then the resulting numbers determine a right triangle.
6. It makes no difference in the distance formula if $(x_1 - x_2)$ and $(y_1 - y_2)$ are used instead of $(x_2 - x_1)$ and $(y_2 - y_1)$, respectively. Because both quantities in the formula are squared, the result is the same whether the difference is positive or negative.
7. a. Move the decimal point twice as many places as in a linear conversion and in the same direction. For example, the area of a square that is 1 m on each side is 1 m^2; $1\text{ m} = 10\text{ dm}$; and $1\text{ m}^2 = 100\text{ dm}^2$. **b.** Move the decimal point 3 times as many places as in a linear conversion and in the same direction; $1\text{ m}^3 = 1000\text{ dm}^3$.

Answer to the Preliminary Problem

It would require $1.5\ \cancel{\text{acres}}\cdot\dfrac{4840\ \cancel{\text{yd}^2}}{1\ \cancel{\text{acre}}}\cdot 16\ \cancel{\text{in.}}\cdot\dfrac{1\ \cancel{\text{yd}}}{36\ \cancel{\text{in.}}}\cdot\dfrac{1\text{ load}}{10\ \cancel{\text{yd}^3}} \approx 323$ loads. Thus, it would require 323 loads.

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Glossary of Symbols

Symbol	Meaning
$=$	is equal to
a_n	nth term of a sequence
a^n	a to the nth power
S_n	sum of the first n terms of a sequence
$p \vee q$	p or q
$\sim p$	negation of p or not p
$p \wedge q$	p and q
$p \equiv q$	p is logically equivalent to q
$p \rightarrow q$	p implies q
$p \leftrightarrow q$	p implies q and q implies p; or p if, and only if, q
4_{five}	4 base five
$E2T_{\text{twelve}}$	base ten meaning $11 \cdot 12^2 + 2 \cdot 12^1 + 10 \cdot 1$
$\{a, b, c\}$	set containing elements a, b, and c
$\{x \mid \ldots\}$	set builder notaion
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$ or $\{\ \}$	empty set
$\subset$	is a proper subset of
$\subseteq$	is a subset of
$A \cup B$	union of sets A and B
$A \cap B$	intersection of sets A and B
U	universal set
$\overline{A}$	the complement of set A
$B - A$	set difference or complement of A relative to B
$A \sim B$	A is equivalent to B
$A \times B$	Cartesian product of sets A and B
$n(S)$	cardinal number of set S
$>$	is greater than
$<$	is less than
$\geq$	is greater than or equal to
$\leq$	is less than or equal to
$f(x)$	f of x, or output of f at x
(a, b)	ordered pair
$g \circ f$	composition of f with g
$(g \circ f)(x)$	$g(f(x))$

Symbol	Meaning
^{-}a	opposite of a or the additive inverse of a
$\lvert a \rvert$	absolute value of a
$a \mid b$	a divides b
$a \nmid b$	a does not divide b
$\sqrt{\ }$	principal square root
GCD	greater common divisor
LCM	least common multiple
$\frac{a}{b}$	fraction "a over b" or ratio, $a \div b$ with $b \neq 0$
$5\frac{3}{4}$	mixed number $5 + \frac{3}{4}$
a^0	$1, a \neq 0$
a^{-n}	$\frac{1}{a^n}, a \neq 0$
$\doteq$	is approximately equal
$\approx$	is approximately equal
$0.\overline{18}$	repeating decimal 0.18181818 . . .
$\sqrt[n]{\ }$	the nth root
$a^{\frac{1}{n}}$	a to the $\frac{1}{n}$ power; or nth root of a
$a^{\frac{m}{n}}$	$\sqrt[n]{a^m}$
%	percent
$P(E)$	probability of an event E
$P(\overline{A})$	probability of the complement of A
$n!$	n factorial which is equal to $n(n-1)(n-2)\cdot \ldots \cdot 3 \cdot 2 \cdot 1$ for $n \geq 1$
0!	zero factorial which equals 1
${}_nP_r$	number of permutations of n objects chosen r at a time
${}_nC_r$	number of combinations of n objects chosen r at a time
$P(B \mid A)$	conditional probability that event B occurs given that event A has occurred
E	mathematical expectation
$\overline{x}$	the arithmetic mean
MAD	mean absolute deviation

Symbol	Meaning
v	variance
s	standard deviation
IQR	interquartile range
Q_n	nth quartile
D_n	nth decile
P_n	nth percentile
$A, B, C, \ldots$	points
l, m, n	lines
$\overleftrightarrow{AB}$	line through A and B
$\overrightarrow{AB}$	ray AB
$\overline{AB}$	segment AB
$\overset{\circ}{AB}\overset{\circ}{}$	segment AB without endpoints
$\overline{AB}\overset{\circ}{}$ or $\overset{\circ}{AB}$	half open segment
AB	length of segment $\overline{AB}$
$m \parallel n$	m is parallel to n
$m \perp n$	m is perpendicular to n
$\angle ABC, \angle A, \angle\alpha,$ or $\angle 1$	an angle
$A\text{-}BC$	half-plane determined by line BC and containing point A
$A\text{-}BC\text{-}D$	dihedral angle determined by points A, D, and line BC
$^\circ$	degree symbol
$'$	minute symbol
$''$	second symbol
$m(\angle BAC)$	angle measure
$\urcorner$	right or 90° angle
$\cong$	is congruent to
$\overset{\frown}{AB}$	minor arc connecting A with B along a circle
$\overset{\frown}{ACB}$	arc connecting A with B and passing through C

Symbol	Meaning
$\triangle ABC$	triangle ABC
$\sim$	is similar to
$y = mx + b$	slope intercept form of a nonvertical line, where m is the slope and b the y-intercept
k	kilo (1000)
c	centi (0.01)
m	milli (0.001)
km	kilometer
m	meter
dm	decimeter (0.1 m)
cm	centimeter (0.01 m)
mm	millimeter (0.001 m)
a	are (100 m^2)
ha	hectare (100 a)
π	pi, the ratio between the circumference and the diameter of a circle
100°C	100 degrees Celsius
32°F	32 degrees Fahrenheit
L	liter
g	gram
kg	kilogram (1000 g)
mg	milligram (0.001 g)
t	metric ton (1000 kg)
V	volume
$S.A.$	surface area
A'	the image of A
$(x, y) \rightarrow (x + a, y + b)$	a translation that moves (x, y) to $(x + a, y + b)$
K	kelvin